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THE BAER SPLITTING PROBLEM IN THE TWENTYFIRST CENTURY

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Abstract. The Baer splitting problem from the 1930s is revisited, after which, using current knowledge about maximal Cohen-Macaulay modules, the structure of Baer modules over regular integral domains of higher Krull dimension is explored. In particular, the countably generated ones in the local case are shown to be free.

1. Introduction

Reinhold Baer was a pioneer in the study of mixed abelian groups. In [2] he began to untangle the mystery of what conditions are necessary/sufficient in order for a mixed abelian group to split, that is, when its subgroup of torsion elements should be a direct summand. The general problem of recognizing when an extension of abelian groups $0 \to T \to M \to G \to 0$ is split, where T is torsion and G is torsion-free, remains a thorny issue to this day. Of course one can determine the existence of nonsplit extensions by computing the abelian group structure of $\mathrm{Ext}^1_{\mathbb{Z}}(G,T)$ in special cases.

Baer's early research in the subject centered around finding necessary and sufficient conditions so that "universal" splitting occurs. For example, Baer [2] characterized the structure of torsion abelian groups T so that $0 \to T \to M \to G \to 0$ would always split regardless of the structure of the torsion-free abelian group G. He found, as did Fomin [6], that such groups T are direct sums $B \oplus D$, where B is bounded (i.e., $nB = 0$ for some $n > 0$) and D is divisible. The corresponding problem for torsion-free abelian groups proved to be a bit more difficult. In [2] Baer managed to show that such a torsion-free abelian group G is necessarily free when it is countable. Rotman [23] referred to such groups as "Baer groups". In Rotman's homological formulation, Baer groups were those groups G such that $\mathrm{Ext}^1_{\mathbb{Z}}(G,T) = 0$ for all torsion groups T.

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As it turns out, in the summer of 1967 I was in a rather lucky position to attack this problem. My advisor Paul Hill and his former Ph.D. student, Charles Megibben, had been successfully using a technique which they called the "back and forth method" (see [13]). I. Kaplansky viewed this set theoretic technique as being derived from "counting principles" of Cantor. Kaplansky had successfully applied a version of this principle in his famous article [17] on projective modules. In essence this technique could be used to establish that certain properties (usually related to countability) of members in a family would carry over to the direct sum of the family. In my case the family consisted of countable abelian groups $\{M_i\}_{i\in I}$ (the size of the index set did not matter) such that

- (†) $M_i/T_i \cong \mathbb{Q}$ (rational numbers), where T_i is the torsion subgroup of M_i, and
- (††) each non-zero torsion-free subgroup of M_i is isomorphic to $\mathbb{Z}$ (integers).

The "method" employed by Hill and Megibben could be used to argue that each torsion-free subgroup of the direct sum $\coprod_{i\in I} M_i$ is free as well. With $T = \coprod_{i\in I} T_i$ and $M = \coprod_{i\in I} M_i$ it was then a simple matter to choose an index set I sufficiently large so that a "Baer group" G would embed into M/T, and then use the property $\mathrm{Ext}^1_{\mathbb{Z}}(G,T) = 0$ in order to lift the embedding $G \hookrightarrow M/T$ to an embedding $G \hookrightarrow M$, thus establishing G must be free (see [9] for details).

Returning to the countable case of Baer groups for the moment, one sees that "modern" proofs that countable Baer groups are free (see [8, Chapter 6]) are a by-product of Baer's investigation [3] into the structure of homogeneous separable abelian groups. In the final analysis, a countable torsion-free abelian group is free provided this is the case for each of its subgroups of finite rank. Applying this same circle of ideas to countably generated modules over the (complete) p-adic integers $\hat{\mathbb{Z}}_{(p)}$ led H. Prüfer [25] (see also Kaplansky's monograph [16, p 48] and Rotman's article [22]) to conclude that countably generated "reduced" (= separated in the p-adic topology) $\hat{\mathbb{Z}}_{(p)}$-modules are free. The work of Raynaud and Gruson [21] in 1971 suggested one might expect generalizations of results on torsion-free modules over DVR's ("discrete valuation rings") to hold over regular local rings of higher Krull dimension, e.g., over rings of the form $V[[X_1,\dots,X_n]]$, where V is a field or DVR. For a regular local ring R and faithfully flat R-module F, Raynaud and Gruson [21, 3.1.3] proved that $\hat{R}\otimes_R F$ is $\hat{R}$-free if and only if F is R-free (here $\hat{R}$ denotes the completion of R with respect to the maximal ideal topology). For Krull $\dim R = 1$, i.e., when R is a DVR, this statement follows from the fact that Baer modules (as defined above) are necessarily free (see the remarks in Section 2 after Proposition 2.1). In addition Raynaud and Gruson observed [21, 3.1.5] that countably generated pure submodules of free modules are again free, thus suggesting the uncomplicated nature of countably generated torsion-free modules. To emphasize this point I will establish in Theorem 4.3 a generalization of Prüfer's theorem [25] for complete regular local rings. In turn this result will be used to prove that countably generated "Baer modules" for regular local rings in higher Krull dimension are necessarily free (Theorem 4.5). Throughout the remainder of this article a Baer module C over a (regular) domain R is one that has the property $\operatorname{Ext}^j_R(C,T) = 0$, for all torsion R-modules T and all $j > 0$.

Having attempted to provide a brief historical perspective regarding the Baer splitting problem and having hinted at modern developments in the theory of torsion-free modules over local rings, I will offer a survey of what can be said about the Baer splitting problem when one replaces the ring of integers $\mathbb{Z}$ by a regular domain. As I have suggested in the preceding discussion there are sufficient results available at the time of this writing so that one can resolve the general problem (in the local case) to the point Baer had come in his 1936 article [2] for the ring of integers.

Terminology and background material that perhaps will facilitate the ease at which one digests the upcoming discussion may be found in Fuchs' volumes [7], [8] on abelian groups, Rotman's book [24] on homological algebra and either of the graduate texts by Atyah-MacDonald [1] or Matsumura [19] for a standard treatment of commutative algebra (especially the chapters on flatness and completion). In addition I recommend that some knowledge of Warfield's article [26], and perhaps my paper [12, Section 3], on purity would be helpful.

2. Reductions of the general Baer problem

Let R be a regular integral domain; so R_P is a regular local ring for each $P \in \operatorname{Spec} R$. If one takes the position that Baer modules as defined in the Introduction must surely turn out to be projective, then one may reduce to the case $\operatorname{pd} C \leq 1$ whenever R has finite Krull dimension. This is an immediate consequence of the general fact that pd $C \leq \dim R$ (see [21, 3.2.6 (p. 84)]) and the observation that syzygies of Baer modules are again Baer modules. In actual fact a much stronger result is true which was noticed by I. Kaplansky [18] in 1962 (an article that was a tribute to Reinhold Baer's sixtieth birthday). Using a simple but clever homological argument Kaplansky showed that a Baer module C satisfies $\operatorname{pd} C \leq 1$ for *any* integral domain R. I repeat his argument below.

Proposition 2.1 (Kaplansky). *Let R be an integral domain and let C be a Baer module. Then* pd $C \leq 1$.

Proof. It suffices to argue that $\operatorname{Ext}^2_R(C, M) = 0$ for any R-module M. For such an R-module M, let the short exact sequence $0 \to M \to E \to T \to$

0 represent an embedding of M into its injective envelope E. Then T is necessarily a torsion module. From the long exact sequence for Ext

$$0 = \operatorname{Ext}^1_R(C,T) \longrightarrow \operatorname{Ext}^2_R(C,M) \longrightarrow \operatorname{Ext}^2_R(C,E) = 0$$

one concludes $\operatorname{Ext}^2_R(C,M) = 0$. $\square$

Corollary 2.2. *With the notation as above, an R-module C is a Baer module if and only if $\operatorname{Ext}^1_R(C,T) = 0$ for all torsion R-modules T.*

Our second reduction of this section is to notice that being a Baer module localizes, that is, one may begin with an analysis in the context of a local domain R. Since the modules involved are not necessarily finitely generated, the claim perhaps needs a formal argument.

Proposition 2.3. *Let R be an integral domain and let C be an R-module. If $\operatorname{Ext}^1_R(C,T) = 0$ for all torsion modules T, then the same is true for C_P over R_P.*

Proof. The key observation here is that a torsion R_P-module T is also torsion as an R-module. Hence, for any extension $0 \to T \to E \to C_P \to$ over R_P, one obtains the R-commutative pullback diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & T & \longrightarrow & E' & \longrightarrow & C & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & T & \longrightarrow & E & \longrightarrow & C_P & \longrightarrow & 0
\end{array}$$

in which $E' = T \oplus C$ since $\operatorname{Ext}^1_R(C,T) = 0$. It follows that $E = T \oplus C_P$ since $E'_P = E$, that is, the top row localizes to the bottom row. $\square$

I will make yet a further reduction as a consequence of some fairly deep results of Raynaud and Gruson [21, pp. 80–84] on faithfully flat and projective modules. This can be motivated by looking at the case in which $\dim R = 1$, i.e., the case in which R is a DVR. Let C be an R-module such that $\hat{R} \otimes_R C$ is free as an $\hat{R}$-module. If T is any torsion R-module, then one observes that $\hat{R} \otimes_R T \cong T$ since T is already a direct limit of $\hat{R}$-torsion modules (keep in mind that R is a DVR here). Thus, given any extension $0 \to T \to E \to C \to 0$ one may apply the functor $\hat{R} \otimes_{\bullet}$ and obtain the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & T & \longrightarrow & E & \longrightarrow & C & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & T & \longrightarrow & \hat{R} \otimes E & \longrightarrow & \hat{R} \otimes C & \longrightarrow & 0.
\end{array}$$

The bottom row splits since $\hat{R} \otimes C$ is assumed $\hat{R}$-free. This yields a commutative triangle

$$\begin{array}{ccccc} 0 & \longrightarrow & T & \longrightarrow & E \\ & & \Big\| & \swarrow & \\ & & T & & \end{array}$$

which implies the original extension $0 \to T \to E \to C \to 0$ is necessarily R-split. It follows that C is a Baer module for R; so C is R-free according to [9]. Raynaud and Gruson [21, 3.1.3] establish a higher dimensional analogue and more general version of this result. They show, if R is any local ring and if C is any R-module such that the completion $\hat{R}$ has the property $\hat{R} \otimes C$ is $\hat{R}$-free, then C is necessarily R-free.

From the above observations one can reduce the general Baer problem for regular domains as follows.

Remark 2.4. Let R be a regular domain and let C be a Baer module.

(a) If one can verify that C is necessarily free in case R is local and $\operatorname{pd} C \leq 1$, then one obtains that, at the very least, Baer modules over a regular domain are locally free.

(b) If R is local and one can show that $\hat{R} \otimes_R C$ is $\hat{R}$-free then one may conclude that C is R-free (Raynaud and Gruson [21, 3.1.3]). The reason that one cannot simply reduce to case (a) above is that, in general $\hat{R}$ has more torsion modules than R, that is, there are prime ideals in $\operatorname{Spec} \hat{R}$ which are not the completion of prime ideals in $\operatorname{Spec} R$. If Baer modules were defined only in terms of the maximal ideals of a regular ring then the "general" problem would reduce to the complete regular local case.

I wish to record a final observation/reduction in the case of modules of projective dimension ≤ 1. Although this reduction has no subsequent application, it seems worthy of note.

Proposition 2.5. *Let R be an integral domain and C an R-module. Then C is a Baer module if and only if $\operatorname{Ext}^1_R(C, \coprod_{i \in I} R/x_iR) = 0$ for all direct sums $\coprod_{i \in I} R/x_iR$ in which $x_i \neq 0$ for each $i \in I$.*

Proof. Of course C is a Baer module precisely when $\operatorname{Ext}^1_R(C, T) = 0$ for each torsion R-module T. However, each torsion module is a homomorphic image of a torsion module of the form $\coprod_{i \in I} R/x_iR$. Since $\operatorname{Ext}^j_R(C, \centerdot) \equiv 0$ for $j \geq 2$, the result follows. ☐

3. Flatness and other properties of Baer modules

In this section I will tackle the property of flatness for Baer modules. It should be noted that in the same article [18] cited in Section 2, I. Kaplansky provides a very elegant and short argument which settles the issue over general integral domains. Kaplansky makes efficient use of a standard duality formula that is derived from properties of the "circle group" over $\mathbb{Z}$. At the risk of overworking the reader I will take a slightly more circuitous route since I believe the "local" criteria for flatness provided in Theorem 3.1 might be of independent interest. In case (R, m, k) is a local ring and C is an R-module that is flat locally on the punctured spectrum, $\operatorname{Spec} R - m$, then perhaps somewhat surprisingly the flatness of C depends on the vanishing of finitely many $\operatorname{Ext}^i_R(C, k)$. After achieving the flatness of Baer modules in Corollary 3.2, I close this section by establishing that Baer modules over regular domains are faithfully flat in a very strong fashion (see Theorem 3.3).

THEOREM 3.1. *Let $(R, \mathfrak{m}, k)$ be a local ring of dimension n and let C be an R-module that is flat locally on* $\operatorname{Spec} R - \mathfrak{m}$. *If* $\operatorname{Ext}^j_R(C, k) = 0$*, for* $1 \leq j \leq n+1$*, then C is necessarily flat.*

Proof. In order to get from the vanishing of Ext to the vanishing of Tor it is necessary to use a duality formula that one can find in Cartan-Eilenberg [4, p. 120]. This formula asserts that

$$\operatorname{Tor}^R_j(C, M)^v \cong \operatorname{Ext}^j_R(C, M^v)$$

where $(\,.\,)^v = \operatorname{Hom}_R(\,.\,, E)$ and where $E = E(k)$, the injective envelope of the residue field. The duality created by $M \longleftrightarrow M^v$ is known as Matlis duality in commutative algebra. When R is complete this duality is a perfect duality for finitely generated R-modules; in particular, $M \cong M^{vv}$ in that case.

Since $k^v \cong k$, the duality cited above gives that $\operatorname{Tor}^R_j(C, k) = 0$ for $1 \leq j \leq n+1$. It follows that $\operatorname{Tor}^R_j(C, L) = 0$, for $1 \leq j \leq n+1$ and all finitely generated R-modules L with $\dim L = 0$ (recall Tor is half exact). I will argue by induction on $\dim M = s$ that $\operatorname{Tor}^R_j(C, M) = 0$, for $1 \leq j \leq n+1-s$ and for all finitely generated R-modules M of dimension s. Since Tor^R_j is half exact and since $\operatorname{Tor}^R_j(C, .)$vanishes in dimension zero, I may consider, when passing by way of induction from dimension s to dimension $s+1$, that $\dim M = s+1$ and depth $M > 0$. This yields a short exact sequence

$$0 \to M \xrightarrow{x} M \to \bar{M} \to 0$$

where $x \in m$ is regular on M. It follows from the induction hypothesis that the multiplication homomorphism $\operatorname{Tor}^R_j(C, M) \xrightarrow{x} \operatorname{Tor}^R_j(C, M)$ is an isomorphism for j in the range $1 \leq j \leq n+1-s-1$. However, $\operatorname{Ass} \operatorname{Tor}^R_j(C, M) \subseteq \{m\}$, for $j > 0$, since C is flat locally on $\operatorname{Spec} R - \mathfrak{m}$. So $x \in \mathfrak{m}$ cannot act as an injective homomorphism should $\operatorname{Tor}^R_j(C, M) \neq 0$. Thus $\operatorname{Tor}^R_j(C, M)$ must vanish in the range $1 \leq j \leq n+1-s-1$. The induction argument is complete. As a consequence of the preceding formula one obtains that $\operatorname{Tor}^R_1(C, M) = 0$ for all finitely generated R-modules M. This statement is a standard equivalent of "being flat". $\square$

Corollary 3.2. *Baer modules are flat over Noetherian integral domains R.*

Proof. This is a local condition. So I may assume $(R, \mathfrak{m}, k)$ is a local integral domain of dimension n and that C is a Baer module over R. If C is assumed not flat I may choose an R and C where $\dim R$ is as small as possible for this occurrence (I'm using that the notion of Baer module "localizes"). Hence C is flat locally on $\operatorname{Spec} R - \mathfrak{m}$ due to the minimality of $\dim R$. Since $\operatorname{Ext}^i_R(C,k) \cong \operatorname{Tor}^R_j(C,k)^v = 0$, for $j > 0$, one achieves a contradiction as a result of Theorem 3.1. $\square$

In the next theorem I argue that C is actually faithfully flat when $C \neq 0$, and R is regular.

Theorem 3.3. *Let $(R, \mathfrak{m}, k)$ be a regular local ring of dimension n and let C be a nonzero Baer module. Then the following properties hold for C.*

(i) *For any nonzero ideal I, C/IC is a nonzero free R/I-module.*
(ii) *The module C is faithfully flat.*
(iii) *For any finitely generated $\hat{R}$-module M, the $\hat{R}$-module $M \otimes_R C$ is $\hat{\mathfrak{m}}$-separated, where $\hat{\mathfrak{m}} = \mathfrak{m}\hat{R}$ (i.e., separated in the $\hat{m}$-adic topology).*

Proof. Let $x \neq 0$ in $\mathfrak{m}$. I claim that $xC \neq C$. Since R is a UFD it suffices to prove this statement when $x = p$ is a prime element of R. If $pC = C$ it would follow that the localization $C_{(p)}$ is a Baer module over $R_{(p)}$ (Proposition 2.1). However, $pC = C$ and C R-flat imply that $C_{(p)}$ is isomorphic to a direct sum of copies of the fraction field of $R_{(p)}$. Since $R_{(p)}$ is a DVR one has from my argument [9, Lemma 2.1] (modified for DVR) that this statement is false. Thus $xC \neq C$ for $x \in \mathfrak{m}$.

If $I \neq 0$ is an ideal in R one considers a free presentation $0 \to K \to F \to C \to 0$, where F is R-free, and the resulting pushout diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K & \longrightarrow & F & \longrightarrow & C & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & K/IK & \longrightarrow & E & \longrightarrow & C & \longrightarrow & 0. \end{array}$$

The bottom row of the above diagram is split exact since K/IK is necessarily a torsion R-module. Thus one obtains a commutative triangle

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K & \longrightarrow & F & \longrightarrow & C & \longrightarrow & 0 \\ & & \downarrow & \swarrow & & & & & \\ & & K/IK & & & & & & \end{array}$$

which further yields the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K/IK & \longrightarrow & F/IF & \longrightarrow & C/IC & \longrightarrow & 0 \\ & & \| & \swarrow & & & & & \\ & & K/IK\,. & & & & & & \end{array}$$

The top row is exact since C is flat (Corollary 3.2). It follows that the top row is split exact, i.e., C/IC is R/IR-free. Moreover, $I \neq 0$ implies there is $x \neq 0$ in I. If $I = (x)$ one gets that C/xC is R/xR-free and nonzero (my initial argument above). But then $\bar{C} \neq J\bar{C}$ where $J = I/(x)$ and $\bar{C} = C/xC$; so $IC \neq C$ even when I properly contains x. This proves (i) and together with 3.2 also proves (ii).

In order to verify part (iii) I consider the induced homomorphism

$$C \xrightarrow{\sigma} \prod_{x \neq 0} C/xC$$

where x ranges over the nonzero elements of the maximal ideal $\mathfrak{m}$. The R-submodule $\operatorname{Ker} \sigma$ is easily shown to be uniquely x-divisible for each $x \neq 0$ (recall C is flat). Hence $C = \operatorname{Ker} \sigma \oplus C'$, where $\operatorname{Ker} \sigma$ is a vector space over the fraction field of R. The initial argument in this proof gives that $\operatorname{Ker} \sigma = 0$ since such a module cannot be a Baer module.

If M is an R-module that is m-separated and x-torsion (i.e., $xM = 0$), then $M \otimes_R C \cong M \otimes_R C/xC \cong \coprod M$ since C/xC is necessarily R/xR-free. Now let M be a finitely generated $\hat{R}$-module and let $T =$ the R-torsion submodule of M. One sees that T is an $\hat{R}$-module as well, and as such, T is a finitely generated $\hat{R}$-module. It follows there is $x \neq 0$ in R with the property $xT = 0$. From above $T \otimes_R C \cong \coprod T$; in particular $T \otimes_R C$ is $\mathfrak{m}$-separated. Therefore the short exact sequence (recall C is flat)

$$0 \to T \otimes_R C \to M \otimes_R C \to M/T \otimes_R C \to 0$$

reduces the question of m-separation to the module $M/T \otimes_R C$ as follows. For each $x \in \mathfrak{m},\ x \neq 0$, one has that applying the functor $R/xR \otimes \boldsymbol{\cdot}$ to the short exact sequence $0 \to T \otimes C \to M \otimes C \to M/T \otimes C \to 0$ yields a short exact sequence "modulo" x since $M/T \otimes C$ is R-torsion-free. It follows there is a commutative diagram

$$\begin{array}{ccccc} T \otimes C & \rightarrowtail & M \otimes C & \twoheadrightarrow & M/T \otimes C \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\sigma} & & \downarrow{\scriptstyle\beta} \\ \prod_x (T \otimes C)/x(T \otimes C) & \rightarrowtail & \prod_x (M \otimes C)/x(M \otimes C) & \twoheadrightarrow & \prod_x (M/T \otimes C)/x(M/T \otimes C) \end{array}$$

Since $\operatorname{Ker} \alpha = 0$ (by the argument above) one gets $\operatorname{Ker} \sigma \subseteq \operatorname{Ker} \beta$. Thus, since it suffices to show $\operatorname{Ker} \sigma = 0$, one may assume $T = 0$ and that M is R-torsion free. Since C is R-flat one has that $M \otimes_R C$ is also R-torsion-free. I now return to the outset of this discussion and consider the map

$$M \otimes_R C \xrightarrow{\sigma} \prod_x (M \otimes_R C)/x(M \otimes_R C).$$

As above, one obtains that $\operatorname{Ker}\sigma$ is uniquely x-divisible for each $x \in m - 0$, and if $\operatorname{Ker}\sigma \neq 0$, then $M \otimes_R C$ contains the fraction field K_R of R as an R-direct summand. Let $\mathfrak{p}$ be any prime ideal of height one in $\operatorname{Spec} R$. Then

$$(M \otimes C)_{\mathfrak{p}} \cong M_{\mathfrak{p}} \otimes C_{\mathfrak{p}} \cong \coprod M_{\mathfrak{p}}$$

since $C_{\mathfrak{p}}$ is necessarily $R_{\mathfrak{p}}$-free. (Recall Baer modules over DVR are known to be free.) The module $M_{\mathfrak{p}} \cong S^{-1}M$ is a finitely generated $S^{-1}\hat{R}$-module, where $S = R - \mathfrak{p}$. The ring $S^{-1}\hat{R}$ is a regular domain of positive (Krull) dimension since $\dim S^{-1}R = 1$. It follows that $S^{-1}M$ has no uniquely π-divisible submodule where $\mathfrak{p} = (\pi)$. So K_R cannot be an R-summand of $\coprod M_{\mathfrak{p}}$. □

There are nonfree modules over regular local rings which have the characteristics of the conclusions of Theorem 3.3; however I suspect they are not Baer modules.

Example 3.4. Let R be a regular local ring with fraction field K_R such that $\mathrm{pd}K_R \geq 2$ (see B. L. Osofsky [20, 6.5] for the existence of such regular local rings). Let C be a first syzygy for K_R, i.e., $0 \to C \to F \to K_R \to 0$ is exact with F an R-free module. Since $K_R = IK_R$ for each nonzero R-ideal I, one has that $C/IC \cong F/IF$ for all such I (I am using here that K_R is R-flat). Of course C is m-separated in this instance, and if R is complete then one may argue $M \otimes_R C \hookrightarrow M \otimes_R F$ for any finitely generated R-module M; so parts (i)–(iii) of Theorem 3.3 hold for such an R-module C.

4. A generalized Prüfer theorem for countably generated $\hat{R}$-modules

In [25] Prüfer showed that a countably generated torsion-free module over a complete DVR is the direct sum of a torsion free divisible R-module (i.e., a K_R-module where K_R denotes the fraction field of R) and a free R-module. For m-separated countably generated torsion free R-modules, this statement simply states that they are free. In [11] it was shown, with a mild strengthening of the "separation" hypothesis, that one could establish a similar result for complete regular local rings of any (Krull) dimension. I will include a proof of this result here since there was an error in that manuscript. As a corollary one sees that countably generated Baer modules over regular local rings are free. This was the state of affairs when I became interested in the Baer Problem for $\mathbb{Z}$ about 35 years ago.

I will begin with some background material concerning the "m-adic" topology and its affect on torsion-free (usually flat) modules. In [10], [11] I studied a particular class of countably generated modules called maximal Cohen-Macaulay modules, abbreviated "MCM" modules. One can deduce powerful theorems in commutative algebra when such modules are present. In a landmark paper [14] Melvin Hochster established their existence over any local ring which contains a field. My special interest was to determine when such modules were free over complete regular local rings. My first observation in [10] was that MCM modules C over a complete regular local ring R contain a free R-submodule F such that (i) F is a pure submodule of C and (ii) $F + \mathfrak{m}^n C = C$ for each $n \geq 1$. That is, C must contain a free submodule which is pure and dense in the $\mathfrak{m}$-adic topology on C. I borrowed a definition from the theory of abelian groups and referred to such submodules as "basic submodules". Next I will discuss the affect of such free submodules on the $\mathfrak{m}$-adic closure of submodules of the form IC where I is an ideal.

4.1. I will refer to an R-module C as *idealwise separated* provided C/IC is $\mathfrak{m}$-adically separated for each ideal I. When $I = 0$ this requirement amounts to stipulating that C itself is separated in the $\mathfrak{m}$-adic topology. Of course one observes that free modules are idealwise separated.

To get a feel for how these notions interact let us consider a torsion-free R-module C having a basic submodule F. In addition I will suppose that C is a submodule of its completion, i.e., C is itself $\mathfrak{m}$-separated. Let $c \in I\hat{C} \cap C = I\hat{F} \cap C$ (one notes $\hat{F} = \hat{C}$ since F is pure and dense). Since F is free one has that the closure of IF in $\hat{F}$ is $I\hat{F}$. So there is a sequence $\{f_n\}_{n=1}$ in IF with $f_n \to c$ in the $\mathfrak{m}$-adic topology on $\hat{F}$. However, since c is in the closure of F in the $\mathfrak{m}$-adic topology on C, it follows that $f_n \to c$ in this topology as well. From this fact one further obtains $c - f_n = v_n \in \mathfrak{m}^{e_n} C$ where $e_n \to \infty$ as $n \to \infty$. Since $f_n \in IF \subseteq IC$ one has $c + IC = v_n + IC$, for each $n \geq 1$, that is, $c + IC$ is in the closure of zero in C/IC. Thus C/IC is $\mathfrak{m}$-adically separated if and only if $I\hat{C} \cap C = IC$.

I need to discuss yet one more topic before getting to the main result of this section.

4.2. An R-module E is called pure injective provided any diagram

$$\begin{array}{ccc} M & \xhookrightarrow{\ i\ } & N \\ {\scriptstyle f}\downarrow & & \\ E & & \end{array}$$

where i is a pure monomorphism can be completed to a diagram

$$\begin{array}{ccc} M & \overset{i}{\hookrightarrow} & N \\ {\scriptstyle f}\downarrow & \swarrow {\scriptstyle h} & \\ E' & & \end{array}$$

such that $f = h \cdot i$. Warfield [26] made a detailed study of pure injectiveness and algebraic compactness. In particular, he showed the existence of pure injective envelopes [26, Proposition 6], and that pure injective envelopes possess the following important property: If the pure injective map $M \hookrightarrow E$ represents the pure injective envelope of M and if $f : M \to E'$ is another pure injective map into a pure injective module E', then there is an injective map $h : E \to E'$ such that the triangle

$$\begin{array}{ccc} M & \overset{i}{\hookrightarrow} & E \\ {\scriptstyle f}\downarrow & \swarrow {\scriptstyle h} & \\ E' & & \end{array}$$

commutes and $h(E)$ is a direct summand of E'. In [12, Section 3] I carried out some computations a bit further in the context of complete local rings $(R, \mathfrak{m}, k)$. It is noted in [12] that $\mathrm{Hom}_R(M, N)$ is pure injective whenever N is pure injective (e.g., if N is actually an injective R-module). It follows that any finitely generated R-module M (when R is local and complete) is a pure injective module since the natural map $M \to \mathrm{Hom}_R(M, \mathrm{Hom}_R(M, E))$ is an isomorphism, where E is the injective envelope of the residue field k. That is, a finitely generated module M is naturally isomorphic to its "double dual" under Matlis duality $(\,\centerdot\,)^v = \mathrm{Hom}_R(\centerdot, E)$. In addition in [12, Proposition 3.9] I noted that the $\mathfrak{m}$-adic closure of the countably infinite direct sum $\coprod R$ in the corresponding direct product $\prod R$ represents the pure injective envelope of $\coprod R$. Put another way, the pure injective envelope of $\coprod R$ is simply its $\mathfrak{m}$-adic completion. This discussion along with the following remark will be used as background for my generalization of Prüfer's theorem that was promised in the Introduction. Namely, a useful way in which to establish that a countably generated torsion-free module is actually free is to show that such a module can be represented as a pure submodule of $\prod R$ (defined above). This fact is established in [10, Corollary 1.6] and follows in part from a result of Jensen [15] which states that countably generated flat modules have projective dimension ≤ 1.

THEOREM 4.3. *Let $(R, \mathfrak{m}, k)$ be a complete regular local ring and let C be a countably generated torsion-free R-module. Then the following are equivalent:*

(a) *C is R-free.*

(b) *C has a basic submodule and is idealwise separated.*
(c) *C is faithfully flat and idealwise separated.*
(d) *C is faithfully flat and is isomorphic to a pure submodule of its $\mathfrak{m}$-adic completion.*

Proof. (a) $\Longrightarrow$ (b) is obvious. The implication (b) $\Longrightarrow$ (c) follows from the discussion in 4.1. Let F be a basic submodule of C. Then $C \hookrightarrow \hat{C} = \hat{F}$ represents a pure monomorphism once one knows that C is idealwise separated (note here that $\hat{C}$ is faithfully flat). To see that (c) $\Longrightarrow$ (d) I appeal to a result of Enochs [5, 3.2.7] that allows one to conclude $\hat{C}$ will be a flat R-module. Once again the discussion surrounding 4.1 gives $I\hat{C} \cap C = IC$ for each ideal I in R (C has a basic submodule by [10, Lemma 2.1]). Since $\hat{C}$ is flat this statement is sufficient to conclude $C \to \hat{C}$ is a pure monomorphism.

In order to argue (d) $\Longrightarrow$ (a) I appeal to [10, Corollary 1.6] in that I will demonstrate C can be realized as a pure submodule of a countable product $\prod R$. Since C is countably generated and flat one has a free resolution (see [15])

$$0 \to F_1 \to F_0 \to C \to 0$$

where $F_i \cong \coprod R$ (countable direct sum) for $i = 1, 2$. Since C is flat the short exact sequence above is pure exact. As a result of this, the sequence of completions is also short exact (the $\mathfrak{m}$-adic topology on F_1 is the same as that induced from F_0), and one obtains the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & F_1 & \longrightarrow & F_2 & \longrightarrow & C & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \hat{F}_1 & \longrightarrow & \hat{F}_0 & \longrightarrow & \hat{C} & \longrightarrow & 0 \end{array}$$

in which the vertical maps are pure injective. Applying facts from the discussion that precedes the statement of Theorem 4.3 one has that $\hat{F}_1 \to \hat{F}_0$ splits since $\hat{F}_1$ is pure injective; the lower sequence is pure exact by Enochs' result [5, 3.2.7]. It follows that $\hat{F}_1 = \hat{F}_0 \oplus \hat{C}$ and that C can be represented as a pure countably generated submodule of $\prod R$. As noted above this statement carries with it the implication that C is free. $\square$

I end this note with the resolution of the Baer problem over regular local rings of higher Krull dimension for the countable case. Since my proof [9] for $R = \mathbb{Z}$ depended heavily on the hereditary property of $\mathbb{Z}$, I have not a clue at this point on how one should approach the general case.

Theorem 4.5. *Let R be a regular domain and let C be a countably generated R-module. If C is a Baer module it must be locally free.*

Proof. From Proposition 2.1 one has that $\mathrm{pd}_R\, C \leq 1$. By Proposition 2.3 I can reduce to the local case, i.e., I may assume $(R, \mathfrak{m}, k)$ is a regular local ring. Let $C' = \hat{R} \otimes_R C$ where $\hat{R}$ denotes the $\mathfrak{m}$-adic completion of R. Since C is necessarily faithfully flat over R by Theorem 3.3(ii), it follows by flat base-change that C' is a countably generated faithfully flat $\hat{R}$-module. From Theorem 4.3 it remains to verify that C'/IC' is $\mathfrak{m}$-separated for each $\hat{R}$-ideal I. However, the isomorphisms

$$C'/IC' \cong \hat{R}/I \otimes_{\hat{R}} C' \cong \hat{R}/I \otimes_{\hat{R}} \hat{R} \otimes_R C \cong \hat{R}/I \otimes_R C$$

together with Theorem 3.3(iii) provide the desired conclusion. □

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