Illinois Journal of Mathematics
Volume 47, Number 1/2, Spring/Summer 2003, Pages 189–205
S 0019-2082

PERIODIC GROUPS WITH NEARLY MODULAR SUBGROUP LATTICE

M. DE FALCO, F. DE GIOVANNI, C. MUSELLA, AND Y. P. SYSAK

Abstract. A theorem of B.H. Neumann states that each subgroup of a group G has finite index in a normal subgroup of G if and only if the commutator subgroup G' of G is finite, i.e., G is finite-by-abelian. As a group lattice version of this theorem for a periodic group G, it is proved that each subgroup of G has finite index in a modular subgroup of G if and only if G is an extension of a finite group by a group with modular subgroup lattice.

1. Introduction

A subgroup of a group G is called *modular* if it is a modular element of the lattice $\mathfrak{L}(G)$ of all subgroups of G. It is clear that every normal subgroup of a group is modular, but arbitrary modular subgroups need not be normal; thus modularity may be considered as a lattice generalization of normality. Lattices in which all elements are modular are also called *modular*. Obviously, the subgroup lattice of any abelian group is modular, and hence groups with modular subgroup lattice naturally arise in the study of lattice isomorphisms of abelian groups; in particular, Baer [2] determined all groups having the same subgroup lattice as an abelian group of prime exponent. The structure of groups with modular subgroup lattice has been completely described by K. Iwasawa [8], [9] and R. Schmidt [12]. For a detailed account of results concerning modular subgroups of groups, we refer the reader to [13].

A subgroup H of a group G is said to be *nearly normal* if it has finite index in its normal closure H^G. A relevant theorem of B.H. Neumann [10] states that all subgroups of a group G are nearly normal if and only if the commutator subgroup G' of G is finite, i.e., if and only if G is a finite-by-abelian group. If φ is a *projectivity* from a group G onto a group $\bar{G}$ (i.e., an

Received October 26, 2001.
2000 *Mathematics Subject Classification.* 20E15.
This work was done while the last author was visiting the Department of Mathematics of the University of Napoli "Federico II". He thanks the "Istituto Nazionale di Alta Matematica" for financial support.

isomorphism from the lattice $\mathfrak{L}(G)$ onto the subgroup lattice $\mathfrak{L}(\bar{G})$ of $\bar{G}$), and N is a normal subgroup of G, then the image N^φ of N is a modular element of the lattice $\mathfrak{L}(\bar{G})$. Furthermore, if H and K are subgroups of G such that $H \leq K$ and the index $|K : H|$ is finite, then H^φ has finite index in K^φ (see [13], Theorem 6.1.7). Thus the image of any nearly normal subgroup of G has finite index in a modular subgroup of $\bar{G}$.

We shall say that a subgroup H of a group G is *nearly modular* if it has finite index in a modular subgroup of G. The definition of nearly modular element can be given in an arbitrary lattice, and a lattice $\mathfrak{L}$ will be called *nearly modular* if all its elements are nearly modular. Thus every projective image of a group whose subgroups are nearly normal is a group with nearly modular subgroup lattice. It was proved in [6] that the commutator subgroup of a locally graded group with this latter property is periodic, and that periodic locally graded groups with nearly modular subgroup lattice are locally finite; in particular every torsion-free locally graded group whose subgroups are nearly modular is abelian. (Here a group G is said to be *locally graded* if every finitely generated non-trivial subgroup of G has a proper subgroup of finite index.)

The aim of this article is to prove the following theorem, that provides a lattice analog of the above quoted result of B.H. Neumann.

Theorem. *A periodic group G has nearly modular subgroup lattice if and only if there exists a finite normal subgroup N of G such that the subgroup lattice $\mathfrak{L}(G/N)$ is modular.*

In our result the assumption that the group is periodic cannot be omitted. In fact, there exists a torsion-free group $G = \langle a, b\rangle$ such that $Z(G) = \langle a\rangle \cap \langle b\rangle$ is infinite cyclic and $G/Z(G)$ is a Tarski group (see [1], proof of Theorem 2); then every non-trivial subgroup X of G has finite index in the modular subgroup $XZ(G)$, and hence the subgroup lattice $\mathfrak{L}(G)$ is nearly modular.

It is well-known that a special role among modular subgroups is played by permutable subgroups; a subgroup H of a group G is said to be *permutable* if $HK = KH$ for each subgroup K of G, and a group is called *quasihamiltonian* if all its subgroups are permutable. It was proved in [3] that if every subgroup of a periodic group G has finite index in a permutable subgroup, then G contains a finite normal subgroup N such that G/N is a quasihamiltonian group. This result will be relevant for our purposes.

Finally, we mention that a complete description of groups with the dual property that every subgroup is modular in a subgroup of finite index has recently been given in [7].

Most of our notation is standard and can be found in [11]. In particular, for a subgroup H of a group G, the *normal closure* H^G and the *core* H_G of H in G are defined as the smallest normal subgroup of G containing H and the largest normal subgroup of G contained in H, respectively. Recall also that if

G is any group, the *finite residual* of G is the intersection of all subgroups of finite index of G and the *locally finite radical* of G is the largest locally finite normal subgroup of G.

We shall use the monograph [13] as a general reference for results on subgroup lattices.

The authors wish to thank the referee for his useful comments.

2. Some preliminaries

Let $\mathfrak{L}$ be a lattice with least element 0 and greatest element I. Recall that an element x of $\mathfrak{L}$ is *covered irreducibly* by elements $x_1, \dots, x_m$ of the interval $[x/0]$ if for each element y of $[x/0]$ such that $[y/0]$ is a distributive lattice with the maximal condition, there is $i \le m$ such that $y \le x_i$, and the set $\{x_1, \dots, x_m\}$ is minimal with respect to such property. Clearly a subgroup H of a group G is covered irreducibly in the lattice $\mathfrak{L}(G)$ by its subgroups $H_1, \dots, H_m$ if and only if H is the set-theoretic union of $H_1, \dots, H_m$ and none of these subgroups can be omitted from the covering.

An element h of the lattice $\mathfrak{L}$ is said to be *cofinite* if there exists a finite chain in $\mathfrak{L}$

$$h = h_0 < h_1 < \cdots < h_t = I$$

such that, for every $i = 0, 1, \dots, t-1$, h_i is a maximal element of the lattice $[h_{i+1}/0]$ and one of the following conditions is satisfied:

- h_{i+1} is covered irreducibly by finitely many elements $k_1, \dots, k_{n_i}$ of $\mathfrak{L}$ such that $k_1 \wedge \cdots \wedge k_{n_i} \le h_i$;
- for every automorphism φ of the lattice $[h_{i+1}/0]$, the element $h_i \wedge h_i^{\varphi}$ is modular in $[h_{i+1}/0]$ and the lattice $[h_{i+1}/h_i \wedge h_i^{\varphi}]$ is finite.

We shall say that an element a of $\mathfrak{L}$ is *nearly modular* if there exists a modular element h of $\mathfrak{L}$ such that $a \le h$ and a is a cofinite element of the lattice $[h/0]$. The lattice $\mathfrak{L}$ is called *nearly modular* if all its elements are nearly modular.

A theorem of R. Schmidt yields that a subgroup H of a group G is cofinite in the lattice $\mathfrak{L}(G)$ if and only if H has finite index in G (see [13], Theorem 6.1.10). Therefore, a subgroup X of G is nearly modular if and only if it is a nearly modular element of the lattice $\mathfrak{L}(G)$, and hence the subject of this article is the structure of groups with nearly modular subgroup lattice.

A group G is called a *P^*-group* if it is the semidirect product of an abelian normal subgroup A of prime exponent by a cyclic group $\langle x \rangle$ of prime-power order such that x induces on A a power automorphism of prime order. (Recall here that a *power automorphism* of a group G is an automorphism mapping every subgroup of G onto itself.) It is easy to see that the subgroup lattice of any P^*-group is modular, and Iwasawa [8], [9] proved that a locally finite group has modular subgroup lattice if and only if it is a direct product

$$G = \underset{i\in I}{\mathrm{Dr}}\, G_i,$$

where each G_i is either a P^*-group or a primary locally finite group with modular subgroup lattice, and elements of different factors have coprime orders. Recall also that a group G is said to be a *P-group* if either it is abelian of prime exponent or $G = \langle x\rangle \ltimes A$ is a P^*-group with the subgroup $\langle x\rangle$ of prime order.

Finally, a subgroup H of a group G is said to be *P-embedded* in G if G/H_G is a periodic group, and the following conditions are satisfied:

- $G/H_G = \Big(\mathrm{Dr}_{i\in I}(S_i/H_G)\Big) \times L/H_G$, where each S_i/H_G is a non-abelian P-group;
- in the above direct decomposition, elements from different factors have coprime orders;
- $H/H_G = \Big(\mathrm{Dr}_{i\in I}(Q_i/H_G)\Big) \times ((H\cap L)/H_G)$, where each Q_i/H_G is a non-normal Sylow subgroup of S_i/H_G;
- $H \cap L$ is a permutable subgroup of G.

All P-embedded subgroups are modular, and it can be proved that every modular subgroup of a locally finite group is either permutable or P-embedded (see [16], Theorem 3.2 and Theorem E).

3. Locally finite groups

It was proved by Stonehewer [14], [15] that a subgroup H of a group G is permutable if and only if H is ascendant in G and it is a modular element of the lattice $\mathfrak{L}(G)$. It follows that modular subgroups coincide with permutable subgroups in locally nilpotent groups. Therefore a locally nilpotent group G has nearly modular subgroup lattice if and only if any subgroup H of G has finite index in a permutable subgroup of G; in particular, periodic locally nilpotent groups with nearly modular subgroup lattice must be finite-by-quasihamiltonian (see [3]).

The first result of this section shows in particular that if G is a locally finite group with nearly modular subgroup lattice and R is the Hirsch-Plotkin radical of G, then all Sylow subgroups of G/R are finite.

Lemma 3.1. *Let G be a locally finite group, and let S be a Sylow p-subgroup of G. If S is nearly modular in G, then $S/O_p(G)$ is finite.*

Proof. If the subgroup S is nearly permutable in G, the statement is already known (see [3], Lemma 3.1). Suppose now that S is not nearly permutable in G, and let X be a modular subgroup of G containing S such that the index $|X : S|$ is finite. Then X is not permutable in G, and so it is a P-embedded subgroup of G (see [13], Theorem 6.2.17). As the set of primes $\pi(X)$ is finite, we have in particular that

$$X/X_G = E/X_G \times Y/X_G,$$

where the factors are coprime, E/X_G is finite and Y is permutable in G. Clearly S is not contained in Y, and hence it is a subgroup of E. It follows that $S \cap X_G$ has finite index in S, and so also in X. Thus the core $(S \cap X_G)_X$ is a subnormal p-subgroup of G and the index $|S : (S \cap X_G)_X|$ is finite, so that $S/O_p(G)$ is also finite. □

LEMMA 3.2. *Let G be a group, and let $(E_n)_{n\in\mathbb{N}}$ be a sequence of periodic subgroups of G such that all subgroups of E_{n+1} are normalized by $\langle E_1, \dots, E_n\rangle$ for each positive integer n and $\pi(E_m) \cap \pi(E_n) = \emptyset$ if $m \neq n$. If every E_n contains a finite non-modular subgroup H_n, then the subgroup $H = \langle H_n \mid n \in \mathbb{N}\rangle$ is not nearly modular in G.*

Proof. Assume by contradiction that H has finite index in a modular subgroup X of G. Clearly H is locally finite, so that also X is locally finite and there exists a positive integer n such that $X \cap E_n$ is contained in H. Therefore $X \cap E_n = H \cap E_n = H_n$, contradicting the assumption that H_n is not modular in E_n. □

The following easy lemma suggests that properties of power automorphisms can be used in the study of groups with nearly modular subgroup lattice.

LEMMA 3.3. *Let G be a group, and let X be a modular subgroup of G. If K is a normal subgroup of G such that $X \cap K = \{1\}$, then every subgroup of K is normalized by X.*

Proof. Let y be any element of K. Then $\langle y\rangle = \langle y, X\rangle \cap K$ is a normal subgroup of $\langle y, X\rangle$, and hence X normalizes all subgroups of K. □

The next two lemmas will be essential for proving the theorem in the case of locally finite groups.

LEMMA 3.4. *Let G be a locally finite group with nearly modular subgroup lattice. Then there exist normal subgroups N and M of G such that $N \leq M$, N and G/M are finite, and the subgroup lattice $\mathfrak{L}(M/N)$ is modular.*

Proof. Let R be the Hirsch-Plotkin radical of G, and suppose first that the factor group G/R is countable. Assume by contradiction that G does not contain any finite normal subgroup N such that the factor group G/N is a finite extension of a group with modular subgroup lattice. Let n be a positive integer for which n finite subgroups $E_1, \dots, E_n$ of G with pairwise coprime orders have been chosen such that every subgroup of E_{i+1} is normalized by $\langle E_1, \dots, E_i\rangle$ for all $i < n$ and the lattices $\mathfrak{L}(E_1), \dots, \mathfrak{L}(E_n)$ are not modular.

Since $\langle E_1, \dots, E_n\rangle$ is nearly modular in G, there exists a finite modular subgroup E of G containing $\langle E_1, \dots, E_n\rangle$. Let π be the set of all prime numbers dividing the order of E, and consider the largest π-subgroup R_π of R. As the Sylow subgroups of G/R are finite by Lemma 3.1, the index $|G/R : O_{\pi'}(G/R)|$ is finite (see [4], Theorem 3.5.15 and Corollary 2.5.13), and hence there exists a π'-subgroup L_1 of G such that $K = L_1R_\pi$ is a normal subgroup of finite index of G (see [4], Theorem 2.4.5). Moreover, the subgroup R_π is finite-by-quasihamiltonian (see [3]), and it is also finite-by-abelian-by-finite because the set π is finite. It follows that the subgroup F, consisting of all elements of R_π having finitely many conjugates in R_π, has finite index in R_π, so that the subgroup $H_1 = L_1F$ has finite index in G. Put $H = (H_1)_G$, so that G/H is finite and $H = LF$, where $L = L_1 \cap H$.

Let X be a modular subgroup of H containing L such that the index $|X : L|$ is finite, so that $X = L(X \cap F)$, where $X \cap F$ is finite. Clearly the product $(X \cap F)F'$ is a normal subgroup of H, and hence $N = ((X \cap F)F')^G$ is a finite normal subgroup of G. Put $\bar{G} = G/N$, so that $\bar{H} = H/N$ is a normal subgroup of finite index of $\bar{G}$ and $\bar{L} = LN/N = XN/N$ is a modular subgroup of $\bar{H}$; in particular, $\bar{L}$ acts as a group of power automorphisms on $\bar{F}$ and hence $\bar{L}/C_{\bar{L}}(\bar{F})$ is finite. It follows that $C_{\bar{H}}(\bar{F}) = C_{\bar{L}}(\bar{F}) \times \bar{F}$ is a subgroup of finite index of $\bar{G}$, so that the normal subgroup $C_{\bar{L}}(\bar{F})$ of $\bar{G}$ is not a finite extension of a group with modular subgroup lattice. Put $C = C_L(F/N)$, so that $C_{\bar{L}}(\bar{F}) = CN/N$, and CN is a normal subgroup of G which is not a finite extension of a group with modular subgroup lattice. As C is a π'-subgroup of finite index of CN, the subgroup $O_{\pi'}(CN)$ has finite index in CN, so that the lattice $\mathfrak{L}(O_{\pi'}(CN))$ is not modular and there exists a finite subgroup E_{n+1} of $O_{\pi'}(CN)$ whose subgroup lattice is not modular. As $O_{\pi'}(CN)$ is normal in G, it follows from Lemma 3.3 that E acts as a group of power automorphisms on $O_{\pi'}(CN)$ and hence also on E_{n+1}. Therefore there exists a sequence $(E_n)_{n\in\mathbb{N}}$ of finite subgroups of G satisfying the hypotheses of Lemma 3.2, and hence G contains a subgroup which is not nearly modular. This contradiction proves the statement when G/R is countable.

We will now prove that the group G/R must be countable. Let V/R be any countable subgroup of G/R. It follows from the first part of the proof that V contains a finite normal subgroup W such that the factor group V/W is a finite extension of a group with modular subgroup lattice, so that in particular V is finite-by-(metabelian-by-finite) and so also soluble-by-finite. On the other hand, the class of soluble-by-finite groups is countably recognizable (see [5], Proposition 2.6), and hence G itself is soluble-by-finite. As the Sylow subgroups of G/R are finite, it follows that G/R is countable. The lemma is proved. □

LEMMA 3.5. *Let the locally finite group $G = AE$ be the product of a normal subgroup A and a finite modular subgroup E. If A has modular subgroup lattice, then $G = M \times K$, where $\mathfrak{L}(M)$ is a modular lattice, the set of primes $\pi(K)$ is finite and $\pi(M) \cap \pi(K) = \emptyset$.*

Proof. It can obviously be assumed that the set of primes $\pi(A)$ is infinite, so that

$$A = \operatorname*{Dr}_{n \in \mathbb{N}} A_n,$$

where each A_n is either a non-trivial primary group with modular subgroup lattice or a P^*-group, and elements of different factors have relatively prime orders (see [13], Theorem 2.4.13). For each positive integer n put

$$B_n = \operatorname*{Dr}_{k \geq n} A_k.$$

Clearly there exists m such that $\pi(B_m) \cap \pi(E) = \emptyset$, and we claim that $[B_n, E] = \{1\}$ for some integer $n \geq m$. If E is permutable in $B_m E$, then E is normal in $B_m E$ and so $[B_m, E] = \{1\}$. Therefore without loss of generality it can be assumed that E is not permutable in $B_m E$, so that E is P-embedded in $B_m E$ (see [13], Theorem 6.2.17). The centralizer $C = C_E(B_m)$ is the core of E in $B_m E$. Hence

$$B_m E/C = S_1/C \times \cdots \times S_t/C \times L/C,$$

where each S_i/C is a non-abelian P-group and elements from different factors have coprime orders,

$$E/C = Q_1/C \times \cdots \times Q_t/C \times (E \cap L)/C,$$

each Q_i/C is a non-normal Sylow subgroup of S_i/C and $E \cap L$ is a permutable subgroup of $B_m E$. In particular, $[B_m, E \cap L] = \{1\}$. Put $S = \langle S_1, \ldots, S_t \rangle$. As the set $\pi(S)$ is finite, there exists an integer $n \geq m$ such that B_n is contained in L. Then

$$[B_n, E \cap S] \leq B_n \cap C = \{1\},$$

and hence

$$[B_n, E] = [B_n, (E \cap S)(E \cap L)] = \{1\}.$$

Put

$$M = B_n \quad \text{and} \quad K = (\operatorname*{Dr}_{k=1}^{n-1} A_k)E.$$

Then K is normal in $G = MK$, the set of primes $\pi(K)$ is finite and $\pi(M) \cap \pi(K) = \emptyset$. The lemma is proved. □

We can now prove the main result of this section.

THEOREM 3.6. *Let G be a periodic locally graded group. Then $\mathfrak{L}(G)$ is a nearly modular lattice if and only if G contains a finite normal subgroup N such that the subgroup lattice $\mathfrak{L}(G/N)$ is modular.*

Proof. The condition of the statement is obviously sufficient. Conversely, suppose that G has nearly modular subgroup lattice, so that it is locally finite (see [6], Theorem 5), and by Lemma 3.4 there exists a finite normal subgroup W of G such that G/W contains a subgroup of finite index with modular subgroup lattice. Without loss of generality it can be assumed that $W = \{1\}$, so that G is a finite extension of a group with modular subgroup lattice. Since every finite subgroup of G is contained in a finite modular subgroup, it follows from Lemma 3.5 that $G = M \times K$, where $\mathfrak{L}(M)$ is a modular lattice, the set of primes $\pi(K)$ is finite and $\pi(M) \cap \pi(K) = \emptyset$. Replacing G by its subgroup K, we may suppose that $\pi(G)$ is finite. Thus G is abelian-by-finite, so that $G = AE$, where A is an abelian normal subgroup and E is a finite modular subgroup of G. It is enough to prove the statement for the factor group G/E_G, so that it can be assumed that E has trivial core in G, and in particular $A \cap E = \{1\}$.

Suppose first that E is permutable in G. Then E^G is locally nilpotent (see [13], Theorem 6.3.1), so that G itself is locally nilpotent, and hence it is finite-by-quasihamiltonian. Assume now that E is not permutable in G, so that it is P-embedded in G (see [13], Theorem 6.2.17). Thus

$$G = S_1 \times \cdots \times S_t \times L,$$

where each S_i is a non-abelian P-group, elements from different factors have coprime orders,

$$E = Q_1 \times \cdots \times Q_t \times (E \cap L),$$

each Q_i is a non-normal Sylow subgroup of S_i and $E \cap L$ is a permutable subgroup of G. Moreover, the core of $E \cap L$ in L is trivial, and $L = (A \cap L)(E \cap L)$. It follows now from the previous case that L contains a finite normal subgroup N such that L/N has modular subgroup lattice. Therefore also the lattice $\mathfrak{L}(G/N)$ is modular. The theorem is proved. □

It follows directly from the above theorem that every periodic locally graded group with nearly modular subgroup lattice is finite-by-metabelian. Moreover, Theorem 3.6 also has the following consequence.

Corollary 3.7. *Let G be a periodic locally graded group with nearly modular subgroup lattice. Then G is metabelian-by-finite.*

Proof. Let N be a finite normal subgroup of G such that the subgroup lattice $\mathfrak{L}(G/N)$ is modular. It is clearly enough to prove that the centralizer $C_G(N)$ is metabelian-by-finite, so that without loss of generality it can be assumed that N is contained in $Z(G)$. Write

$$G/N = H/N \times K/N,$$

where $\pi(H/N)$ is finite and $\pi(N) \cap \pi(K/N) = \emptyset$. Then K contains a normal subgroup L such that $K = N \times L$, so that L is metabelian and $G = H \times L$.

As $\pi(H/N)$ is finite, the group H/N is abelian-by-finite and hence G is metabelian-by-finite. □

4. Periodic groups

A group G is called an *extended Tarski group* if it contains a cyclic non-trivial normal subgroup N with prime-power order such that G/N is a Tarski group and $H \leq N$ or $N \leq H$ for every subgroup H of G. It was proved by R. Schmidt that a periodic group G has modular subgroup lattice if and only if $G = M \times T$, where $\pi(M) \cap \pi(T) = \emptyset$, M is a locally finite group with modular subgroup lattice and the group $T = \mathrm{Dr}_i\, T_i$ is a direct product of Tarski and extended Tarski groups such that $\pi(T_i) \cap \pi(T_j) = \emptyset$ if $i \neq j$ (see [13], Theorem 2.4.16). The first lemma of this section shows that Tarski sections also occur in the structure of arbitrary periodic groups with nearly modular subgroup lattice.

LEMMA 4.1. *Let $G = \langle E, g \rangle$ be an infinite periodic group generated by a finite subgroup E and an element g whose order is a power of a prime number p. If the subgroup lattice $\mathfrak{L}(G)$ is nearly modular, then G contains a finite normal subgroup N such that G/N is a Tarski group.*

Proof. Assume that the statement is false, and choose a counterexample $G = \langle E, g \rangle$ such that the element g has minimal order. Since E is contained in a finite modular subgroup of G, we may suppose that E itself is modular in G, so that the lattices $[G/E]$ and $[\langle g \rangle / \langle g \rangle \cap E]$ are isomorphic; in particular $[G/E]$ is finite, and so every locally finite subgroup of G containing E is finite. Thus it can also be assumed that E is a maximal locally finite subgroup of G, because all such subgroups are modular in G. If $g^p \in E$, then E is a maximal subgroup of G, and hence G/E_G is a Tarski group by a result of Stonehewer (see [16], Theorem B), contradicting the choice of G. Therefore $g^p \notin E$, so that the infinite group $H = \langle E, g^p \rangle$ contains a finite normal subgroup L such that H/L is a Tarski group, and $K = \langle g^p \rangle L$ is a maximal subgroup of H. Let X be any finite modular subgroup of G containing K. Clearly $H \cap X = K$, so that the lattices $[\langle H, X \rangle / X]$ and $[H/K]$ are isomorphic, and the subgroup X is maximal in $\langle H, X \rangle$. The above quoted result of Stonehewer yields that X contains a normal subgroup X_0 of $\langle H, X \rangle$ such that the factor group $\langle H, X \rangle / X_0$ is a Tarski group. Since E is a maximal locally finite subgroup of G, the subgroup X_0 must be contained in E. On the other hand, HX_0 is infinite, so that $\langle H, X \rangle = HX_0 = H$ and $X = H \cap X = K$. It follows that K is a maximal locally finite subgroup of G, and in particular it is modular in G. As H is a proper subgroup of G, the element g does not belong to K and so the subgroup $V = \langle K, g \rangle$ is infinite. Moreover, the lattices $[V/K]$ and $[\langle g \rangle / \langle g^p \rangle]$ are isomorphic, so that K is a maximal subgroup of V and $H \cap V = K$. As above we obtain that V/K_V is a Tarski group.

Since E and K are modular subgroups of G, also $H = \langle E, K\rangle$ is modular and so it is a maximal subgroup of G. Moreover, $K = H \cap V$ is not normal in V, and so $H^g \neq H$; it follows that for every integer s with $1 \leq s < p$ the intersection $H \cap H^{g^s}$ is a maximal subgroup of H. Assume that all these subgroups are finite. Thus $L \leq H \cap H^{g^s}$ for each s, so that the subgroup $L^{\langle g\rangle}$ is finite, and hence also $V = \langle L, g\rangle$ must be finite, a contradiction. Therefore $H \cap H^{g^s}$ is infinite for some s and so $H = (H \cap H^{g^s})L$. Thus the finite residual J of H coincides with the finite residual of $H \cap H^{g^s}$ and the index $|H : J|$ is finite because H satisfies the minimal condition on subgroups. In particular, J has no proper subgroups of finite index, so that it is contained in the finite residual J^{g^s} of H^{g^s}, and hence $J^{g^s} = J$. It follows that $J^g = J$, and so J is a normal subgroup of $G = \langle H, g\rangle$. Clearly $H = JL$, so that $J/J \cap L$ is a Tarski group, and $J \cap L$ is a finite normal subgroup of G. Replacing G by the factor group $G/J \cap L$, it can be assumed that $J \cap L = \{1\}$, so that $H = J \times L$ and J is a Tarski group. Then $K = (K \cap J) \times L$ is contained in the normal subgroup $(V \cap J)C_V(J)$ of V, and hence $V = (V \cap J)C_V(J)$. It follows that

$$G = \langle H, V\rangle = JV = J \times C_V(J),$$

so that $C_V(J)$ is infinite and the intersection $C_V(J) \cap K_V$ is a finite normal subgroup of G such that $G/C_V(J) \cap K_V$ is the direct product of two Tarski groups. Thus we may suppose that G itself is a direct product of two Tarski groups, so that the cyclic subgroup $\langle g\rangle$ has order p and hence E is a maximal subgroup of G. Therefore G/E_G is a Tarski group, and this last contradiction completes the proof of the lemma. □

We shall say that a perfect group G is a *generalized Tarski group* if the centre $Z(G)$ of G is finite and $G/Z(G)$ is a Tarski group. It is clear that if G is any generalized Tarski group, then the subgroup lattice $\mathfrak{L}(G)$ is nearly modular and every proper subgroup of G is finite and abelian. Note also that Tarski groups and extended Tarski groups are obvious examples of generalized Tarski groups. In order to prove our theorem, a careful analysis of the behaviour of generalized Tarski subgroups is needed.

LEMMA 4.2. *Let G be a periodic group with nearly modular subgroup lattice, and let L be the locally finite radical of G. If G/L is a Tarski group, then G contains a generalized Tarski subgroup T such that $G = LT$, $L \cap T = Z(T)$ and $[L, T] = \{1\}$.*

Proof. Clearly G contains a subgroup E generated by two elements of prime power order such that $G = LE$, and the factor group $E/E \cap L$ is isomorphic to the Tarski group G/L. The intersection $K = E \cap L$ is finite by Lemma 4.1, and so $E = KC_E(K)$. Let T be the finite residual of E. Since E satisfies the minimal condition on subgroups, the index $|E : T|$ is finite, so that $E = KT$ and T is a perfect subgroup of G with finite centre; moreover, T is contained in $C_E(K)$ and $K \cap T = Z(T)$, so that $T/Z(T)$ is a Tarski group and T is a generalized Tarski group. Therefore $G = LT$ and $L \cap T = Z(T)$. Let H be any finite subgroup of L containing $Z(T)$, and let X be a finite modular subgroup of G containing H. As the lattices $[\langle X, T\rangle/X]$ and $[T/X \cap T]$ are isomorphic, it follows that the group $\langle X, T\rangle$ satisfies the maximal condition on subgroups, so that $\langle X, T\rangle \cap L$ is finite and hence also $T/C_T(\langle X, T\rangle \cap L)$ is a finite group. Thus

$$[H, T] \leq [\langle X, T\rangle \cap L, T] = \{1\},$$

and so $[L, T] = \{1\}$ since L is covered by its finite subgroups containing $Z(T)$. □

COROLLARY 4.3. *Let G be a periodic group with nearly modular subgroup lattice. If G is not locally finite, then it contains a generalized Tarski subgroup.*

Proof. Choose an infinite finitely generated subgroup $K = \langle x_1, \dots, x_n\rangle$ of G, where $x_1, \dots, x_n$ are elements of prime power order and n is minimal with respect to this condition. Then $H = \langle x_1, \dots, x_{n-1}\rangle$ is finite, and it follows from Lemma 4.1 that $K = \langle H, x_n\rangle$ contains a finite normal subgroup N such that K/N is a Tarski group. Application of Lemma 4.2 yields that K contains a generalized Tarski subgroup. □

LEMMA 4.4. *Let G be a periodic group with nearly modular subgroup lattice, and let T be a generalized Tarski subgroup of finite index of G. Then there exists a finite normal subgroup K of G such that $G = KT$, $K \cap T = Z(T)$ and $[K, T] = \{1\}$. In particular, T is normal in G.*

Proof. Clearly T is the finite residual of G, and in particular T and $Z(T)$ are normal subgroups of G. Let K be a normal subgroup of G which is maximal with respect to the condition $K \cap T = Z(T)$, so that K is finite and TK/K is a Tarski group. Put $\bar{G} = G/K$; then $\bar{T} = TK/K$ is the unique minimal normal subgroup of $\bar{G}$, and hence $C_{\bar{G}}(\bar{T}) = \{1\}$. Assume that $\bar{T}$ is a proper subgroup of $\bar{G}$, and let $\bar{x} \neq 1$ and $\bar{g}$ be elements of $\bar{T}$ and $\bar{G} \setminus \bar{T}$, respectively. If the subgroup $\langle \bar{g}, \bar{x}\rangle$ is infinite, then $\langle \bar{g}\rangle\bar{T} = \langle \bar{g}, \bar{x}\rangle$ since $\bar{T}$ has finite index in $\bar{G}$; on the other hand, if $\langle \bar{g}, \bar{x}\rangle$ is finite, we have $\langle \bar{g}\rangle\bar{T} = \langle\langle \bar{g}, \bar{x}\rangle, \bar{y}\rangle$ for some element y of T. In both cases, it follows from Lemma 4.1 that $\langle \bar{g}\rangle\bar{T} = \langle \bar{h}\rangle \times \bar{T}$, contrary to the condition $C_{\bar{G}}(\bar{T}) = \{1\}$. This contradiction shows that $G = KT$. In particular, K is the locally finite radical of G, and an application of Lemma 4.2 yields that $[K, T] = \{1\}$. The lemma is proved. □

LEMMA 4.5. *Let G be a periodic group with nearly modular subgroup lattice, and let T be a generalized Tarski subgroup of G. Then T is normal in G.*

Proof. Assume that the statement is false. Without loss of generality we may suppose that $G = \langle g, T\rangle$, where g is an element of G of minimal order for which $T^g \neq T$. Then the order of g is a power of a prime number p, and g^p normalizes T. The subgroup $\langle g^p\rangle T$ has finite index in a modular subgroup X of G, and it follows from Lemma 4.4 that X contains a finite normal subgroup K such that $X = KT$, $K \cap T = Z(T)$ and $[K, T] = \{1\}$, so that in particular T is normal in X. Since the lattices $[G/X]$ and $[\langle g\rangle/\langle g\rangle \cap X]$ are isomorphic, the subgroup X is maximal in G. If $X^g = X$, then $G = \langle g\rangle X$, so that the index $|G : T|$ is finite and T is normal in G by Lemma 4.4, contrary to the assumption. It follows that $X \cap X^g$ is a maximal subgroup of X. If $X \cap X^g$ is infinite, then the intersection $T \cap T^g$ is also infinite, so that $T^g = T$ and T is normal in G, a contradiction. Thus the maximal subgroup $M = X \cap X^g$ of X is finite. On the other hand, M is also maximal in X^g, and hence there exists a maximal subgroup L of X such that $L^g = M$. If $L \neq M$, then $X = \langle L, M\rangle$ is contained in $\langle M, g\rangle$ and so $G = \langle M, g\rangle$; on the other hand, if $L = M$, the subgroup $\langle M, g\rangle$ is finite and $G = \langle\langle M, g\rangle, x\rangle$ for some element x of X having prime power order. In both cases it follows from Lemma 4.1 that G contains a finite normal subgroup N such that G/N is a Tarski group, and hence T is normal in G by Lemma 4.2, a final contradiction. □

Corollary 4.6. *Let G be a periodic group with nearly modular subgroup lattice, and let T_1 and T_2 be distinct generalized Tarski subgroups of G. Then $[T_1, T_2] = \{1\}$.*

Proof. The subgroups T_1 and T_2 are normal in G by Lemma 4.5, so that in particular $[T_1, T_2] \leq T_1 \cap T_2 = Z(T_1) \cap Z(T_2)$. Thus T_1 acts trivially on $T_2/Z(T_2)$. If y is any element of T_2, the finite subgroup $\langle y, Z(T_2)\rangle$ is normalized, and so even centralized by T_1. Therefore $[T_1, T_2] = \{1\}$. □

Lemma 4.7. *Let the group $G = A \times B$ be the direct product of two Tarski groups A and B. If the subgroup lattice $\mathfrak{L}(G)$ is nearly modular, then $\pi(A) \cap \pi(B) = \emptyset$.*

Proof. Assume by contradiction that there exists a prime number $p \in \pi(A) \cap \pi(B)$. Then $A = \langle a, x\rangle$ and $B = \langle b, y\rangle$, where all elements a, x, b, y have order p. Put $H = \langle ab, xy\rangle$, and let X be a modular subgroup of G containing H such that the index $|X : H|$ is finite. Since the elements ab and xy also have order p, it follows from Lemma 4.1 that H contains a finite normal subgroup K such that H/K is a Tarski group, and so by Lemma 4.2 there exists in X a generalized Tarski subgroup of finite index. In particular, X is a proper subgroup of G, and hence X contains neither A nor B because $AX = BX = G$. Since $\langle a\rangle^{xy} = \langle a\rangle^x \neq \langle a\rangle$, the subgroup $\langle a\rangle$ is not normalized by X, and so the proper subgroup $A \cap X$ of A is not trivial by Lemma 3.3. On the other hand, $A \cap X$ is normal in $G = XB$, and this contradiction completes the proof. □

Our next result will be crucial for our purposes.

Lemma 4.8. *Let G be a periodic group with nearly modular subgroup lattice. If G is generated by generalized Tarski subgroups, then there exists a finite subgroup Z of $Z(G)$ such that the subgroup lattice $\mathfrak{L}(G/Z)$ is modular.*

Proof. Let $\{T_i \mid i \in I\}$ be a collection of generalized Tarski subgroups of G such that $G = \langle T_i \mid i \in I\rangle$. It follows from Lemma 4.5 and Corollary 4.6 that every T_i is normal in G and $[T_i, T_j] = \{1\}$ for $i \neq j$, so that $Z(G) = \langle Z(T_i) \mid i \in I\rangle$, and the factor group $G/Z(G)$ is isomorphic to the direct product of the Tarski groups $T_i/Z(T_i)$, with $i \in I$. Thus by Lemma 4.7 we have that $\pi(T_i/Z(T_i)) \cap \pi(T_j/Z(T_j)) = \emptyset$ if $i \neq j$. Let I_0 be the subset of I consisting of all indices i such that T_i neither is a Tarski group nor an extended Tarski group; for each $i \in I_0$ the subgroup lattice $\mathfrak{L}(T_i)$ is not modular (see [13], Theorem 2.4.16), and hence T_i contains a cyclic non-modular subgroup $\langle x_i\rangle$ whose order is a power of a prime number p_i. Clearly $\langle x_i\rangle$ is not contained in the centre $Z(T_i)$ and $x_i^{p_i} \in Z(T_i)$; in particular, $p_i \neq p_j$ if $i \neq j$. Put $A = \langle x_i \mid i \in I_0\rangle$, and let X be a modular subgroup of $G_0 = \langle T_i \mid i \in I_0\rangle$ such that the index $|X : A|$ is finite. Clearly the subgroup A is abelian and $A \cap T_i = \langle x_i\rangle$ for each $i \in I_0$. Moreover, $AZ(G_0)$ is a maximal locally finite subgroup of G_0, so that X is contained in $AZ(G_0)$ and hence $X = AE$, where E is a finite subgroup of $Z(G_0)$. Put $\bar{G} = G/E$; then $\langle \bar{x}_i\rangle = \bar{X} \cap \bar{T}_i$ is a modular subgroup of $\bar{T}_i$ for all $i \in I_0$. Clearly $\bar{T}_i = \langle \bar{x}_i, \bar{x}_i^{\bar{g}_i}\rangle$ for some element $\bar{g}_i$ of $\bar{T}_i$, and the lattice $[\bar{T}_i/\langle \bar{x}_i\rangle]$ is isomorphic to the interval

$$[\langle \bar{x}_i^{\bar{g}_i}\rangle / \langle \bar{x}_i\rangle \cap \langle \bar{x}_i^{\bar{g}_i}\rangle] = [\langle \bar{x}_i^{\bar{g}_i}\rangle / \langle \bar{x}_i^{p_i}\rangle].$$

Thus $\langle \bar{x}_i\rangle$ is a maximal subgroup of $\bar{T}_i$, and hence $Z(\bar{T}_i) = \langle \bar{x}_i^{p_i}\rangle$. It follows that $\pi(\bar{T}_i) \cap \pi(\bar{T}_j) = \emptyset$ for all i, j in I such that $i \neq j$. Since the factor group

$$\bar{G} = \mathop{\mathrm{Dr}}_{i \in I} \bar{T}_i$$

has nearly modular subgroup lattice, application of Lemma 3.2 yields that all but finitely many $\bar{T}_i$'s have modular subgroup lattice. Therefore $\bar{G}$ contains a finite central subgroup $\bar{Z} = Z/E$ such that the subgroup lattice of $\bar{G}/\bar{Z}$ is modular, and Z is a finite central subgroup of G such that $\mathfrak{L}(G/Z)$ is modular. □

Lemma 4.9. *Let the group $G = A \times T$ be the direct product of a periodic abelian group A and a Tarski group T such that $\pi(A) \subseteq \pi(T)$. If the subgroup lattice $\mathfrak{L}(G)$ is nearly modular, then A is finite.*

Proof. Assume by contradiction that A is infinite, and let H be a subgroup of prime order of T. Since $\mathfrak{L}(G)$ is nearly modular, there exists a finite modular subgroup X of G containing H. Clearly $AX = AH$ and $B = A \cap X$ is a finite normal subgroup of G, so that replacing G by the factor group G/B we may suppose that $A \cap X = \{1\}$, and hence $H = X$ is modular in G. By hypothesis there exist elements $a \in A$ and $x \in T \setminus H$, with the same prime order p, such that $H \cap \langle a \rangle\langle x \rangle = \{1\}$. Thus

$$\langle H, ax \rangle \cap \langle a \rangle\langle x \rangle = \langle ax, H \cap \langle a \rangle\langle x \rangle\rangle = \langle ax \rangle.$$

On the other hand, $\{1\} \neq [x, H] = [ax, H] \leq T$, so that $\langle H, ax \rangle = \langle a \rangle T$, and hence

$$\langle H, ax \rangle \cap \langle a \rangle\langle x \rangle = \langle a \rangle T \cap \langle a \rangle\langle x \rangle = \langle a \rangle\langle x \rangle,$$

a contradiction because $\langle ax \rangle \neq \langle a \rangle\langle x \rangle$. Therefore A must be finite. □

LEMMA 4.10. *Let the group $G = T \times H$ be the direct product of a Tarski group T and an infinite P-group $H = \langle x \rangle \ltimes A$, where A is an abelian group of prime exponent $q \notin \pi(T)$ and x has prime order $p \in \pi(T)$. Then the subgroup lattice $\mathfrak{L}(G)$ is not nearly modular.*

Proof. Assume by contradiction that $\mathfrak{L}(G)$ is a nearly modular lattice. Let y be an element of prime order of T, and let X be a finite modular subgroup of G containing $\langle y, x \rangle$. Clearly $X = \langle y, x \rangle E$, where E is a finite subgroup of A. As every subgroup of A is normal in G, and the factor group G/E is also a counterexample, replacing G by G/E we may suppose that $\langle y, x \rangle$ is a modular subgroup of G. Let $z \in T \setminus \langle y \rangle$ be an element of order p, and let $a \neq 1$ be an element of A. Then the product zx has order p and $\langle x \rangle \cap \langle x \rangle^a = \{1\}$. Moreover, $[y, zx] = [y, z] \neq 1$, so that $T = \langle y, [y, zx] \rangle$ and hence

$$T\langle x \rangle^a = \langle y, zx \rangle^a = \langle y, (zx)^a \rangle \leq \langle \langle y, x \rangle, (zx)^a \rangle.$$

As $\langle y, x \rangle$ is modular in G, we have

$$\langle \langle y, x \rangle, (zx)^a \rangle \cap \langle z, x^a \rangle = \langle zx \rangle^a (\langle y, x \rangle \cap \langle z, x^a \rangle) = \langle zx \rangle^a.$$

Therefore

$$\langle z, x^a \rangle = T\langle x \rangle^a \cap \langle z \rangle\langle x \rangle^a \leq \langle \langle y, x \rangle, (zx)^a \rangle \cap \langle z, x^a \rangle = \langle zx \rangle^a,$$

and this contradiction proves the lemma. □

LEMMA 4.11. *Let G be a periodic non-trivial group whose locally finite radical is trivial. If the subgroup lattice $\mathfrak{L}(G)$ is nearly modular, then G is generated by its Tarski subgroups.*

Proof. Let T be the subgroup generated by all Tarski subgroups of G. It follows from Lemma 4.5, Corollary 4.6 and Lemma 4.7 that $T = \mathrm{Dr}_i\, T_i$, where each T_i is a Tarski group and $\pi(T_i) \cap \pi(T_j) = \emptyset$ if $i \neq j$. Assume by contradiction that T is properly contained in G, and let x be an element of $G \setminus T$ whose order is a power of a prime number p. If $p \in \pi(T_i)$, by Lemma 4.4 there exists an element y such that $\langle x, T_i \rangle = \langle y \rangle \times T_i$, and the same lemma also yields that $[y, T_j] = \{1\}$ for all $j \neq i$, so that y belongs to $C_G(T)$. On the other hand, $C_G(T) \cap T = \{1\}$, so that the normal subgroup $C_G(T)$ of G is locally finite by Corollary 4.3, and hence $C_G(T) = \{1\}$. Thus $y = 1$, and this contradiction proves the lemma. □

Proof of the Theorem. Let G be a group whose subgroup lattice $\mathfrak{L}(G)$ is nearly modular, and assume by contradiction that G does not contain any finite normal subgroup N such that $\mathfrak{L}(G/N)$ is modular. The locally finite radical L of G is a proper subgroup by Theorem 3.6, and Lemma 4.11 yields that the factor group G/L is generated by its Tarski subgroups, so that it follows from Lemma 4.2 that $G = LT$, where T is the subgroup generated by all generalized Tarski subgroups of G. The same lemma also gives that $[L, T] = \{1\}$. Moreover, by Theorem 3.6 the locally finite group L contains a finite normal subgroup E such that L/E has modular subgroup lattice, while it follows from Lemma 4.8 that there exists a finite subgroup Z of $Z(T)$ such that the lattice $\mathfrak{L}(T/Z)$ is modular. Clearly, EZ is a finite normal subgroup of G, and replacing G by G/EZ it can be assumed without loss of generality that

$$T = \mathop{\mathrm{Dr}}_{n \in \mathbb{N}} T_n,$$

where each T_n either is trivial or a Tarski or an extended Tarski group with $\pi(T_m) \cap \pi(T_n) = \emptyset$ if $m \neq n$, and

$$L = \mathop{\mathrm{Dr}}_{n \in \mathbb{N}} L_n,$$

where each L_n either is a primary group with modular subgroup lattice or a P^*-group and $\pi(L_m) \cap \pi(L_n) = \emptyset$ if $m \neq n$. Let K be the direct product of all subgroups L_n such that $\pi(L_n) \cap \pi(T) = \emptyset$. Then K is a direct factor of G and $\pi(K) \cap \pi(G/K) = \emptyset$, so that we may also suppose that $K = \{1\}$, and hence $\pi(L_n) \cap \pi(T) \neq \emptyset$ for all n such that $L_n \neq \{1\}$. For each positive integer n, let I_n be the set of all $j \in \mathbb{N}$ such that $\pi(L_j) \cap \pi(T_n) \neq \emptyset$ and $\pi(L_j) \cap \pi(T_m) = \emptyset$ for any $m < n$, and put $M_n = \mathrm{Dr}_{j \in I_n} L_j$ and $G_n = T_n M_n$. For every $j \in I_n$, there exists an abelian non-trivial subgroup A_j of L_j such that $\pi(A_j) \subseteq \pi(T_n)$. Since $[M_n, T_n] \leq [L, T] = \{1\}$, we have

$$G_n/Z(T_n) = T_n/Z(T_n) \times M_n Z(T_n)/Z(T_n),$$

so that Lemma 4.9 yields that the subgroup $\langle A_j \mid j \in I_n \rangle$ is finite and in particular the set I_n is finite. It follows that the subgroup M_n must be finite for every n. In fact, if M_n would be infinite, for some $j \in I_n$ the subgroup L_j should be an infinite P^*-group of the form $L_j = \langle x \rangle \ltimes A$, where A is an infinite abelian normal subgroup of prime exponent $q \notin \pi(T_n)$ and x is an element of order p^k for some prime $p \in \pi(T_n)$ and $k \geq 1$; thus the subgroup lattice $\mathfrak{L}(L_j T_n / \langle x^{p^{k-1}} \rangle)$ is not nearly modular by Lemma 4.10, contrary to the hypothesis of the theorem.

Now let $\mathcal{S}$ be the set of all subgroups G_n such that the lattice $\mathfrak{L}(G_n)$ is not modular, and assume that $\mathcal{S}$ is infinite. Since every M_n is finite, there exists a subsequence $(G_{r_n})_{n\in\mathbb{N}}$, consisting of elements of $\mathcal{S}$, such that $\pi(G_{r_m}) \cap \pi(G_{r_n}) = \emptyset$ if $m \neq n$. Therefore

$$\langle G_{r_n} \mid n \in \mathbb{N}\rangle = \operatorname*{Dr}_{n\in\mathbb{N}} G_{r_n}$$

and hence the subgroup lattice $\mathfrak{L}(\langle G_{r_n} \mid n \in \mathbb{N}\rangle)$ is not nearly modular by Lemma 3.2. This contradiction shows that $\mathcal{S}$ is finite and so the normal subgroup

$$M = \langle M_n \mid G_n \in \mathcal{S}\rangle$$

of G is also finite. Put $\bar{G} = G/M$ and use bars for homomorphic images modulo M. Then $\bar{G}_n$ has modular subgroup lattice for every positive integer n and so $\bar{G}_n = \bar{T}_n \times \bar{H}_n$, where $\bar{H}_n$ is finite and $\pi(\bar{T}_n) \cap \pi(\bar{H}_n) = \emptyset$ (see [13], Theorem 2.4.16). This implies that $\bar{L}_j = (\bar{L}_j \cap \bar{T}_n) \times (\bar{L}_j \cap \bar{H}_n)$ for any $j \in I_n$, so that $\bar{L}_j \cap \bar{T}_n \neq \{1\}$ and hence $\bar{L}_j \leq \bar{T}_n$. Thus $\bar{G}_n = \bar{T}_n$ for every n and the group

$$\bar{G} = \operatorname*{Dr}_{n\in\mathbb{N}} \bar{G}_n$$

has modular subgroup lattice. This last contradiction completes the proof. □

References

[1] I.S. Ashmanov and A.Y. Ol'shanskiĭ, *Abelian and central extensions of aspherical groups*, Soviet. Math. (Iz. VUZ) **29** (1985), 65–82.

[2] R. Baer, *The significance of the system of subgroups for the structure of the group*, Amer. J. Math. **61** (1939), 1–44.

[3] M. De Falco, F. de Giovanni, C. Musella, and Y.P. Sysak, *Groups in which every subgroup is nearly permutable*, Forum Math., to appear.

[4] M.R. Dixon, *Sylow theory, formations and Fitting classes in locally finite groups*, World Scientific, Singapore, 1994.

[5] S. Franciosi, F. de Giovanni, and Y.P. Sysak, *Groups with many polycyclic-by-nilpotent subgroups*, Ricerche Mat. **48** (1999), 361–378.

[6] F. de Giovanni and C. Musella, *Groups with nearly modular subgroup lattice*, Colloq. Math. **88** (2001), 13–20.

[7] F. de Giovanni, C. Musella, and Y.P. Sysak, *Groups with almost modular subgroup lattice*, J. Algebra **243** (2001), 738–764.

[8] K. Iwasawa, *Über die endlichen Gruppen und die Verbände ihrer Untergruppen*, J. Fac. Sci. Imp. Univ. Tokyo **4** (1941), 171–199.

[9] ———, *On the structure of infinite M-groups*, Jap. J. Math. **18** (1943), 709–728.

[10] B.H. Neumann, *Groups with finite classes of conjugate subgroups*, Math. Z. **63** (1955), 76–96.

[11] D.J.S. Robinson, *Finiteness conditions and generalized soluble groups*, Springer, Berlin, 1972.

[12] R. Schmidt, *Gruppen mit modularem Untergruppenverband*, Arch. Math. (Basel) **46** (1986), 118–124.

[13] ———, *Subgroup lattices of groups*, de Gruyter, Berlin, 1994.

[14] S.E. Stonehewer, *Permutable subgroups of infinite groups*, Math. Z. **125** (1972), 1–16.

[15] ______, *Modular subgroups of infinite groups*, Symposia Math., vol. XVII (INDAM, Rome, 1973), Academic Press, London, 1976, pp. 207–214.

[16] ______, *Modular subgroup structure in infinite groups*, Proc. London Math. Soc. (3) **32** (1976), 63–100.

M. De Falco, F. de Giovanni, C. Musella, Dipartimento di Matematica e Applicazioni, Università di Napoli "Federico II", Complesso Universitario Monte S. Angelo, Via Cintia, I-80126, Napoli, Italy

E-mail address, M. De Falco: defalco@matna2.dma.unina.it

E-mail address, F. de Giovanni: degiova@matna2.dma.unina.it

E-mail address, C. Musella: musella@matna2.dma.unina.it

Y. P. Sysak, Institute of Mathematics, Ukrainian National Academy of Sciences, vul. Tereshchenkivska 3, 01601 Kiev, Ukraine

E-mail address: sysak@imath.kiev.ua