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GROUPS OF CENTRAL TYPE AND SCHUR MULTIPLIERS WITH LARGE EXPONENT

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In commemoration of Reinhold Baer

Abstract. It is shown that finite groups with Schur multipliers of large exponent lead to groups of central type.

1. Introduction

Let G be a group with finite central factor group $G/Z(G)$. Then the commutator subgroup G' of G is finite by a well known theorem of Schur (cf. Huppert [Hu, IV.2.3], and see Baer [B] for related results). Let $\mathrm{M}(G) = \mathrm{H}_2(G, \mathbb{Z})$ denote the Schur multiplier of G. If we describe $\mathrm{M}(G)$ by means of a free presentation of G (the Hopf-Schur formula), this theorem implies that if G is finite, then so is $\mathrm{M}(G)$. Then, if e is the exponent of $\mathrm{M}(G)$, by another result of Schur e^2 is a divisor of $|G|$ ([Hu, V.23.9]). What does it mean when we have equality $e^2 = |G|$ here?

Theorem 1. *Let $G/Z(G)$ be finite and let e be the exponent of $M = G' \cap Z(G)$. Then:*

(a) *e^2 is a divisor of $|G : Z(G)|$.*
(b) *If $e^2 = |G : Z(G)|$ then $M = Z(G') \cong \mathrm{M}(G/Z(G))$ is cyclic and $|G'' \cap M|^2 = |G' : M|$ is relatively prime to $|G/G'Z(G)|$. Also, $G/Z(G)$ is solvable with derived length at most 3.*

Thus the hypothesis on G in (b) carries over to G', G'' and so on. We see that $G''M/M$ is a Hall subgroup of G'/M and that $G'' \cap M \cong \mathrm{M}(G'/M)$, etc.. Since the p-component of the Schur multiplier of a finite group, for any prime p, is isomorphic to a subgroup of the multiplier of a Sylow p-subgroup, we may also read off that all nontrivial Sylow subgroups of $G/Z(G)$ are abelian of rank 2, and even homocyclic. Indeed, the Schur multiplier of $G/Z(G)$ agrees with that of the direct product over a Sylow system of $G/Z(G)$ (in view of the

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Künneth theorem). Simple examples show that $G/Z(G)$ need not be abelian; there are examples where $G/Z(G)$ has derived length 3.

The proof of Theorem 1 reduces at once to the case where G is finite. Solvability of G in (b) is easily proved by a transfer argument in case $G/Z(G)$ has odd order. Then $G/Z(G)$ is even metabelian, i.e., has derived length at most 2. In the general case we make use of the fact that G must be a group of *central type*. This means that there is an irreducible (complex) character $\chi \in \mathrm{Irr}(G)$ such that $\chi(1)^2 = |G : Z(G)|$. Using the classification of the finite simple groups it has been shown by Howlett and Isaacs [HI] that groups of central type are solvable. In our situation we may avoid the classification theorem but we must still appeal to Walter's theorem [W] describing the finite simple groups with abelian Sylow 2-subgroups.

Let $Z^*(G)$ denote the (central) characteristic subgroup of G which is minimal subject to being the image in G of the centre of some central extension of G. The group $Z^*(G)$ is the image in G of the centre of any Schur cover of G (see [BFS] for a detailed discussion).

Theorem 2. *Let G be finite and e be the exponent of* $\mathrm{M}(G)$. *Then:*

(a) *e^2 is a divisor of $|G : Z^*(G)|$.*
(b) *If $e^2 = |G : Z^*(G)|$ then $\mathrm{M}(G) \cong \mathrm{M}(G/Z^*(G))$ is cyclic of order $|G : Z^*(G)|^{1/2}$ and G' is metabelian with $Z^*(G') = 1$ and with $\mathrm{M}(G')$ being isomorphic to the $\pi(G')$-component of $\mathrm{M}(G)$ (which has order $|G'|^{1/2}$).*

Here $\pi(G')$ denotes the set of primes dividing $|G'|$. Theorem 2 follows from Theorem 1 by considering a Schur cover of G; in (b) the Schur covers of G will be groups of central type again. Recall that any central group extension $Z \rightarrowtail G \twoheadrightarrow G/Z$ gives rise to a natural exact homology sequence

$$Z \otimes G/G' \to \mathrm{M}(G) \to \mathrm{M}(G/Z) \to G' \cap Z \to 1.$$

Here the map on the left is the Ganea (commutator) mapping, and the map on the right the co-transgression. One knows that $Z \subseteq Z^*(G)$ if and only if the Ganea mapping is the zero map (see Theorem 4.2 in [BFS]). We shall refer to this homology sequence several times.

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2. Groups of central type

In this section G is a finite group. We summarize some basic facts on groups of central type.

Lemma 1. *Let $\chi \in \mathrm{Irr}(G)$ be an irreducible character. Then $\chi(1)^2 \leq |G : Z(G)|$. Equality holds if and only if χ vanishes outside $Z(G)$.*

For a proof see Isaacs [I, (2.30)]. Note that if $\chi(1) = e$ and $e^2 = |G : Z(G)|$, then the restriction $\chi_{Z(G)}$ equals $e\varphi$ for some unique linear character φ of $Z(G)$, and the induced character is $\varphi^G = e\chi$ by Frobenius reciprocity. So G is of central type provided some irreducible character of $Z(G)$ is *fully ramified* in G.

Lemma 2. *G is of central type if and only if all Sylow subgroups P of G are of central type, with $P \cap Z(G) = Z(P)$.*

This is Theorem 2 in [DJ]. Theorem 3 in [DJ] gives the following.

Lemma 3. *Suppose G is a p-group of central type for some prime p. If $Z(G)$ is cyclic and $Z_0/Z(G)$ is a normal subgroup of $G/Z(G)$ of order p, then $G_0 = C_G(Z_0)$ is a group of central type with $Z(G_0) = Z_0$.*

Lemma 4. *If G is a group of central type and $G/Z(G)$ has abelian Sylow 2-subgroups, then G is solvable.*

This is true without the assumption on the Sylow 2-subgroups [HI]. The lemma may be proved along the lines given in [HI] by referring to Walter [W].

3. Symplectic actions

Let p be a prime. Let P be a finite group such that $P/Z(P)$ is abelian of type (p^a, p^a) for some integer $a \geq 1$ and such that P' is cyclic of order p^a. Examples of such groups are the Schur covers of abelian groups of type (p^a, p^b) with $b \geq a$ (see Baer's result as stated in Proposition 7.3 of [BFS]). Suppose H is a finite p'-group acting on P and centralizing $Z(P)$.

Lemma 5. *Either H acts trivially on P or $[P, H]$ covers $P/Z(P)$.*

Proof. Suppose that H acts nontrivially on P. Since H is a p'-group and P' is a p-group contained in the Frattini subgroup of the (nilpotent) group P, the group H acts nontrivially on P/P'. If H centralized the p-group $P/Z(P)$, it would act as a p-group on P/P', because it centralizes $Z(P)/P'$. It follows that H, being a p'-group, acts nontrivially even on the Frattini factor group V of $P/Z(P)$. It suffices to show that $[V, H] = V$.

Since H centralizes $Z(P)$, it respects the symplectic form on V induced by the (bilinear) commutator mapping $P/Z(P) \times P/Z(P) \to P'$. It follows that H acts on V as a p'-subgroup of the symplectic group $\mathrm{Sp}(V) = \mathrm{Sp}_2(p)$. We have $V = [V, H] \times C_V(H)$ (Maschke), with $[V, H] \neq 1$. If $[V, H] \neq V$ then V acts as a group of diagonal matrices on V having at least one entry

1. But all these matrices have determinant 1. Thus H must centralize V, a contradiction. □

Lemma 6. *Suppose H' is an abelian Hall subgroup of H. If H' is non-trivial on P, the exponent of H/H' is divisible by* 4.

Proof. By Lemma 5 we know that $[P, H']$ covers $P/Z(P)$. As before we consider the action of H on the Frattini factor group V of $P/Z(P)$. So H acts on V symplectically. We may identify $X = H/C_H(V)$ with a p'-subgroup of $\mathrm{Sp}_2(p)$. By hypothesis $X' \neq 1$ is an abelian Hall subgroup of X. This forces p to be odd (and even $p \geq 5$). Now H is an M-group ([Hu, V.18.4]). Enlarging the field of scalars, if necessary, we may likewise describe X as a group of monomial 2×2-matrices (with determinant 1). It follows that X has a cyclic subgroup of index 2.

The Sylow 2-subgroups of $\mathrm{Sp}_2(p)$ are generalized quaternion groups. The unique (central) involution of $\mathrm{Sp}_2(p)$ must belong to X. We conclude that $X' \neq 1$ has odd order and that X/X' is cyclic of order divisible by 4. □

Example. Suppose p is odd and q is an odd prime dividing $p^2 - 1$. Let P be a Schur cover of an abelian p-group of type (p^a, p^a) for some integer $a \geq 1$, and let Q be a Schur cover of an abelian q-group of type (q^b, q^b) for some integer $b \geq 1$. Then there is a symplectic action of Q on P such that $C_Q(P) \supseteq Q'$ has index q in Q. The semidirect product PQ is a Schur cover of $(P/P')(Q/Q')$ with $Z(PQ) = P' \times Q'$.

Let R be a Schur cover of an abelian 2-group of type $(2^c, 2^c)$ with $c \geq 2$. There is a symplectic action of R on Q such that $C_R(Q)$ has index 2. Thus R acts on Q through the central involution in $\mathrm{Sp}_2(q)$ inverting the elements of Q/Q'. The semidirect product $H = QR$ has a homomorphic image in $\mathrm{Sp}_2(p)$ of order $4q$, the kernel in R being a subgroup $C \subset C_R(Q)$ containing R'. Of course, R/C is cyclic of order 4 and $C_R(Q)/C$ maps onto the centre of $\mathrm{Sp}_2(p)$. Let $G = PH$ be the semidirect product with respect to the resulting symplectic action of H on P. This is a Schur cover of $(P/P')[(Q/Q')(R/R')]$ with

$$Z(G) = P' \times Q' \times R'.$$

Moreover, $G'' = PQ'$ and $G''' = P'$.

4. The primary case

The crucial step in proving Theorem 1 is to handle the situation where $G/Z(G)$ is a p-group for some prime p. Here we have the following result.

Proposition. *Let $G/Z(G)$ be a finite p-group and let e be the exponent of $M = G' \cap Z(G)$. Then:*

(a) *e^2 is a divisor of $|G : Z(G)|$.*

(b) *If* $e^2 = |G : Z(G)|$ *then* $G' = M \cong \mathrm{M}(G/Z(G))$ *is cyclic and* $G/Z(G) \cong M \times M$.

Proof. Let $Z = Z(G)$. We know that M is finite. Let $\varphi : M \to \mathbb{Q}/\mathbb{Z}$ be a linear character (homomorphism). Since $\mathbb{Q}/\mathbb{Z}$ is divisible, there is an extension of φ to Z, say $\widehat{\varphi} : Z \to \mathbb{Q}/\mathbb{Z}$. By construction $\widehat{\varphi}$ has finite order; we may choose $\widehat{\varphi}$ such that its order is a p-power. Let χ be an irreducible character of $G/\operatorname{Ker}(\widehat{\varphi})$ occurring in the induced character $\widehat{\varphi}^G$.

The determinantal character of χ, when restricted to M, is $\varphi^{\chi(1)}$. Thus $\varphi^{\chi(1)} = 1$ as $M \subseteq G'$. Since $G/\operatorname{Ker}(\widehat{\varphi})$ is a finite p-group whose centre contains $Z/\operatorname{Ker}(\widehat{\varphi})$, by Lemma 1 and a familiar property of irreducible character degrees $\chi(1)^2$ is a divisor of $|G : Z|$. Thus the order $o(\varphi)$ of φ divides $|G : Z|$. This gives (a).

Now suppose $e = p^a$ and $|G : Z| = p^{2a}$. The result is obvious for $a = 0$, while if $a = 1$, the group G/Z is necessarily elementary abelian. So we may assume that $a \geq 2$. Once it has been proved that G/Z is homocyclic of type (p^a, p^a), the bilinear commutator mapping $(Zx, Zy) \mapsto [x, y]$ will show that $G' = M$ is cyclic of order p^a.

Let $\varphi : M \to \mathbb{Q}/\mathbb{Z}$ be a linear character of order $e = p^a$. As before choose an extension $\widehat{\varphi}$ to Z of (finite) p-power order ($\geq p^a$), and let χ be an irreducible constituent of $\widehat{\varphi}^G$. Then $\varphi^{\chi(1)} = 1$ and $\chi(1)^2$ is a divisor of $|G/\operatorname{Ker}(\widehat{\varphi}) : Z(G/\operatorname{Ker}(\widehat{\varphi}))|$, which in turn divides $|G : Z| = p^{2a}$. We conclude that Z maps onto $Z(G/\operatorname{Ker}(\widehat{\varphi}))$ and that $\chi(1) = p^a$. Hence $G/\operatorname{Ker}(\chi)$ is a group of central type, and without loss we may assume that $\operatorname{Ker}(\chi) = 1$. Then $Z = Z(G)$ is finite and cyclic. By construction $G' \cap Z$ still has order p^a (and is cyclic).

By Lemma 3 there is a normal subgroup G_0 of G of index p such that $|Z(G_0) : Z| = p$, with G_0 of central type. Thus $|G_0 : Z(G_0)| = p^{2(a-1)}$. Applying the transfer from G to G_0 shows that the exponent of $G_0' \cap Z(G_0)$ is (at least) p^{a-1}. Arguing by induction on a we thus may assume that $G_0/Z(G_0)$ is abelian of type (p^{a-1}, p^{a-1}). It follows that G_0' is the (unique) subgroup of order p^{a-1} of the cyclic group $Z = Z(G)$.

Of course, $G' \cap Z$ contains G_0' with index p. It follows that $G' = [G_0, y]$ for any $y \in G \smallsetminus G_0$. The map $x \mapsto G_0'[x, y]$ being a homomorphism $G_0 \to G_0/G_0'$, there is $x \in G_0$ such that

$$G' \cap Z = \langle [x, y] \rangle.$$

Now consider the subgroup $\widetilde{G}$ of G generated by x, y and Z. Since $[x, y]$ is in the centre of G (and of $\widetilde{G}$), we have $[x^n, y^m] = [x, y]^{nm}$ for all integers n, m. Since $[x, y]$ has order p^a, we see that both Zx and Zy have order (at least) p^a in G/Z. Similarly, we must have $\langle x \rangle \cap \langle y \rangle \subseteq Z$. Thus $G = \widetilde{G} = \langle Z, x, y \rangle$ and G/Z is homocyclic of type (p^a, p^a), as desired.

Note finally that the Schur multiplier of an abelian p-group of type (p^a, p^a) is cyclic of order p^a. □

5. Proof of Theorem 1

Let $Z = Z(G)$. Let M_p be the p-component of $M = G' \cap Z$ for some prime p, and let e_p be the exponent of M_p. Assume further that P/Z is a Sylow p-subgroup of G/Z. The transfer from G to P shows that $M^{|G:P|} \subseteq P' \cap M$. Hence $M_p \subseteq P'$. We even have

$$M_p = P' \cap Z,$$

because $P' \cap Z$ is a p-group. Indeed, $P' \cap Z$ is the image of the p-group $\mathrm{M}(P/Z)$ under the co-transgression resulting from the central extension $Z \rightarrowtail P \twoheadrightarrow P/Z$. By the proposition e_p^2 is a divisor of $|P : Z(P)|$, which in turn divides $|P : Z|$. We infer that e^2 is a divisor of $|G : Z|$.

Now assume that $e^2 = |G : Z|$. Then $e_p^2 = |P : Z|$ and so necessarily $Z(P) = Z = Z(G)$. By the Proposition P/Z is homocyclic of type (p^a, p^a) for some integer $a \geq 0$, and M_p is cyclic of order p^a. In particular, $M_p \cong \mathrm{M}(P/Z)$. Since this holds true for all primes, we see that $M \cong \mathrm{M}(G/Z)$ is cyclic. (Notice that the p-component of $\mathrm{M}(G/Z)$ is isomorphic to a subgroup of $\mathrm{M}(P/Z)$.)

Arguing as in the proof of the Proposition, we may assume now that G is finite and that $Z = Z(G)$ is a cyclic group whose order is divisible only by primes dividing $|M|$. (Extend a faithful linear character of M suitably to Z, and pass to the quotient group modulo the kernel of such an extended character.) Note that

$$|G : Z| = |M|^2.$$

Writing $P = P_0 Z$ for some Sylow p-subgroup P_0 of G, we may infer from $Z(P) = Z = Z(G)$ that $Z(P_0) = Z \cap P_0$. It follows from the proof of the Proposition that P_0 is a group of central type. Thus Lemma 2 yields that G is a group of central type. In particular, G is solvable (Lemma 4). Since all Sylow subgroups of G/Z are abelian of rank at most 2, the derived length of G/Z is at most 3 by Satz VI.14.18 in [Hu].

Let H be a p-complement in the normalizer $N_G(P) = N_G(P_0)$ (Schur-Zassenhaus). By Lemma 5 either $[P, H] = 1$ or $[P, H]$ maps onto P/Z. In the former case $N_G(P) = P_0 H$ centralizes P/Z and so

$$G' \cap P = G' \cap Z = M$$

by Burnside's transfer theorem. Then $M_p = G' \cap P_0$ is the Sylow p-subgroup of G'. In the latter case $P \subseteq G'Z$ and so $(G' \cap P)/M \cong P/Z$ is abelian of type (p^a, p^a). From $Z(P) = Z$ it follows that $Z(G' \cap P) = M$. (If $|G/Z|$ is odd and p is the smallest prime divisor of $|G/Z|$, the former case must happen in view of the order of $\mathrm{Sp}_2(p)$. In this way we obtain that G/Z is solvable by induction.)

We deduce that $G'Z/Z \cong G'/M$ is a Hall subgroup of G/Z. Also $Z(G') = M$ as $Z(P_1) = M_p$ for any Sylow p-subgroup P_1 of G' (and any p; observe that $P_1 = G' \cap P_0$ covers $(G' \cap P)/M$.) If p is a divisor of $|G'/M|$ then

$M = Z(G' \cap P)$ and so

$$(G' \cap P)' = P' = M_p.$$

In this case $P_1' = M_p = Z(P_1)$, that is, P_1 is a Schur cover of an abelian group of type (p^a, p^a). The proof is complete.

6. Proof of Theorem 2

Let $Z^* = Z^*(G)$. Recall that $Z^* = Z(E)/M$ for any Schur cover $M \rightarrowtail E \twoheadrightarrow G$ of G. Thus $G/Z^* \cong E/Z(E)$. So large portions of Theorem 2 follow from Theorem 1 (with E in place of G). Observe that $M \cong \mathrm{M}(G))$ and that $M \subseteq E' \cap Z(E)$. In particular, $E'/M \cong G'$ and the exponent e of $\mathrm{M}(G)$ is a divisor of $\widetilde{e} = \exp(E' \cap Z(E))$.

By Theorem 1 (a) $\widetilde{e}^2$ is a divisor of $|E : Z(E)| = |G : Z^*|$. Suppose we have $e^2 = |G : Z^*|$. Then $E' \cap Z(E)$ must be cyclic of exponent $\widetilde{e} = e$ by Theorem 1 (b). Thus $E' \cap Z(E) = M$ and so $G' \cap Z^* = 1$. The exact homology sequence for $Z^* \rightarrowtail G \twoheadrightarrow G/Z^*$ yields $\mathrm{M}(G) \cong \mathrm{M}(G/Z^*)$ (in view of Theorem 4.2 in [BFS]). We also know from Theorem 1 that G/Z^* is solvable with derived length at most 3. It follows that G' is metabelian. From $Z(E') = M$ we infer that $Z^*(G') = 1$.

The remainder is straightforward.

Corollary. *Suppose we have $e^2 = |G : Z^*(G)|$ for the exponent e of the Schur multiplier of the finite group G. If $|G/G'|$ is not divisible by 2^4, then G is metabelian.*

Proof. Again let again $Z^* = Z^*(G)$, and assume that $G'' \neq 1$. We know that G'' is an abelian Hall subgroup of G' and that $G'Z^*/Z^* \cong G'$ is a Hall subgroup of G/Z^*. Thus G'' is the nilpotent residual of G' and so $[G'', G'] = G''$. Let P be the Sylow p-subgroup of G'' for some prime p dividing $|G''|$. Then P is normal in G and $C_G(P) \supseteq G''Z^*$. So $H = G/C_G(P)$ is a p'-group, H' is abelian, and $[P, H'] = P$. From Lemma 6 it follows that the exponent of H/H' is divisible by 4. Hence the exponent of $G/G'Z^*$ is divisible by 4. But the Sylow 2-subgroup of $G/G'Z^*$ is of type $(2^a, 2^a)$ for some integer a, which must satisfy $a \geq 2$ now. Consequently 2^4 divides $|G/G'|$ and we are done. □

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