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# GROUPS WITH THE MAXIMAL CONDITION ON NON FC-SUBGROUPS

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*To the memory of Reinhold Baer on the 100th anniversary of his birth*

Abstract. A group $G$ satisfies the maximal condition on non FC-subgroups if every ascending chain of non FC-subgroups terminates in finitely many steps. In this paper the authors obtain some structural results for locally FC-groups with the maximal condition on non FC-subgroups.

## 1. Introduction

The maximal and minimal conditions are some of the oldest, classical finiteness conditions in algebra. In group theory the maximal condition has played an important role in a number of fundamental papers of R. Baer [3], [4], [5], [6], [7], and its influence has been exhibited in various problems in group theory.

In this paper we consider another generalization of the maximal condition. Let $\mathcal{P}$ be a subgroup theoretical property. This property can be internal to the group as, for example, in the cases when $\mathcal{P}$ denotes the property of being a normal subgroup or a subnormal subgroup, or it can be external to the group, as in the case when $\mathcal{P}$ is the property of belonging to some class of groups $\mathfrak{X}$. If $G$ is a group then let $\mathcal{L}_{\overline{\mathcal{P}}}(G) = \{H \leq G \mid H \text{ does not have } \mathcal{P}\}$. Many papers have been written concerning properties $\mathcal{P}$ in which the set $\mathcal{L}_{\overline{\mathcal{P}}}(G)$ is small in some sense. For example, when $\mathcal{L}_{\overline{\mathcal{P}}}(G) = \{G\}$, $G$ will be a group in which all proper subgroups are $\mathcal{P}$-groups. The starting point of such research lies in the paper of R. Dedekind [12], in which all finite groups with all subgroups normal are classified. The general structure of such groups was described by Baer in [1].

For infinite group theory various finiteness conditions have been useful, particularly the maximal and minimal conditions. A group $G$ is said to satisfy max-(non-$\mathcal{P}$) if the set, $\mathcal{L}_{\overline{\mathcal{P}}}(G)$, ordered by inclusion, satisfies the maximal

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condition. Note that if every proper subgroup of a group $G$ is finite then $G$ satisfies max-(non-$\mathcal{P}$) for each property $\mathcal{P}$. There are rather exotic examples of such groups (see the book of A. Yu Ol'shanskii [27, Chapter 9]), so the study of groups with the conditions max-(non-$\mathcal{P}$) usually requires some additional restrictions. Since locally soluble groups with all proper subgroups finite have been classified, the condition of local solubility, and some of its generalizations, have been typical appropriate such restrictions. A number of papers have been concerned with the problem of classifying groups satisfying max-(non-$\mathcal{P}$) for various different properties $\mathcal{P}$. Groups with the maximal condition on non-abelian subgroups were considered by Kurdachenko and Zaicev [24] and groups with the maximal condition on non-nilpotent subgroups were considered by the authors in [13] and [14]. Groups with the maximal condition on non-normal subgroups were discussed in [11], [22] and groups with the maximal condition on non-subnormal subgroups have been considered in [23].

We recall that a group $G$ is called an FC-group (a term due to Baer) if every element of $G$ has only finitely many conjugates. This class, which is an important natural extension of the classes of abelian and finite groups, was first introduced by R. Baer in [2], and now the theory of FC-groups is a well-established part of infinite group theory. An important subclass of the class of FC-groups is the class of BFC-groups, where a group $G$ is a called a BFC-group if it has boundedly finite conjugacy classes or, equivalently, it is finite-by-abelian. In [15] and [16] we studied groups with the maximal condition on non BFC-groups. In this paper we study groups satisfying the maximal condition on non FC-subgroups. The group $G$ satisfies the condition max-(non-FC) if the set $\{H \leq G \mid H \text{ is not an FC-group}\}$ satisfies the maximal condition or, equivalently, if every ascending chain of non FC-subgroups terminates in finitely many steps. Note that the so-called minimal non FC-groups, those groups in which every proper subgroup is an FC-group, have been studied by a number of authors (see, for example, [8], [9], [25], [26]), and such groups satisfy max-(non-FC).

In [15] we discussed the class of locally FC-groups satisfying the maximal condition on non BFC-subgroups and were able to establish a number of structure theorems for such groups. In the current paper our results are necessarily less complete. This is because minimal non BFC-groups, those groups with infinite derived subgroup having all proper subgroups BFC, are well-understood, as the results of [10] show. On the other hand it is still unknown whether there exists a perfect minimal non FC-group. Such groups of course satisfy max-(non-FC).

In Section 2 we obtain a number of elementary results that are analogous to results from [15]. In Section 3 we obtain our main results. A key role is played by residually finite groups. We show, for example, that a locally finite, residually finite group satisfying max-(non-FC) is an FC-group. It is not difficult to show that a group with max-(non-FC) either is finitely generated or is a locally FC-group and in this paper we consider groups of the latter type. If $G$ is a locally FC-group then the set of elements of finite order forms a subgroup, the torsion subgroup of $G$, which we sometimes denote by $T(G)$. The derived subgroup of an FC-group is well-known to be periodic (see [30, Theorem 1.6]), so if $G$ is a locally FC-group then $G/T(G)$ is torsion-free abelian. There is therefore a good deal of interest in the form that this factor group can take. However the structure of $G/T(G)$ is rather limited. For example, we obtain the following theorem and we note that the structure of locally FC-groups with max-(non-BFC) was elucidated in [15].

Theorem 3.7. *Let $G$ be a locally FC-group satisfying max-(non-FC) and let $T$ be the torsion subgroup of $G$. If $G/T$ is not finitely generated then either $G$ is an FC-group or $G$ satisfies max-(non-BFC).*

When $G/T(G)$ is finitely generated the structure of $G$ is more problematic and we do not have a complete theory. For soluble groups, or locally nilpotent groups, however, much can be said. As a sample of our results we shall prove:

Theorem 3.13. *Let $G$ be a locally FC-group satisfying max-(non-FC). If $G$ is soluble then either $G$ is an FC-group or $G$ satisfies max-(non-BFC).*

The notation used in this paper is generally that in standard use and the reader is referred to [28] for such notation, when it is not immediately explained.

## 2. Elementary results

It is clear that the class of groups satisfying max-(non-FC) is closed under taking subgroups and factor groups. If $G$ is a group satisfying max-(non-FC) and $N \triangleleft G$ is a non-FC normal subgroup of $G$ then $G/N$ must satisfy the maximal condition on subgroups and we shall use this fact repeatedly without specifically mentioning it. One easy consequence of this is that if $G$ is a group satisfying max-(non-FC) and if $N \triangleleft G$, where $G/N$ is an infinite direct product of non-trivial groups, then $N$ is an FC-subgroup of $G$. We shall now obtain some less obvious results. Some of the results we obtain are analogous to results occurring in [15]; where possible we use the corresponding result from [15] to obtain our result in the more general case. We shall require the following information about finitely generated finite-by-abelian groups. Such groups are of course BFC-groups.

Lemma 2.1. *Let $G$ be a finitely generated finite-by-abelian group and let $Q$ be a periodic subgroup of $\mathrm{Aut}\,G$. Then $Q$ is finite.*

*Proof.* Since $G/G'$ is a finitely generated abelian group there is a finite normal subgroup $T$ of $G$ such that $G/T$ is finitely generated and torsion-free abelian. Then $\mathrm{Aut}\,(G/T) \cong GL_n(\mathbb{Z})$, where $n$ is the torsion-free rank of $G/T$, and $\operatorname{Aut} T$ is finite. Since $Q$ is periodic [31, Corollary 4.8] implies that $Q/C_Q(G/T)$ is finite and of course $Q/C_Q(T)$ is also finite. Let $L = C_Q(T) \cap C_Q(G/T)$. Then $L$ is a stability group of the chain $1 \leq T \leq G$ and it is easily seen using [19, 1.C.3] that $L$ is also finite. Thus $Q$ is finite since $Q/L$ embeds in $Q/C_Q(G/T) \times Q/C_Q(T)$. □

We shall let $FC(G)$ denote the FC-center of $G$. As is well-known this is the subgroup of elements of $G$ that have only finitely many conjugates.

LEMMA 2.2. *Let $G$ be a group satisfying max-(non-FC) and suppose that $H$ is a normal subgroup of $G$ such that $G/H$ is a direct product of infinitely many cyclic groups of prime order. Then $G$ is an FC-group.*

*Proof.* First we show that $H \leq FC(G)$. Clearly $H$ is an FC-group. Suppose, for a contradiction, that $x \in H \setminus FC(G)$ and let $X = \langle x \rangle^H$. Since $H$ is an FC-group $X$ is a finitely generated, center-by-finite group and hence $X'$ is finite. Now $H/C_H(X)$ is finite and $G/H$ is periodic, so $N_G(X)/C_G(X)$ is also periodic and hence Lemma 2.1 implies that $N_G(X)/C_G(X)$ is finite. By the choice of $x$ it follows that $G/N_G(X)$ is an infinite direct product of non-trivial cyclic groups. Hence there exists $U \leq G$ such that $N_G(X) \leq U$ and both $G/U$ and $U/N_G(X)$ are infinite direct products of cyclic groups. Since $G$ has max-(non-FC) it follows that $U$ is an FC-group and in particular $|U : C_U(X)|$ is finite. Thus $U/N_U(X)$ is finite which is the contradiction sought. Thus $H \leq FC(G)$.

Now let $g \in G$ and let $V = \langle H, g \rangle$. Then $V \lhd G$, $V/H$ is finite and $G/V$ is a direct product of cyclic groups of prime order. Thus $V \leq FC(G)$ and the result follows. □

Our next result is of a similar type.

LEMMA 2.3. *Let $G$ be a group satisfying max-(non-FC) and suppose that $H$ is a normal subgroup of $G$ such that $G/H = C_1/H \times C_2/H$ where, for $i = 1, 2$, $C_i/H$ is a Prüfer $p_i$-group, for the prime $p_i$. Then $G$ is an FC-group.*

*Proof.* First we show that $H \leq FC(G)$. Clearly each $C_i$ is an FC-group. Let $x \in H$ and let $X = \langle x \rangle^H$. Since $H$ is an FC-group, $X$ is finitely generated and hence center-by-finite. Since $C_i$ is an FC-group we have $|C_i : C_{C_i}(X)|$ finite for each $i$. Since $H \leq N_G(X)$ and $C_i/H$ is a Prüfer group it follows that $X \lhd C_i$ for $i = 1, 2$ and hence $X \lhd G$. Now $H/C_H(X)$ is finite, so $G/C_G(X)$ is periodic and since $X'$ is finite Lemma 2.1 implies that $G/C_G(X)$ is finite. Thus $x \in FC(G)$ and $H \leq FC(G)$ as required.

Now let $g \in G$ and let $V = \langle H, g \rangle$. Then $V \lhd G$, $V/H$ is finite and as above $V \leq FC(G)$. The result follows. □

We can now amalgamate Lemmas 2.2 and 2.3 to obtain the possible periodic abelian factor groups of a non FC-group with max-(non-FC).

Lemma 2.4. *Let $G$ be a group satisfying max-(non-FC) and suppose that $H$ is a normal subgroup of $G$ such that $G/H$ is a periodic abelian group. If $G$ is not an FC-group then either $G/H$ is finite or $G/H = F/H \times D/H$, where $F/H$ is finite and $D/H \cong C_{p^\infty}$ for some prime $p$.*

*Proof.* Let $\pi = \pi(G)$ so that $G/H = \underset{p\in\pi}{\mathrm{Dr}}\, S_p/H$, where $S_p/H$ is the $p$-component of $G/H$. If $S_p/S_p^pH$ is infinite for some prime $p$ then Lemma 2.2 implies that $G$ is an FC-group, contrary to the hypothesis. Thus $S_p/S_p^pH$ is finite for every prime $p$. It follows from [21, Lemma 7] that, for each prime $p$, $S_p/H = F_p/H \times D_p/H$, where $F_p/H$ is finite and $D_p/H$ is divisible. Let $\sigma = \{p \in \pi \mid F_p \neq H\}$ and $\rho = \{p \in \pi \mid D_p \neq H\}$. Then $\sigma$ is finite by Lemma 2.2 and $|\rho|$ is at most 1 by Lemma 2.3. If $|\rho| = \{p\}$ then Lemma 2.3 also implies that $D_p/H = D/H \cong C_{p^\infty}$, as required. □

We have already remarked that in a locally FC-group $G$ the factor group $G/T(G)$ is abelian. We can now be a little more specific as to the structure of this quotient group.

Lemma 2.5. *Let $G$ be a locally FC-group satisfying max-(non-BFC). Suppose that $H \lhd G$ and that $G/H$ is a BFC-group and suppose that $G$ is not an FC-group. Then $G/H$ has finite rank and either*

(i) *$G/H$ is finitely generated, or*
(ii) *$G/H$ contains a finitely generated normal subgroup $L/H$ such that $G/L \cong C_{p^\infty}$, for some prime $p$.*

*Proof.* By hypothesis there is a subgroup $K \lhd G$ such that $G/K$ is abelian and $K/H$ is finite. If $G/K$ has infinite torsion-free rank then it is easy to construct a subgroup $L/K$ such that $G/L$ is periodic and $\pi(G/L)$ is infinite, which contradicts Lemma 2.4. Thus $G/K$ has finite torsion-free rank and there is a finitely generated torsion-free subgroup $M/K$ such that $G/M$ is periodic. By Lemma 2.4 again, it follows that either $G/M$ is finite or $G/M = L/M \times D/M$, for some finite subgroup $L/M$ and subgroup $D/M \cong C_{p^\infty}$, for some prime $p$. Hence (i) or (ii) hold and clearly $G/H$ has finite rank. □

The following lemma is no doubt well-known but we include the proof anyway.

Lemma 2.6. *Let $G$ be an abelian-by-finitely generated FC-group. Then $G$ is a BFC-group*

*Proof.* Let $A \lhd G$ be abelian such that $G/A$ is finitely generated. Since $G$ is an FC-group, $\langle x\rangle^G$ is finite-by-cyclic, for each $x \in G$, and hence there is a finitely generated normal subgroup $X$ such that $G = AX$. Then $G' \leq X$ is finitely generated and periodic and hence is finite. The result follows. $\square$

We often need to consider groups with certain types of abelian normal subgroups. The next few lemmas take care of the situations we need to understand.

LEMMA 2.7. *Let $G$ be a group satisfying max-(non-FC) and suppose that $A$ is an infinite normal elementary abelian $p$-subgroup of $G$, for some prime $p$, such that $G/A$ is finitely generated. If $G$ is a locally FC-group, then $G$ contains a normal subgroup $H$ such that $G/H$ is an infinite elementary abelian $p$-group. Moreover $G$ is a BFC-group.*

*Proof.* The group $G/A$ is a finitely generated FC-group and it follows from Lemma 2.6 that if $H$ is an FC-subgroup of $G$ then $H$ is a BFC-group. Thus $G$ satisfies max-(non-BFC) and the result follows from [15, Lemmas 5.3 and 2.3]. $\square$

LEMMA 2.8. *Let the group $G$ contain an infinite normal elementary abelian $q$-subgroup $A$ such that $G/A$ is a Prüfer $p$-group, where $p, q$ are primes. If $G$ satisfies max-(non-FC) then $G$ is abelian.*

*Proof.* We claim that such a group satisfies max-(non-BFC). If $H$ is an FC-subgroup of $G$ then either $HA/A$ is finite or $HA/A \cong C_{p^\infty}$. In the former case Lemma 2.6 implies that $H$ is a BFC-group. In the latter case $H/(H \cap A) \cong C_{p^\infty}$. If $p \neq q$ it follows from [19, 1.D.4] that there is a $p$-subgroup $D$ such that $H = (H \cap A)D$. Since $p \neq q$ we have $D \cong C_{p^\infty}$. If $p = q$ then by [20, Corollary to Lemma 1] $H = (H \cap A)D$ for some divisible group $D$. In either case we have $D \leq Z(H)$, by [30, Theorem 1.9], and $H$ is abelian, from which the claim follows. The result can now be read off from the analogous results [15, Lemma 5.4 and Corollary 4.2]. $\square$

LEMMA 2.9. *Let $G$ be a locally FC-group and suppose that $A \leq L$ are normal subgroups of $G$ satisfying the following conditions:*

(i) *$A$ is an infinite elementary abelian $q$-subgroup, for some prime $q$,*
(ii) *$L/A$ is finitely generated,*
(iii) *$G/L$ is a Prüfer $p$-group.*

*If $G$ satisfies max-(non-FC) then $G$ contains a normal subgroup $D$ such that $G/D$ is an infinite elementary abelian $p$-group. In particular, $G$ is an FC-group.*

*Proof.* By Lemma 2.7 $L$ contains a normal subgroup $B_1$ such that $L/B_1$ is an infinite elementary abelian $q$-group. Hence if $B_2 = L^q L'$ then $L/B_2$ is an infinite elementary abelian $q$-group. By Lemma 2.8 $G/B_2$ is abelian, so $G/B_2 = (L/B_2)(D/B_2)$, where $D/B_2$ is a Prüfer group, by [20, Lemma

1]. Since $(L/B_2) \cap (D/B_2)$ is finite, $G/D$ is an infinite elementary abelian $q$-group. The result now follows from Lemma 2.2. $\square$

## 3. Locally FC-groups satisfying max-(non-FC)

The key to the results we obtain lies in the residually finite case and is embodied in the next two propositions.

PROPOSITION 3.1. *Let $G$ be a residually finite, locally finite group satisfying max-(non-FC). Then $G$ is an FC-group.*

*Proof.* Suppose, for a contradiction, that there exists an element $g \in G$ such that $|G : C_G(g)|$ is infinite. We claim that if $A$ is a finite subgroup of $G$ containing $g$ then there exists $M \triangleleft G$ such that $G/M$ is finite, $A \cap M = 1$ and there exists $x \in M \setminus C_M(g)$ such that $g^x \neq g^a$ for all $a \in A$. To prove the claim note that it is clear that there are such subgroups $A$. Also since $G$ is residually finite there exists $M \triangleleft G$ such that $A \cap M = 1$ and $G/M$ is finite. Also $|M : C_M(g)|$ is infinite whereas the set $\{g^a \mid a \in A\}$ is finite, so the existence of such an element $x$ is also clear.

Now choose a finite subgroup $B_1 = A_1$ containing $g$ and such that $g$ does not centralize $A_1$. Then by the claim there exists $M_1 \triangleleft G$ and $a_1 \in M_1 \setminus C_{M_1}(g)$ such that $M_1 \cap A_1 = 1$, $G/M_1$ is finite and $g^{a_1} \neq g^a$ for all $a \in A_1$. Let $A_2 = \langle a_1 \rangle^{A_1}$ and $B_2 = A_1 A_2$. Then $A_2 \leq M_1$ and $B_2$ is finite. In general, having constructed $A_1, B_1, \ldots, A_r, B_r, M_1, \ldots, M_{r-1}$, $a_1, \ldots, a_{r-1}$ with $B_r = A_1 \ldots A_r$ finite, there exists $M_r \triangleleft G$ such that $M_r \leq M_{r-1}$, $G/M_r$ is finite and $B_r \cap M_r = 1$. By the claim there exists $a_r \in M_r \setminus C_{M_r}(g)$ such that $g^{a_r} \neq g^a$ for all $a \in B_r$. Let $A_{r+1} = \langle a_r \rangle^{B_r}$ and $B_{r+1} = B_r A_{r+1}$. Then $A_{r+1} \leq M_r$, so $B_r \cap A_{r+1} = 1$ and the construction proceeds.

Let $H = \bigcup_{i \geq 1} B_i$ and let $K = A_1 A_3 A_5 \ldots$. By construction, $K$ contains $g, g^{a_2}, g^{a_4}, \ldots$ and these are all distinct. Thus $K$ is not an FC-group. On the other hand $K \lneqq KA_2 \lneqq KA_2A_4 \lneqq \cdots$ is a strictly ascending chain of non-FC-subgroups yielding the desired contradiction. $\square$

PROPOSITION 3.2. *Let $G$ be a locally FC-group satisfying max-(non-FC) and suppose that $H$ is a normal subgroup of $G$ such that $G/H$ is an infinite periodic residually finite group. Then $G$ is an FC-group.*

*Proof.* Note that $G/H$ is locally finite, so Proposition 3.1 implies that $G/H$ is an FC-group. Suppose for a contradiction that there exists $g \in G$ such that $|G : C_G(g)|$ is infinite. Now $\langle g \rangle^G H/H$ is finite and hence $G/\langle g \rangle^G H$ is also an infinite residually finite group. In particular there exists $N \triangleleft G$ such that $g \in N$ and $G/N$ is an infinite residually finite, locally finite group. Now $N$ must be an FC-group, for otherwise $G/N$ has the maximum condition and hence is finite. Consequently $g$ has only finitely many conjugates in $N$. Since

$g$ has infinitely many conjugates we can choose $a_1 \in G$ such that $g^{a_1} \neq g^a$ for all $a \in N$. Since $a_1 \notin N$ but $G/N$ is a locally finite FC-group there exists $N_1/N \lhd G/N$ such that $a_1 \in N_1$ and $N_1/N$ is finite. Again, $N_1$ is an FC-group and there exists $M_1/N \lhd G/N$ such that $G/M_1$ is finite, $M_1/N \cap N_1/N = 1$ and $|M_1 : C_{M_1}(g)|$ is infinite. In general we construct normal subgroups $N_1, N_2, \dots N_r, M_1 \dots, M_{r-1}$ and elements $a_i \in N_i$ (for $i = 1, \dots, r$) satisfying:

(i) $N_i/N$ is finite,
(ii) $N_1 \dots N_i/N \cap N_{i+1}/N = 1$,
(iii) the elements $g^{a_i}$ are all distinct,
(iv) $N_1 \dots N_i$ is an FC-group,
(v) $|M_i : C_{M_i}(g)|$ is infinite.

Since $N_1 \dots N_r/N$ is finite there exists $M_r/N \leq M_{r-1}/N$ such that $G/M_r$ is finite and $N_1 \dots N_r/N \cap M_r/N = 1$. Clearly $M_r$ satisfies condition (v). Since $N_1 \dots N_r$ is an FC-group there exists $a_{r+1} \in M_r$ such that $g^{a_{r+1}} \neq g^a$ for all $a \in N_1 \dots N_r$ and $N_{r+1} = \langle a_{r+1} \rangle^G N/N$ is finite. Clearly $N_{r+1} \leq M_r$ and so $N_1 \dots N_r/N \cap N_{r+1}/N = 1$. Furthermore, $N_1 \dots N_{r+1}$ is an FC-group, $g^{a_{r+1}}$ is distinct from $g^{a_1}, \dots, g^{a_r}$ and $|M_r : C_{M_r}(g)|$ is infinite, by the choice of $g$. The construction therefore proceeds and we can inductively construct the elements and groups satisfying (i)–(v).

Now let $K = N_1 N_2 \dots$ and $L = N_1 N_3 \dots$. By construction $L$ is not an FC-group and $L \lneqq LN_2 \lneqq LN_2N_4 \lneqq \cdots$ is an infinite ascending chain of non-FC-subgroups, which is a contradiction. The proposition follows. □

LEMMA 3.3. *Let $G$ be a locally FC-group satisfying max-(non-FC). Suppose that $G$ contains normal subgroups $H, R$ satisfying the following conditions:*

(i) *$G/H \cong C_{p^\infty}$ for some prime $p$;*
(ii) *$R$ contains an $H$-invariant subgroup $L$ such that $R/L$ is either a finite non-abelian simple group or a finite elementary abelian $q$-group for some prime $q$;*
(iii) *$R \leq H$ and $H/R$ is finite.*

*Then $L$ is $G$-invariant.*

*Proof.* Since $G/H$ does not satisfy the maximal condition, $H$ is an FC-group. Let $L_0 = \operatorname{core}_G L$. If $R/L_0$ is finite then $G/L_0$ is a Černikov group and hence $G/L_0 = H/L_0 \cdot C/L_0$, where $C/L_0 \cong C_{p^\infty}$. Since $H/L_0$ is finite we have $C/L_0 \leq Z(G/L_0)$ and hence $L/L_0$ is normal in $G/L_0$. Thus $L = L_0$ in this case.

Thus we may suppose that $R/L_0$ is infinite. Factoring by $L_0$ we may assume that $L_0 = 1$, so that $R$ is then of finite exponent and $G$ is locally finite. If $R/L$ is a finite elementary abelian $q$-group then $R$ embeds in the Cartesian product of the groups $R/L^g$, for $g \in G$, and it follows that $R$ is an infinite elementary abelian $q$-group. If $P/R$ is the divisible part of the Černikov group

$G/R$ then $P/R \cong C_{p^\infty}$ and, using Lemma 2.8, $P$ is abelian. Since $G/P$ is finite, $L$ has only finitely many conjugates in $G$ which gives the contradiction that $R$ is finite. The result follows in this case.

We may therefore suppose that $R/L$ is a finite non-abelian simple group. It follows by [19, 1.D.4] that $G$ contains a $p$-subgroup $P$ such that $G = PH$. If $p \notin \pi(H)$ then $P \cong C_{p^\infty}$. If $p \in \pi(H)$ then $P \cap H$ is a residually finite FC $p$-group of finite exponent. Let $X = P \cap H$. If $X/X'X^p$ is infinite then, by Lemma 2.8, $P/X'X^p$ is abelian and Lemma 2.4 implies that $P$ is an FC-group. On the other hand if $X/X'X^p$ is finite then by [18, Lemma 2] $X$ is nilpotent and $X/X'$ is of finite exponent, so $X/X'$ is finite, whence $X$ is also finite. It is easy to see that $P$ is an FC-group in this case also. Thus, in any case, $P$ is an FC-group and it follows from [20, Corollary to Lemma 1] that $G$ contains a subgroup $C \cong C_{p^\infty}$ such that $G = HC$.

Since $R$ is a periodic FC-group it is generated by its finite normal subgroups. If $F$ is a finite normal subgroup of $R$ then $F^H$ is also finite, since $H/R$ is finite. Thus we may assume that $F$ is $H$-invariant. Since $\operatorname{core}_G L = 1$ there exists a finite subset $S$ of $G$ such that $F \cap (\bigcap_{g\in S} L^g) = 1$. Thus $F$ can be embedded in a finite direct product of finite simple groups, so $F$ itself is a direct product of such subgroups. Let $E$ be a minimal $H$-invariant subgroup of $F$ and suppose that $E^G = E^C$ is infinite. Since $E^c$ is a normal $H$-simple subgroup of $H$, for each $c \in C$, it follows from [29, 3.3.11] that $E^G$ is an infinite direct product of the subgroups $E^c$ for certain $c \in C$. If $a \in E$ has prime order $q$ then $Q = \langle a\rangle^C$ is an infinite elementary abelian $q$-group and it follows using Lemma 2.8 that $QC$ is abelian. This gives the contradiction that $a^c = a$ for all $c \in C$. It follows that $E^G$ is finite and hence $C \le C_G(E^G)$. Since $F$ is the direct product of such minimal $H$-invariant subgroups it follows that $C$ also centralizes $F$ and hence $C \le C_G(R)$. In particular, since $L$ is $H$-invariant, it is also $G$-invariant and the result follows. □

Corollary 3.4. *Let $G$ be a locally FC-group satisfying max-(non-FC). Suppose that $H \triangleleft G$ satisfies the following conditions:*

(i) *$G/H \cong C_{p^\infty}$ for some prime $p$;*
(ii) *$H$ contains a normal subgroup $L$ such that $H/L$ is finite.*

*Then $L$ is $G$-invariant.*

*Proof.* Since $H/L$ is finite it has an $H$-chief series, so the result follows by induction on the length of this series, using Lemma 3.3. □

Lemma 3.5. *Let $G$ be a locally FC-group satisfying max-(non-FC). Suppose that $H$ is a normal periodic subgroup of $G$ satisfying the following conditions:*

(i) *$G/H$ is finitely generated;*
(ii) *$H$ contains a normal subgroup $L$ such that $H/L$ is finite.*

*Then $H/\mathrm{core}_G\, L$ is finite.*

*Proof.* Let $L_0 = \mathrm{core}_G\, L$ and suppose for a contradiction that $H/L_0$ is infinite. Since $H/L_0$ embeds in the Cartesian product of the groups $H/L^g$ as $g$ runs through $G$ it follows that $H/L_0$ is a periodic infinite residually finite group. By Proposition 3.2 $H$ is an FC-group and it follows that $H = FL$ for some finite normal subgroup $F$ of $H$.

Since $G/H$ is finitely generated, $G = KH$ for some finitely generated subgroup $K$ and $\langle F, K\rangle$ is a finitely generated FC-group, so is center-by-finite. Thus $|G : N_G(F)|$ is finite and hence $F^G$ is also finite. Let $G_1 = C_G(F^G), H_1 = G_1 \cap H$ and $L_1 = G_1 \cap L = H_1 \cap L$. Let $g_1, g_2 \in H_1L$ so that $g_1 = xu_1$, where $x \in H_1, u_1 \in L$. Since $H = F^G L$ write $g_2 = yu_2$, where $y \in F^G$ and $u_2 \in L$. Using the commutator laws we have $[g_1, g_2] = [x, u_2]^{u_1}[u_1, u_2][u_1, y]^{u_2}$. Since $L \lhd H$, $[g_1, g_2] \in L$ and hence $H_1L/L$ is abelian. Since $H_1/L_1 \cong H_1L/L$ it follows that $H_1/L_1$ is also abelian and since $H_1 \lhd G$, $H_1/L_1^g$ is abelian for all $g \in G$. Thus if $Q = \mathrm{core}_G\, L_1$ then $H_1/Q$ is a bounded abelian group. Since $F^G$ is a finite normal subgroup of $G$, $G/C_G(F^G)$ is finite, so $H/H_1$ is also finite. Thus $H_1/Q$ is infinite since $H/L_0$ is infinite and $Q \leq L_0$. Now $H_1/Q$ is bounded and hence contains a characteristic subgroup $H_2/Q$ of finite index such that $(H_2/Q)/(H_2/Q)^p$ is infinite for some prime $p$. Also $G/H_2$ is finitely generated, so by Lemma 2.7 there exists $M \lhd G$ such that $G/M$ is an infinite elementary abelian $p$-group and by Lemma 2.2 $G$ is an FC-group.

Hence $G/Z(G)$ is periodic and therefore $KZ(G)/Z(G)$ is finite. Consequently there exists a natural number $m$ such that $K^m \leq Z(G)$. However $K/K^m$ is also finite. It follows that $L$ has only finitely many conjugates in $G$, which is a contradiction. □

Our next result has a number of interesting consequences for the structure of the groups we are considering.

PROPOSITION 3.6. *Let $G$ be a locally FC-group satisfying max-(non-FC) and let $R$ be a normal subgroup of $G$ such that $G/R \cong C_{p^\infty}$, for some prime $p$. If $G$ is not an FC-group then $R$ is an abelian-by-finite FC-group and hence is finite-by-abelian.*

*Proof.* Clearly $R$ is an FC-group. Let $H$ be the hypercenter of $R$ and suppose that $R/H$ is infinite. By a theorem of Gorčakov [17] $R/H$ can be embedded in a direct product of finite groups, each with trivial center, and hence $R$ contains normal subgroups $H_\lambda$, for $\lambda \in \Lambda$, such that $R/H_\lambda$ is finite with trivial center and $\bigcap_{\lambda\in\Lambda} H_\lambda = H$. By Corollary 3.4 $H_\lambda \lhd G$ for each $\lambda \in \Lambda$. Let $D_\lambda/H_\lambda$ denote the divisible part of the Černikov group $G/H_\lambda$. Since $G/C_G(R/H_\lambda)$ is finite $D_\lambda \leq C_G(R/H_\lambda)$. Since $Z(R/H_\lambda) = 1$ we have $D_\lambda \cap R = H_\lambda$ and $G/H_\lambda = R/H_\lambda \times D_\lambda/H_\lambda$.

Let $D = \bigcap_{\lambda\in\Lambda} D_\lambda$. Then $G/D$ is residually finite and $D \cap R = \bigcap_{\lambda\in\Lambda}(D_\lambda \cap R) = \bigcap_{\lambda\in\Lambda} H_\lambda = H$. In particular $RD/D \cong R/(D\cap R) = R/H$ so, by assumption, $G/D$ is infinite also. Of course $G/D$ is periodic, so it follows from Proposition 3.2 that $G$ is an FC-group, which is a contradiction. Consequently, $R/H$ is finite.

Let $Z$ be the center of $H$. Since $H$ is an FC-group $H/Z$ is a periodic, residually finite, hypercentral group. Let $H_q/Z$ be a Sylow $q$-subgroup of $H/Z$ for the prime $q$ and suppose that $(H_q/Z)/(H_q/Z)'(H_q/Z)^q$ is infinite. Then $(H/Z)/(H/Z)'(H/Z)^q$ is also infinite and by Lemma 2.9 $G$ is then an FC-group, contrary to our assumption. It follows that $(H_q/Z)/(H_q/Z)'(H_q/Z)^q$ is finite for each prime $q$ and hence by [18, Lemma 2] $H_q/Z$ is finite.

Suppose that $\pi(H/Z)$ is infinite. Then there is a $G$-invariant subgroup $S \leq H$, containing $Z$, such that every Sylow $q$-subgroup of $H/S$ is elementary abelian, $\pi(H/S)$ is infinite and $\pi(H/S)\cap\pi(G/H) = \emptyset$. If $E/H$ is the divisible part of $G/H$ then $E/H \cong C_{p^\infty}$ and clearly $H/S \leq Z(E/S)$, so $E/S$ is center-by-locally cyclic and hence is abelian. If $P/S$ is a Sylow $p$-subgroup of $E/S$ then $P/S \cong C_{p^\infty}$ and $E/P \cong H/S$. Hence $G/P$ is an infinite periodic residually finite group. By Proposition 3.2 $G$ is then an FC-group, a contradiction. Thus the set $\pi(H/Z)$ is finite, so $H/Z$ is also finite. Hence $R/Z$ is finite and $R$ is abelian-by-finite. Lemma 2.6 now gives the result. $\square$

The structure of locally FC-groups with max-(non-BFC) is given in [15]. The following result therefore gives the structure of certain locally FC-groups with max-(non-FC).

THEOREM 3.7. *Let $G$ be a locally FC-group satisfying max-(non-FC) and let $T$ be the torsion subgroup of $G$. If $G/T$ is not finitely generated then either $G$ is an FC-group or $G$ satisfies max-(non-BFC).*

*Proof.* Since $G/T$ is abelian and not finitely generated it follows from Lemma 2.5 that $G/T$ contains a subgroup $R/T$ such that $G/R \cong C_{p^\infty}$ for some prime $p$. By Proposition 3.6 $R$ is a BFC-group.

Let $L$ be an FC-subgroup of $G$. Then $L/(L\cap R) \cong LR/R$ either is finite or is isomorphic to $C_{p^\infty}$. In the former case $L$ is a BFC-group, by Lemma 2.6, since $R$ is abelian-by-finite. In the latter case $(L\cap R)'$ is finite and if $x \in L\cap R$ then $L/(L\cap R)'/C_{L/(L\cap R)'}(x(L\cap R)')$ is finite and $L\cap R/(L\cap R)' \leq C_{L/(L\cap R)'}(x(L\cap R)')$. Thus $L/(L\cap R)'$ is center-by-locally cyclic and hence abelian, so again $L'$ is finite. Thus every FC-subgroup of $G$ is a BFC-group and consequently $G$ satisfies max-(non-BFC), as required. $\square$

In order to study the case when $G/T(G)$ is finitely generated we shall require the following easily proven lemma.

LEMMA 3.8. *Let $G$ be a group satisfying max-(non-FC) and suppose that $H$ is a subgroup of $G$. Suppose that $U \leq V$ are $H$-invariant subgroups of*

*$G$ and that $H \cap V \leq U$. If $UH$ is not an FC-group then $V/U$ satisfies the maximal condition on $H$-invariant subgroups.*

In the case when $G/T(G)$ is finitely generated things are not always as smooth as in the contrary case.

LEMMA 3.9. *Let $G$ be a locally FC-group satisfying max-(non-FC) and let $T$ be the torsion subgroup of $G$. If $G/T$ is finitely generated and $G$ is not an FC-group then $T/T^{\mathfrak{F}}$ is finite.*

*Proof.* Suppose on the contrary that $T/T^{\mathfrak{F}}$ is infinite. Then $T$ is an FC-group by Proposition 3.2. Let $g \in G \setminus T$, $L = \langle g, T\rangle$ and let $H \triangleleft T$ be finite. Then $\langle H, g\rangle$ is an FC-group, so $g^m \in C_G(H)$, for some natural number $m$, and hence $H^{\langle g\rangle}$ is also finite. We claim that there is a natural number $t$ such that $g^t \in C_G(T/T^{\mathfrak{F}})$. Suppose not and let $H_1/T^{\mathfrak{F}}$ be a finite $L$-invariant subgroup of $T/T^{\mathfrak{F}}$. Then there exists a natural number $r_1$ such that $\langle g^{r_1}\rangle = C_{\langle g\rangle}(H_1/T^{\mathfrak{F}})$. Since $T/T^{\mathfrak{F}}$ is residually finite there exists $E_1 \triangleleft T$ such that $T/E_1$ is finite and $H_1 \cap E_1 = T^{\mathfrak{F}}$. By Lemma 3.5 $|T : \bigcap_{x\in\langle g\rangle} E_1^x|$ is finite, so we may assume that $E_1$ is $L$-invariant. Since $T/E_1$ is finite there is a finite $L$-invariant subgroup $K_1/T^{\mathfrak{F}}$ such that $T/T^{\mathfrak{F}} = (K_1/T^{\mathfrak{F}})(E_1/T^{\mathfrak{F}})$ and hence there is a natural number $m_1$ such that $g^{m_1} \in C_G(K_1/T^{\mathfrak{F}})$. If there is a natural number $t_1$ such that $g^{t_1} \in C_G(H/T^{\mathfrak{F}})$ for every $L$-invariant subgroup $H/T^{\mathfrak{F}}$ of $E_1/T^{\mathfrak{F}}$ then $g^{t_1m_1} \in C_G(T/T^{\mathfrak{F}})$, and the claim follows. Hence we may assume that $E_1/T^{\mathfrak{F}}$ contains a finite $L$-invariant subgroup $H_2/T^{\mathfrak{F}}$ such that $\langle g^{r_2}\rangle = C_{\langle g\rangle}(H_2/T^{\mathfrak{F}})$ with $r_2 > r_1$. Again there is an $L$-invariant subgroup $E_2/T^{\mathfrak{F}}$ such that $T/E_2$ is finite and $H_1H_2 \cap E_2 = T^{\mathfrak{F}}$. In this way we construct a family of finite $L$-invariant subgroups $\{H_n/T^{\mathfrak{F}} \mid n \in \mathbb{N}\}$ such that $\langle H_n/T^{\mathfrak{F}} | n \in \mathbb{N}\rangle = \operatorname{Dr}_{n\in\mathbb{N}} H_n/T^{\mathfrak{F}}$ and if $\langle g^{r_n}\rangle = C_{\langle g\rangle}(H_n/T^{\mathfrak{F}})$ then $r_1 < r_2 < \cdots$. Let $U/R = \operatorname{Dr}_{n\in\mathbb{N}} H_{2n+1}/T^{\mathfrak{F}}$. Then $C_{\langle g\rangle}(U/T^{\mathfrak{F}}) = 1$ and hence $\langle U, g\rangle$ is not an FC-group. However $L/U$ has a strictly ascending series $H_2U/U < H_2H_4U/U < \cdots$ of $\langle g\rangle$-invariant subgroups, contradicting Lemma 3.8. The claim follows.

Since $G/T$ is finitely generated there is a finitely generated subgroup $F/T^{\mathfrak{F}}$ such that $G/T^{\mathfrak{F}} = (F/T^{\mathfrak{F}})(T/T^{\mathfrak{F}})$. Then $F/T^{\mathfrak{F}}$ is center-by-finite and hence $Z(F/T^{\mathfrak{F}})$ contains a torsion-free subgroup $C/T^{\mathfrak{F}}$ such that $C/T^{\mathfrak{F}} \leq C_G(T/T^{\mathfrak{F}})$ and $F/C$ is finite. Thus $C/T^{\mathfrak{F}} \triangleleft G/T^{\mathfrak{F}}$. Now $TC/C \cong (TC/T^{\mathfrak{F}})/(C/T^{\mathfrak{F}})$ and since $T/T^{\mathfrak{F}}$ is periodic and $C/T^{\mathfrak{F}}$ is torsion-free we have $TC/C \cong T/T^{\mathfrak{F}}$. However $T/T^{\mathfrak{F}}$ is infinite, so $G/C$ is an infinite locally finite, residually finite group, so by Proposition 3.2 $G$ is an FC-group, which is the contradiction sought. $\square$

Lemma 3.10. *Let $G$ be a locally FC-group satisfying max-(non-FC) and suppose that $G$ is not an FC-group. If $T$ is the torsion subgroup of $G$ and $G/T$ is finitely generated then $T^{\mathfrak{F}} = G^{\mathfrak{F}}$.*

*Proof.* By Lemma 3.9 $T/T^{\mathfrak{F}}$ is finite and hence $G/T^{\mathfrak{F}}$ is a finitely generated FC-group, so is residually finite. Thus $G^{\mathfrak{F}} \leq T^{\mathfrak{F}}$. Since it is clear that $T^{\mathfrak{F}} \leq G^{\mathfrak{F}}$ the result follows. □

Lemma 3.11. *Let $G$ be a locally FC-group satisfying max-(non-FC) and suppose that $G$ is not an FC-group. Suppose that $T$ is the torsion subgroup of $G$, $G/T$ is finitely generated and that $T^{\mathfrak{F}}$ is an FC-group. Then $G$ satisfies max-(non-BFC).*

*Proof.* By Lemma 3.9 $T/T^{\mathfrak{F}}$ is finite and $T^{\mathfrak{F}}$ is $\mathfrak{F}$-perfect. However an $\mathfrak{F}$-perfect FC-group is divisible abelian, so $G$ is abelian-by-finitely generated. It is now easy to see that $G$ has max-(non-BFC), using Lemma 2.6 and the fact that a finitely generated FC-group has the maximal condition. □

Thus the case when $T^{\mathfrak{F}}$ is an FC-group poses no problems. However problems may arise when $T^{\mathfrak{F}}$ is not an FC-group. It is conceivable that $T^{\mathfrak{F}}$ is a minimal non FC-group, for instance. We have the following result.

Lemma 3.12. *Let $G$ be a locally FC-group satisfying max-(non-FC), let $T$ be the torsion subgroup of $G$ and suppose that $G/T$ is finitely generated. If $T^{\mathfrak{F}}$ is not an FC-group then $T^{\mathfrak{F}}$ is perfect.*

*Proof.* Suppose, for a contradiction, that $R = T^{\mathfrak{F}}$ is not perfect. Since $R$ is $\mathfrak{F}$-perfect, by Lemma 3.9, it follows that $R/R' \cong C_{p^\infty}$, for some prime $p$. We assume that $R' \neq R''$. Suppose that $R'/R'' \neq (R'/R'')^q$ for some prime $q$. If $R'/(R')^qR''$ is infinite then $R/(R')^qR''$ is abelian, using Lemma 2.8, and then $R' = (R')^qR''$, contrary to our assumption. Thus $R'/(R')^qR''$ is finite. Then $R/(R')^qR''$ is Černikov with divisible part $D/(R')^qR'' \cong C_{p^\infty}$, say. It is easy to see that $R/(R')^qR''$ is then abelian anyway and we again obtain the contradiction that $R' = (R')^qR''$. It follows that $R'/R''$ is divisible. Let $R/R' = \langle a_nR' \mid a_1^p \in R', a_{n+1}^pR' = a_nR', n \in \mathbb{N}\rangle$. Then for each natural number $n$, $R/\langle a_n, R'\rangle$ does not satisfy the maximal condition, so $\langle a_n, R'\rangle$ is an FC-group. By [30, Theorem 1.9] $R'/R'' \leq Z(\langle a_n, R'\rangle/R'')$ for all $n$ and hence $R'/R'' \leq Z(R/R'')$. Thus $R/R''$ is center-by-locally cyclic and hence is abelian, and we obtain the contradiction that $R' = R''$.

Thus $R'$ is perfect and by Proposition 3.6 $R'$ is therefore finite, so $R$ is a BFC-group contrary to our hypothesis. This contradiction gives the result. □

As an immediate corollary we obtain the following theorem, once more noting that locally FC-groups with max-(non-BFC) are reasonably well-understood.

Theorem 3.13. *Let $G$ be a locally FC-group satisfying max-(non-FC). If $G$ is soluble then $G$ either is an FC-group or $G$ satisfies max-(non-BFC).*

*Proof.* Suppose that $G$ is not an FC-group and let $T$ be the torsion subgroup of $G$. If $G/T$ is not finitely generated then, by Theorem 3.7, $G$ satisfies max-(non-BFC). Thus we may assume that $G/T$ is finitely generated. By Lemma 3.9 $T/T^{\mathfrak{F}}$ is therefore finite, so $T^{\mathfrak{F}}$ is $\mathfrak{F}$-perfect. Note that $T^{\mathfrak{F}}$ is an FC-group, by Lemma 3.12, and hence is a divisible abelian group, by [30, Theorem 1.9]. Now, if $H$ is an FC-subgroup of $G$ then $H$ is abelian-by-finitely generated and Lemma 2.6 implies that $H$ is a BFC-group. Thus $G$ satisfies max-(non-BFC). □

For locally nilpotent groups our results become particularly pleasing. We record these results in the following theorem. We remark that the structure of locally nilpotent groups with max-(non-BFC) is well-understood (see [15]). If $H$ is a perfect, locally finite minimal non FC-group then results of Belyaev [8] and Kuzucuoglu and Phillips [25] show that $H$ is a $p$-group. Parts of the result below could appeal to these theorems but the proofs are easy enough not to warrant this.

Theorem 3.14. *Let $G$ be a locally nilpotent group satisfying max-(non-FC). Then $G$ is a group of one of the following types:*

(i) *$G$ is an FC-group;*
(ii) *$G$ is finitely generated;*
(iii) *$G$ satisfies max-(non-BFC);*
(iv) *$G$ is hyperabelian and $G^{\mathfrak{F}} \neq 1$ satisfies the following properties:*
    (a) *$G^{\mathfrak{F}}$ is a perfect, $\mathfrak{F}$-perfect, $p$-subgroup of $G$;*
    (b) *$G^{\mathfrak{F}}$ is a non-FC-subgroup of $G$;*
    (c) *Every proper subgroup of $G^{\mathfrak{F}}$ is an FC-group, and hence is hypercentral;*
    (d) *$G/G^{\mathfrak{F}}$ is a finitely generated FC-group.*

*Proof.* Suppose that $G$ is neither an FC-group nor finitely generated. Then $G$ is a locally FC-group and has a torsion subgroup, T, say. If $G/T$ is not finitely generated then Theorem 3.7 implies that $G$ satisfies max-(non-BFC). Thus we may suppose that $G/T$ is finitely generated. Lemma 3.10 implies that $T^{\mathfrak{F}} = G^{\mathfrak{F}}$ and Lemma 3.9 shows that $T/T^{\mathfrak{F}}$ is finite. Hence $T^{\mathfrak{F}}$ is $\mathfrak{F}$-perfect. Lemma 3.11 shows that if $T^{\mathfrak{F}}$ is an FC-group then $G$ satisfies max-(non-BFC). We may therefore suppose that $T^{\mathfrak{F}}$ is not an FC-group, so Lemma 3.12 implies that $T^{\mathfrak{F}}$ is perfect.

Clearly $G/T^{\mathfrak{F}}$ is finitely generated and by Proposition 3.1 $T^{\mathfrak{F}}$ is non-trivial. If $H$ is a proper subgroup of $T^{\mathfrak{F}}$ then $H$ is not maximal, since a maximal subgroup of a locally nilpotent group is well-known to have finite index. Hence it is possible to find a proper subgroup $K$ of $T^{\mathfrak{F}}$ containing $H$ properly. In this way we can construct a strictly ascending chain $H = H_0 \subsetneqq H_1 \subsetneqq H_2 \subsetneqq \dots$ of subgroups of $T^{\mathfrak{F}}$ and it follows from the condition max-(non-FC) that $H$ is an FC-group. Hence $T^{\mathfrak{F}}$ is a minimal non FC-group. Moreover, $T^{\mathfrak{F}}$ is a $p$-group for some prime $p$, for otherwise it is the direct product of two proper FC-subgroups, and hence is an FC-group, which is a contradiction.

Finally, $G$ is a hyperabelian group. By Mal'cev's theorem [28, Corollary 1 to Theorem 5.27] every $G$-chief factor of $T^{\mathfrak{F}}$ is of prime order, so $T^{\mathfrak{F}}$ contains a proper non-trivial $G$-invariant subgroup $U$. Then $U$ is hypercentral, so $Z(U)$ is also non-trivial. Since $G/T^{\mathfrak{F}}$ is nilpotent it is now easy to see that $G$ is hyperabelian. □

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