

THE EMBEDDING OF A CYCLIC PERMUTABLE SUBGROUP IN A FINITE GROUP

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ABSTRACT. In earlier work, the authors described the structure of the normal closure of a cyclic permutable subgroup of odd order in a finite group. As might be expected, the even order case is considerably more complicated and we have found it necessary to divide it into two parts. This part deals with the situation where we have a finite group G with a cyclic permutable subgroup A satisfying the additional hypothesis that X is permutable in A_2X for all cyclic subgroups X of G (where A_2 is the 2-component of A).

1. Introduction and statement of main results

A subgroup A of a group G is said to be *permutable* if $AX = XA$ for all subgroups X of G . Clearly this is equivalent to the product AX itself being a subgroup. In [3] we proved that when G is finite and A is a cyclic permutable subgroup of G , then, provided A has odd order,

$[A, G]$ is abelian and A acts on it as a group of power automorphisms.

This was achieved by reducing to the case where G is a p -group for an odd prime p . From then on, most of our arguments failed when p was replaced by 2. The purpose of the present work is to show how, using different methods, the requirement that A has odd order can be dropped. While our main theorems here include the principal results of [3] (viz. Theorem 1.1) as a special case, it turns out that in general $[A, G]$ is *not* abelian and A does *not* act on it as a group of power automorphisms. However, as we shall see, the difference between the conclusions of Theorems A and B here and Theorem 1.1 of [3] is minimal in a literal sense.

As in [3], we begin by reducing to the situation where G has prime power order, in this case a power of 2. Then in the remainder of Section 2 and in Section 3 we prove a succession of preliminary results, many of which have key significance in relation to establishing our main theorems. Section 4 contains

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the proofs of these theorems reduced to the 2-group case. Finally in Section 5 we give examples showing how, when A has even order, the results of the earlier work in [3] fail to extend.

Let G be a finite p -group which is the product of two cyclic subgroups A and X . When p is odd, Huppert showed in [5, Hauptsatz I] that G is metacyclic and in [5, Satz 15] that every subgroup of G is permutable. On the other hand, when $p = 2$, then G is not metacyclic in general (see [5, §3]); and clearly not every subgroup of G need be permutable, as can be seen from the dihedral group of order 8. However, if A is permutable in G , then

$$(1) \quad AX \text{ is metacyclic,}$$

by [12, Theorem 5.2.13]. But again the dihedral group of order 8 shows that this is not sufficient to guarantee that every subgroup of G is permutable. When p is odd, then AX has a modular subgroup lattice, which is not always the case when $p = 2$, even when A is a permutable subgroup. Thus in passing from p odd to $p = 2$, we encounter a lack of symmetry in these products AX of cyclic subgroups. This is particularly evident right at the start of our previous work in [3], where in Lemma 2.3(v), with $A = \langle a \rangle$, we were able to show that each element of the derived subgroup of AX has the form $[a, x]$ for some element x in X . This had far-reaching applications in [3], where we showed that if G is a p -group, with p odd, and if $A = \langle a \rangle$ is a cyclic permutable subgroup of G , then every element of $[A, G]$ has the form $[a, g]$, for some element g of G . If G is the dihedral group of order 16, with A the normal cyclic subgroup of order 8, and if X is a non-central subgroup of order 2, then $G = AX$. But not every element of $[A, G]$ has the form $[a, g]$.

Thus in order to extend our earlier work [3] to include cyclic permutable subgroups of even order, we shall adopt an additional hypothesis ($(*)$ below). But first we recall the following fundamental result.

LEMMA 1.1 ([12, Lemma 5.2.11]). *Let A be a cyclic permutable subgroup of a group G . Then every subgroup of A is also permutable in G .*

Throughout, for any finite group G and prime p , we shall use the notation G_p to denote a Sylow p -subgroup of G . Then when A is a cyclic permutable subgroup of G , our additional hypothesis is

$$(*) \quad X \text{ is permutable in } A_2X, \text{ for all cyclic subgroups } X \text{ of } G.$$

This is automatically satisfied whenever A has odd order, hence our new results will include those of [3]. Since a permutable subgroup of a finite group is always subnormal [9], it follows that a cyclic permutable 2-subgroup will be centralised by all elements of odd order. Thus $(*)$ is really only required when X is a 2-group. When A is also a 2-group, then, by (1), AX is metacyclic and so has a normal subgroup $K = \langle k \rangle$, say, with AX/K cyclic. Therefore

$AX = KY$ with $Y = \langle y \rangle$, say, also cyclic. Then $(*)$ is equivalent to

(2) *either AX is quaternion of order 8 or $k^y = k^r$ with $r \equiv 1 \pmod{4}$.*

For,

(3) *if X is also permutable in AX , then AX satisfies (2),*

by [8] (see also [12, 5.2.14]). Conversely, (2) is equivalent to AX having a modular subgroup lattice, by a theorem of Iwasawa ([6]; see also [12, Theorem 2.3.1]); and then $(*)$ follows from [12, Lemma 2.3.2]. The important point about $(*)$ is that dihedral actions cannot occur in the metacyclic groups A_2X .

We shall see in Section 5 that even when A is a cyclic permutable subgroup of a finite group G satisfying $(*)$, then $[A, G]$ is *not* always abelian and A does *not* always act on it as a group of power automorphisms. So our results here are best possible. In the absence of $(*)$, we can still get strong structure theorems for $[A, G]$ and the A -action, but they are less precise than Theorems A and B. Thus it is more convenient and appropriate to publish that work separately. For the present, therefore, we set our sights on proving the following results.

THEOREM A. *Let A be a cyclic permutable subgroup of a finite group G satisfying $(*)$ and let N be the derived subgroup of $[A, G]$. Then*

- (i) *N has order at most 2 and lies in A ; and*
- (ii) *A acts on $[A, G]/N$ as a group of universal power automorphisms.*

THEOREM B. *Let A be a cyclic permutable subgroup of a finite group G satisfying $(*)$. Then $[A, G]$ is abelian if and only if A acts on $[A, G]$ as a group of power automorphisms.*

By a well-known result of Cooper [2], all power automorphisms of a finite abelian group are universal, i.e., all elements map to the *same* power.

If H is a subgroup of a group G , then H_G is the *core* and H^G the *normal closure* of H in G . For any prime p , the set of all primes different from p is represented by p' . The centre of a group G will be denoted by $Z(G)$ and the second centre by $Z_2(G)$. The intersection of the normalisers of all the subgroups of G is called the *norm of G* and is denoted by $\text{norm}(G)$. In a p -group G , $\Omega(G)$ is the subgroup generated by the elements of order p , and $\Omega_2(G)$ is the subgroup generated by the elements of order at most p^2 . Finally C_n denotes a cyclic group of order n . All other notation is standard.

2. Reduction to 2-groups and other preliminary results

We begin by showing that the proofs of Theorems A and B both reduce to the case when G is a 2-group.

THEOREM 2.1. *Let A be a cyclic permutable subgroup of a finite 2-group G satisfying $(*)$ and let N be the derived subgroup of $[A, G]$. Then*

- (i) N has order at most 2 and lies in A ; and
- (ii) A acts on $[A, G]/N$ as a group of universal power automorphisms.

THEOREM 2.2. *Let A be a cyclic permutable subgroup of a finite 2-group G satisfying $(*)$. Then $[A, G]$ is abelian if and only if A acts on $[A, G]$ as a group of power automorphisms.*

Proof of Theorem A. Let p be a prime and let $P = A_p$. Then P is permutable in G , by Lemma 1.1, and P is subnormal in G , by [9]. Therefore $P^G = P[P, G]$ is a p -group. All elements of G , of order relatively prime to p , normalise P . Also P is contained in each Sylow p -subgroup G_p . We claim that, when p is odd,

- (4) $[P, G]$ is abelian and P acts on it as a group of power automorphisms.

For, if P is normal in G , then (4) is trivially true. So suppose that $P_G < P$. By [7], P/P_G lies in the hypercentre of G/P_G . Therefore elements in G , of order relatively prime to p , centralise P/P_G and hence also centralise P . Thus $[P, G] = [P, G_p]$ and (4) follows from [3, Theorem 2.1].

Since A is subnormal in G , $A^G = A[A, G]$ is nilpotent. Let $A = P_1 \times \cdots \times P_s$ be the decomposition of A into its primary components. Then

$$(5) \quad [A, G] = [P_1, G] \times \cdots \times [P_s, G].$$

Hence, by (4) and Theorem 2.1, $N = [A, G]'$ has order at most 2. If $N \neq 1$, then $|A|$ must be even and we may assume that P_1 is the 2-component of A . By Theorem 2.1, $N = [P_1, G]' \leq P_1 \leq A$ and (i) follows.

For (ii), again let $P = \langle x \rangle$ be any one of the P_i corresponding to the prime p . Then

$$(6) \quad P \text{ acts as a group of power automorphisms on } [A, G]/N.$$

For, write any element of $[A, G]$ in the form uv , with u a p -element and v a p' -element. So u lies in $[P, G]$; and $u^x = u^n$ if p is odd, by (4). If $p = 2$, then $u^x \equiv u^n \pmod{N}$, by Theorem 2.1. Also $v^x = v$ in both cases. As in the proof of [3, Theorem 1.1], it follows that there is an integer r such that $(uv)^x \equiv (uv)^r \pmod{N}$. Then (6) is true and so (ii) follows. \square

Proof of Theorem B. Suppose that $[A, G]$ is abelian. Then Theorem A(ii) shows that

$$(7) \quad A \text{ acts on } [A, G] \text{ as a group of power automorphisms.}$$

Conversely, suppose that (7) is true. The factors on the right hand side of (5), corresponding to odd primes, are abelian, by [3, Theorem 1.1]; while if one of the factors is a 2-group, then it is abelian, by Theorem 2.2. Therefore $[A, G]$ is abelian. \square

From now on we can restrict our attention to 2-groups G with a cyclic permutable subgroup $A = \langle a \rangle$ satisfying (*). Even without the hypothesis (*), there is a good analogue to [3, Lemma 2.3].

LEMMA 2.3. *Let $G = AX$ be a finite 2-group, where $A = \langle a \rangle$ and $X = \langle x \rangle$ are cyclic subgroups and A is permutable in G . Then*

- (i) G is metacyclic;
- (ii) $G' = \langle [a, x] \rangle$;
- (iii) for each integer i , $\langle [a^i, x] \rangle = \langle [a, x]^i \rangle$;
- (iv) a conjugates $[a, x]$ to a power congruent to 1 modulo 4; and
- (v) each element of G' has the form $[a^i, x]$, for some integer i .

Proof. (i) This is (1) above.

(ii) Let $N = \langle [a, x] \rangle$. Then $N \leq G'$ and N is normal in G , by (i). Since G/N is abelian, we must have $N = G'$.

(iii) Without loss of generality we may assume that $i \geq 1$. Let 2^t be the highest power of 2 dividing i . So $\langle a^{2^t} \rangle = \langle a^i \rangle$. Thus by Lemma 1.1 and (ii) above,

$$\langle [a^i, x] \rangle = \langle [a^{2^t}, x] \rangle.$$

Also $\langle [a, x]^i \rangle = \langle [a, x]^{2^t} \rangle$ and so we may assume that $i = 2^t$. Then, by induction on t , it suffices to establish the case $t = 1$. Modulo N^2 , N is central in G , so $[a^2, x] \in N^2$. Conversely, suppose that $\langle [a^2, x] \rangle < N^2$. So $[a, x]^2 \neq 1$. By (i), there is a cyclic normal subgroup K of G with G/K cyclic. Then we may assume that $N < K$. Also we may assume that $[a, x]^4 = [a^2, x] = 1$, and so $a^2 \in Z(G)$. If $[a, x] \in A$, then $[a^2, x] = [a, x]^2 = 1$. Therefore suppose that $[a, x] \notin A$. Hence $A \cap K \leq N^2 \leq K^4$. Now a must centralise K/K^4 for A to be permutable in G . Thus a must centralise N (of order 4). Therefore $1 = [a^2, x] = [a, x]^2$, a contradiction. So (iii) follows.

(iv) It is sufficient to assume that $N^4 = 1$ and to show that a centralises N . Since $[a, x, a] \in N^2$, we may assume that $|N| = 4$. But a cannot invert $[a, x]$, otherwise $[a^2, x] = 1$ and then $[a, x]^2 = 1$, by (iii). Therefore a centralises N .

(v) This is clear if $|G'| = 2$. Thus suppose that $|G'| = 2^n$, $n \geq 2$, and proceed by induction on $|G'|$. By (iii), $[a^{2^{n-1}}, x]$ has order 2 and lies in $Z(G)$. By induction, each element of G' has the form

$$[a^i, x][a^{2^{n-1}}, x]^\varepsilon,$$

where $\varepsilon = 0$ or 1. But when $\varepsilon = 1$, this product is $[a^j, x]$, $j = i + 2^{n-1}$, and so (v) follows. \square

From part (iii) we immediately obtain:

COROLLARY 2.4. *Let $A = \langle a \rangle$ be a cyclic permutable subgroup of a finite 2-group G . Then $[A, G] = \langle [a, g] \mid g \in G \rangle$.*

Our objective is to find the structure of A^G when $A = \langle a \rangle$ is a cyclic permutable subgroup of a finite 2-group G . When G is a p -group with p an odd prime, then, as we proved in [3], each element of $[A, G]$ has the form $[a, g]$. This is not the case when $p = 2$, as is shown by taking G to be the dihedral group of order 16 and A the normal cyclic subgroup of order 8. There, however, each element of $[A, G]$ has the form $[a^i, g]$, and this would have been sufficient, had it been generally true, to prove that $[A, G]$ is always abelian with A acting on it as a group of power automorphisms. But we shall see in Section 5 that this is not the case, even when $(*)$ is satisfied. Nevertheless the key to finding the structure of A^G is to get information about the form of the elements of $[A, G]$ in terms of simple commutators. The subgroup A^2 will play a vital rôle and our starting point is the following.

LEMMA 2.5. *Let A be a cyclic permutable subgroup of a finite 2-group G . Then*

$$(i) [A^2, G] = [A, G]^2.$$

If in addition $()$ is satisfied, then*

$$(ii) [A^2, G] = [A, G^2].$$

Proof. Let $A = \langle a \rangle$. For each $g \in G$, Lemma 2.3(iii) gives $\langle [a^2, g] \rangle = \langle [a, g]^2 \rangle$. Thus $[A^2, G] \leq [A, G]^2$ (using Corollary 2.4). Conversely, we may assume that $[A^2, G] = 1$. Then $[A, G]$ is generated by commutators $[a, g]$ of order at most 2 and centralised by a , by Lemma 2.3(iv). Therefore $[A, G]$ is centralised by A and hence also by A^G . Thus $[A, G]$ is abelian and even elementary. So $[A, G]^2 = 1$ and (i) follows.

Now we suppose that $(*)$ is satisfied. Assume first that $[A^2, G] = 1$. Then by Lemma 2.3(iii) and $(*)$, for each $g \in G$ we have $\langle [a, g^2] \rangle = \langle [a, g]^2 \rangle = \langle [a^2, g] \rangle = 1$. Therefore $[A, G^2] = 1$. Hence $[A^2, G] \geq [A, G^2]$. Conversely, $[A^2, G]$ is generated by cyclic subgroups $\langle [a^2, g] \rangle = \langle [a, g^2] \rangle \leq [A, G^2]$. Hence $[A^2, G] \leq [A, G^2]$ and (ii) follows. \square

REMARK. Again the dihedral group of order 16 shows that (ii) can fail if $(*)$ is not satisfied.

Let A be a permutable subgroup of prime order p in a group G . In [12, Theorem 5.2.9], it is shown that A^G is elementary abelian. Moreover, if A is not normal in G , then $[A, G]$ lies in $Z(G)$. For the case $p = 2$, there is a better result.

LEMMA 2.6. *Let A be a cyclic permutable subgroup of order 2 or 4 in a finite group G . Then*

$$(i) A^G \text{ is abelian, elementary if } |A| = 2 \text{ and of exponent 4 if } |A| = 4.$$

If in addition $()$ holds, then*

- (ii) $A^G \leq \text{norm}(G) \leq Z_2(G)$; and
- (iii) $[A, G] \leq Z(G)$.

Proof. Let $A = \langle a \rangle$. Of course the second inclusion in (ii) is Schenkman's result [11]. If $|A| = 2$, then [12, Theorem 5.2.9] and easy arguments suffice. Therefore suppose that $|A| = 4$. Since elements of odd order centralise A , we may assume that G is a 2-group.

(i) By Lemma 1.1, A^2 is permutable in G and so again Schmidt's result shows that A^G has exponent 4. Let $g \in G$. By Lemma 2.3(iv), a centralises $[a, g]$. It follows that a commutes with all its conjugates and this proves (i).

(ii) Let X be a cyclic subgroup of G . To show that A normalises X , we may assume that $A \cap X = 1$. Put $X_1 = \Omega(X)$. Then AX_1 is a subgroup and so X_1 normalises A . By (*), AX_1 is abelian and therefore X_1 is normal in AX . By induction on $|X|$, we may assume that A normalises X modulo X_1 . Thus A normalises X and (ii) follows.

(iii) This is an immediate consequence of (ii). □

Another application of Schmidt's results gives us a particularly useful expression for elements of $[A, G]$.

LEMMA 2.7. *Let $A = \langle a \rangle$ be a cyclic permutable subgroup of a finite 2-group G satisfying (*). Then each element of $[A, G]$ has the form $a^i[a, g]$, for some integer i and element g in G .*

Proof. We proceed by induction on $|G|$. Let $A_1 = \Omega(A)$ and suppose first that A_1 is normal in G . By induction, each element of $[A, G]$ has the form $a^i[a, g]$ modulo A_1 and therefore has the required form in G . Now suppose that $A_G = 1$. By Lemma 1.1, A_1 is permutable in G and so, by Lemma 2.6, $[A_1, G] \leq Z(G)$. It follows from Lemma 2.3(v) (and (*)) that there is a central element $[a, g_1]$ of order 2, for some $g_1 \in G$. Therefore, again by induction, each element of $[A, G]$ has the form $a^i[a, g]$ or $a^i[a, g][a, g_1] = a^i[a, g_1g]$, as required. □

For elements in $[A^2, G]$, it turns out that we can take $i = 0$ above, and this is one of our key results.

LEMMA 2.8. *Let $A = \langle a \rangle$ be a cyclic permutable subgroup of a finite 2-group G satisfying (*). Then each element of $[A^2, G]$ has the form $[a, u]$, where $u \in G^2$.*

Proof. By Lemma 1.1, A^2 is permutable in G , so if $|A| \leq 8$, then $[A^2, G] \leq Z(G)$, by Lemma 2.6(iii). Thus, by Lemma 2.5(ii), $[A^2, G]$ is generated by central elements of the form $[a, u]$ with $u \in G^2$. But products of such elements have the same form, as required.

Therefore suppose that $|A| \geq 16$. We proceed by induction on $|A|$. Thus we may assume that each element w of $[A^4, G]$ has the form $[a^2, u_1]$, with $u_1 \in G^2$. Since

$$w \in (A^2 \langle u_1 \rangle)' \leq (A \langle u_1 \rangle)',$$

we have $w = [a, u_1^i]$, for some integer i , again by Lemma 2.3 and (*). Here $u_1^i \in G^2$. Also $[A^4, G] \triangleleft G$ and so there is a central series of G passing through $[A^4, G]$. Therefore, using the identity

$$(8) \quad [a, xy] = [a, y][a, x]^y,$$

a simple induction allows us to assume that

$$(9) \quad [A^4, G] = 1.$$

Then it follows, from Lemma 2.3(iii), that $|[a, g]| \leq 4$, for all g in G . Thus a centralises $[a, g]$, by Lemma 2.3(iv), and hence A centralises $[A, G]$. Then A^G also centralises $[A, G]$ and we see that

$$(10) \quad [A, G] \text{ is abelian.}$$

In particular,

$$(11) \quad [A, G] \text{ has exponent at most 4.}$$

Let $|A| = 2^n$, $n \geq 4$. From (9), $A^4 \triangleleft G$ and therefore, by Lemma 2.6(iii),

$$[A, G, G] \leq A^4 \cap [A, G].$$

Thus, by (11),

$$(12) \quad [A, G, G] \leq \Omega_2(A).$$

Let $x, y \in G$. Then

$$\begin{aligned} [a, x^2, y] &= [[a, x]^2, y] \quad (\text{by Lemma 2.3(iii) and } (*)) \\ &= [a, x, y]^2 \quad (\text{by (10)}) \\ (13) \quad &\in \Omega(A) \quad (\text{by (12)}) \end{aligned}$$

$$(14) \quad \leq Z(G) \quad (\text{by (9)}).$$

Therefore $[a, x^2, y^2] = [a, x^2, y]^2 = 1$, by (14) and (13), respectively.

It follows that $[a, x^2, G^2] = 1$ and so $[a, g_1^2 \dots g_n^2] = [a, g_1^2] \dots [a, g_n^2]$, for all g_1, \dots, g_n in G . Thus $[A, G^2]$ is generated by elements of the form $[a, g^2]$ and therefore each element of $[A, G^2]$ has this form. But $[A^2, G] = [A, G^2]$, by Lemma 2.5(ii), and so the proof is complete. \square

Since a^i in the statement of Lemma 2.7 is in $[A, G]$, we see that if $A \cap [A, G] = 1$, then each element of $[A, G]$ has the form $[a, g]$. In this case we can easily show that $[A, G]$ is abelian with A acting on it as a group of power automorphisms.

LEMMA 2.9. *Let $A = \langle a \rangle$ be a cyclic permutable subgroup of a finite 2-group G such that each element of $[A, G]$ has the form $[a, g]$, for some element g in G . Then $[A, G]$ is abelian and A acts on it as a group of power automorphisms.*

Proof. By Lemma 2.3(ii), A normalises every subgroup of $[A, G]$ and therefore so also does A^G ($\geq [A, G]$). Thus by a famous result of Dedekind [4] and Baer [1], $[A, G]$ is either abelian or isomorphic to a direct product of the quaternion group of order 8 and an elementary abelian 2-group (see also [12, Theorem 2.3.12]). In the latter case, each element $[a, g]$ has order at most 4 and so it is centralised by a , by Lemma 2.3(iv). Then A centralises $[A, G]$ and hence A^G does the same. But this implies that $[A, G]$ is abelian, a contradiction. The lemma follows. \square

REMARK. Of course A acts as a group of *universal* power automorphisms in Lemma 2.9, by Cooper's result [2].

If $A \cap [A, G] = 1$ in Lemma 2.7, then the hypotheses of Lemma 2.9 are satisfied and we have $[A, G]$ abelian with A acting as a group of power automorphisms. It follows that we have to investigate the case $A \cap [A, G] = B$ (say) $\neq 1$. If $G \neq 1$, then clearly $B < A$. In fact, the case $|A : B| = 2$ occurs very rarely, as we now see.

LEMMA 2.10. *Let A be a cyclic permutable subgroup of a finite 2-group G satisfying $(*)$ and suppose that $A \cap [A, G] = A^2$. Then*

- (i) $A^2 \triangleleft G$; and
- (ii) $|A| \leq 4$.

Proof. (i) We proceed by induction on $|A|$. If $|A| \leq 4$, then $[A, G] \leq Z(G)$, by Lemma 2.6. Then $A^2 \triangleleft G$. Therefore suppose that $|A| \geq 8$. Let N be the normal closure of $\Omega(A)$ in G . By induction, $A^2N \triangleleft G$. But

$$N = \Omega(A)[\Omega(A), G]$$

and $[\Omega(A), G]$ is elementary abelian and central in G , by Lemma 2.6. Therefore

$$(A^2N)^2 = (A^2[\Omega(A), G])^2 = A^4 \triangleleft G.$$

But modulo A^4 , A has order 4 and so $A^2/A^4 \triangleleft G/A^4$ by the above. Therefore $A^2 \triangleleft G$.

(ii) Let $A = \langle a \rangle$. By Lemma 2.6(iii), $[A, G, G] \leq (A^4)^G = A^4$, by (i). Thus, modulo A^4 , the elements of $[A, G]$ have the form $[a, g]$. In particular, $a^2 = [a, g]a^{4i}$, for some element g in G and integer i . Therefore $a^2 = [a, x]$, for some x in G , by Lemma 2.3(v) and $(*)$. Hence $a^x = a^3$. If $|A| \geq 8$, then $a^2 \notin \langle x \rangle = X$, say. Thus $A \cap X \leq A^4$ and so AX/A^4X^2 is the dihedral group of order 8, contradicting $(*)$. Therefore $|A| \leq 4$. \square

In Lemma 2.8 we deduced that $[A, G]$ had exponent at most 4 from the facts that all commutators $[a, g]$ had order at most 4 and $[A, G]$ was abelian (see (11), (9) and (10), respectively). In fact, (10) was not really required there. Rather (11) is a consequence of (9) for more general reasons, which we now present (and will require later).

LEMMA 2.11. *Let $A = \langle a \rangle$ be a cyclic permutable subgroup of a finite 2-group G and let $[A, G]$ have exponent 2^n . Then there is a commutator $[a, g]$ of order 2^n .*

Proof. We proceed by induction on $|[A, G]|$. Let the maximum order of an element of the form $[a, g]$ be 2^m . So $m \leq n$ and we may assume that $m \geq 1$. Let $N = [A^{2^{m-1}}, G]$. Then, by Lemma 2.3, N is generated by elements of order 2 and centralised by A . Therefore A^G centralises N and hence N is elementary abelian. By induction $[A, G]/N$ has exponent 2^{m-1} and so $[A, G]$ has exponent at most 2^m . Then $n \leq m$ and therefore $n = m$, as required. \square

3. The structure of A^G

In this section we shall find the precise structure of $[A, G]$ and the way in which A acts (as a cyclic permutable subgroup of a finite 2-group satisfying (*)). Already from the previous section we have sufficient information to give a good global picture of what is going on. Thus let $A = \langle a \rangle$ and assume the hypotheses of Theorems 2.1 and 2.2. Then Lemmas 2.3 and 2.8 show that A normalises every cyclic and hence every subgroup of $[A^2, G]$. Therefore A^G does the same and so $[A^2, G]$ has all its subgroups normal. As in the proof of Lemma 2.9, we see that

$$[A^2, G] \text{ is abelian.}$$

Since power automorphisms of finite abelian groups are universal, it follows that a induces by conjugation a universal power automorphism of $[A^2, G]$, i.e., there is a positive integer r such that a conjugates each element of $[A^2, G]$ to its r th power. Let $u \in [A^2, G]$ and $g \in G$. Then $u^{[a, g]} = u^{a^{-1}g^{-1}ag} = ((u^{a^{-1}g^{-1}})^r)^g = (u^r)^{a^{-1}} = u$. Thus

$$(15) \quad [A^2, G] \leq Z([A, G]).$$

By Lemma 2.3(iii), $[A, G]/[A^2, G] = B$ (say) is generated by elements of order at most 2, all centralised by A . Thus B is centralised by A and therefore by A^G . It follows that B is elementary abelian. Hence, by (15), we have

$$[A, G] \text{ has nilpotency class at most 2.}$$

Therefore $[A, G]'$ is elementary abelian. Summarising these results, we have:

LEMMA 3.1. *Let A be a cyclic permutable subgroup of a finite 2-group G satisfying (*). Then*

- (i) $[A^2, G]$ is abelian and A acts on it as a group of power automorphisms;
- (ii) $[A^2, G] \leq Z([A, G])$;
- (iii) $[A, G]$ has class at most 2; and
- (iv) $[A, G]'$ is elementary abelian.

Suppose that A is a cyclic permutable subgroup of a finite p -group G . We know that when p is odd, $[A, G]$ is abelian and A acts on it as a group of power automorphisms. But we shall see in Section 5 (Example 5.2) that there is a group G of order 2^{17} with a cyclic permutable subgroup A of order 2^7 and satisfying (*) such that $[A, G]$ is not abelian. In that example, $[A, G]'$ has order 2. Of course Theorem A shows that $[A, G]'$ always has order at most 2 when A is a cyclic permutable subgroup of a finite group G satisfying (*). The key result now in proving Theorem A is the following.

LEMMA 3.2. *Let $A = \langle a \rangle$ be a cyclic permutable subgroup of a finite 2-group G satisfying (*). Let $[A, G]$ have exponent 2^n ($n \geq 1$) and let $x, y \in G$. Then there are integers r and s congruent to 1 modulo 4 such that $[a, x]^a = [a, x]^r$, $[a, y]^a = [a, y]^s$ and $r \equiv s$ modulo 2^{n-1} . In particular, all commutators of the form $[a, g]$ of order at most 2^{n-1} map under conjugation by a to the same power.*

Proof. We may assume that $n \geq 3$, otherwise a centralises $[A, G]$ and we can take $r = s = 1$. By Lemma 2.11, there is an element $g \in G$ such that $|[a, g]| = 2^n$. Let $[a, g]^a = [a, g]^q$. By Lemma 2.3(iv), $q \equiv 1 \pmod{4}$. Let $|[a, x]| = 2^m$, so $m \leq n$, and let $[a, x]^a = [a, x]^r$, with $r \equiv 1 \pmod{4}$. We show, by induction on m , that whenever $m < n$, then

$$(16) \quad r \equiv q \pmod{2^m}.$$

If $m \leq 2$, then certainly (16) is true. Therefore suppose that $m \geq 3$ and that (16) holds for smaller values of m . Put $u = [a, g]^{2^{n-m}}$. Then u belongs to $[A, G]^2 = [A^2, G]$ (by Lemma 2.5) and u has order 2^m .

Since there is a central series of G passing through $[A^2, G]$, it follows from Lemma 2.8, the commutator identity (8) and a simple induction argument, that every element of the coset $[A^2, G][a, x]$ has the form $[a, h]$, for some element h of G . Therefore we may assume that

$$(17) \quad u[a, x] = [a, h].$$

Here $|[a, h]| \leq 2^m$, because u and $[a, x]$ both have order 2^m and $[A, G]$ has class ≤ 2 with $[A, G]'$ elementary abelian (Lemma 3.1).

Replacing x by x^2 , we see by induction that $r \equiv q \pmod{2^{m-1}}$. Thus $r = q + k2^{m-1}$, for some integer k . We may assume that k is odd, otherwise (16) follows. Therefore

$$(18) \quad r \equiv q + 2^{m-1} \pmod{2^m}.$$

Let $[a, h]^a = [a, h]^t$. If $|[a, h]| < 2^m$, then induction gives $t \equiv q \pmod{|[a, h]|}$ and we can take $t = q$. On the other hand, if $|[a, h]| = 2^m$, then the argument above shows that $t \equiv q \pmod{2^{m-1}}$ and $t = q + \ell 2^{m-1}$. Either ℓ is even, in which case again we can take $t = q$; or ℓ is odd, and then we can take $t = q + 2^{m-1}$. However, if $t = q$, then since $u^a = u^q$, (17) and Lemma 3.1 show that a conjugates $[a, x]$ to its q th power, contradicting (18). Similarly if $t = q + 2^{m-1}$, then in the same way we see that a conjugates u to its $(q + 2^{m-1})$ -th power, which is not the case. Therefore the induction goes through and (16) is true.

The lemma now follows if either $|[a, x]| < 2^n$ or $|[a, y]| < 2^n$. Therefore suppose that $|[a, x]| = |[a, y]| = 2^n$. By Lemma 3.1, a conjugates $[a, x]^2$ and $[a, y]^2$ to the same power, because they both lie in $[A^2, G]$ ($= [A, G]^2$), on which a acts as a universal power automorphism. Thus if $[a, x]^a = [a, x]^r$ and $[a, y]^a = [a, y]^s$, then $r \equiv s \pmod{2^{n-1}}$ and the lemma follows in this case also. \square

We shall need another result about the form of elements in $[A, G]$, this time without the hypothesis (*).

LEMMA 3.3. *Let $A = \langle a \rangle$ be a cyclic permutable subgroup of a finite 2-group $G = \langle a, x, y \rangle$. Then*

$$(19) \quad [A, G] = (A \cap [A, G])\langle [a, x] \rangle \langle [a, y] \rangle.$$

Proof. We argue by induction on $|G|$. Clearly we may assume that $[A, G] \neq 1$. Let N be a minimal normal subgroup of G lying in A or in $[A, G]$. By induction

$$(20) \quad [A, G]N = N(A \cap [A, G])\langle [a, x] \rangle \langle [a, y] \rangle.$$

Let $A_1 = \Omega(A)$ and suppose that $A_1 \triangleleft G$. In this case we can take $N = A_1$ in (20). If $N \leq [A, G]$, then $N \leq A \cap [A, G]$ and (20) becomes (19). On the other hand, if $N \cap [A, G] = 1$, then intersecting both sides of (20) with $[A, G]$ gives (19).

Finally suppose that $A_G = 1$. Then without loss of generality $L = [A_1, \langle x \rangle] \neq 1$ and by Lemma 2.3(iii) and [12, Theorem 5.2.9], L is a central subgroup of order 2 in G . (Note that we cannot use Lemma 2.6(iii) here, because we are not assuming that (*) is satisfied.) So we can take $N = L$ in (20) and then $N \leq \langle [a, x] \rangle$. Again (20) becomes (19) and we have the result. \square

REMARK. Of course Lemma 3.3 extends naturally to the general case in which G is generated by a and any number of other elements. Also it is not necessary for G to be a 2-group. But in proving Theorem 2.2 we reduce to the case covered by Lemma 3.3 and so we restrict to that notationally simpler situation here.

In the proof of Theorem 2.1 we shall need the extension of Lemma 2.5 to powers other than the square. Care has to be taken here, because given a group G and integers m and n , G^{mn} is smaller than $(G^m)^n$ in general. For example, the Burnside group B of exponent 4 generated by 2 elements has order 2^{12} and B^2 also has exponent 4. In our case we can avoid this difficulty. Again we focus on our particular needs.

LEMMA 3.4. *Let H be a finite 2-group of class at most 2 and suppose that H' has exponent at most 2. Let m and n be integer powers of 2. Then $(H^m)^n = H^{mn}$.*

Proof. We may assume that $m, n \geq 2$. Certainly $H^{mn} \leq (H^m)^n$. For the reverse inclusion, it suffices to show that $x^n \in H^{mn}$ for each element $x \in H^m$. We have

$$x = h_1^m \dots h_\ell^m,$$

where each $h_i \in H$. So $x = (h_1 \dots h_\ell)^m u$, for some $u \in H'$. Put $h = h_1 \dots h_\ell$. Thus $x^n = (h^m u)^n = h^{mn} u^n$, since $u \in Z(H)$. Therefore $x^n = h^{mn}$, since $u^2 = 1$, and so $x^n \in H^{mn}$, as required. \square

4. Proofs of the main results

We establish the second of our main theorems first.

Proof of Theorem 2.2. Here G is a finite 2-group and $A = \langle a \rangle$ is a cyclic permutable subgroup satisfying (*). We suppose that the element a acts on $[A, G]$ as a power automorphism. Then A normalises every subgroup of $[A, G]$ and therefore so also does A^G . As in the proof of Lemma 2.9, it follows that $[A, G]$ is abelian.

Conversely, suppose that $[A, G]$ is abelian. It is sufficient to show that a conjugates any two commutators $[a, x]$ and $[a, y]$ to the same power. For then a will conjugate *all* commutators $[a, g]$ to the same power and a will act as a power automorphism on $[A, G]$. Thus we may assume that

$$G = \langle a, x, y \rangle.$$

Let $[A, G]$ have exponent 2^n . Since $[A, G] = \langle [a, x], [a, y] \rangle^G$, we may assume without loss of generality that $|[a, x]| = 2^n$. Suppose that $|[a, y]| < 2^n$. By Lemma 3.2 there are integers r and s such that $[a, x]^a = [a, x]^r$ and $[a, y]^a = [a, y]^s$ and $r \equiv s \pmod{2^{n-1}}$. But then $[a, y]^s = [a, y]^r$ and we are finished. Therefore suppose that $|[a, x]| = |[a, y]| = 2^n$.

Put $B = A \cap [A, G]$. If $|B| \leq 4$, then by Lemma 2.7, each element u of $[A, G]$ has the form $a^i [a, g]$, with $|a^i| \leq 4$. But a conjugates $[a, g]$ to its k th power, where $k \equiv 1 \pmod{4}$, by Lemma 2.3(iv). Thus a conjugates u to its k th power, i.e., a acts as a power automorphism on $[A, G]$. Therefore we may assume that $|B| \geq 8$.

By Lemma 2.6(iii),

$$(21) \quad \begin{aligned} [A, G, G] &\leq (A^4)^G \cap [A, G] = A^4[A^4, G] \cap [A, G] \\ &= B[A^4, G] \triangleleft G, \end{aligned}$$

since $B = A \cap [A, G] = A^4 \cap [A, G]$, by Lemma 2.10(ii). We claim that we may assume that

$$(22) \quad [A, G, G] \text{ is not contained in } B^2[A^4, G] = K,$$

say. For $K \leq [A^2, G]$, by Lemma 2.5(i). So if (22) is false, then every element of $[A, G, G]$ has the form $[a, u]$, where $u \in G^2$, by Lemma 2.8. However, there is a central series of G passing through $[A, G, G]$; and modulo $[A, G, G]$ each element of $[A, G]$ has the form $[a, g]$ with $g \in G$. Then a simple induction using (8) shows that every element of $[A, G]$ has the form $[a, g]$. Therefore a induces a power automorphism on $[A, G]$, necessarily universal by [2]. Thus we may assume that (22) holds.

Put $X = \langle [a, x] \rangle$ and $Y = \langle [a, y] \rangle$. By Lemma 3.3, $[A, G] = BXY$. Therefore

$$[A, G, G] = [BXY, G] = [B, G][X, G][Y, G] \leq [A^4, G][X, G][Y, G],$$

since $B \leq A^4$. By (22) we may assume, without loss of generality, that

$$(23) \quad [X, G] \text{ is not contained in } K.$$

Note that $K = B^2[A^4, G] = (B[A^4, G])^2[A^4, G] \triangleleft G$, by (21). Now by Lemma 2.3(iv) and (*), $[a, x, a]$ and $[a, x, x]$ both belong to $X^4 = \langle [a^4, x] \rangle \leq [A^4, G] \leq K$. Therefore by (23) we must have

$$(24) \quad [a, x, y] \notin K.$$

Let $[a, x]^a = [a, x]^r$. Then since $[A, G]$ is abelian, we have

$$(25) \quad [a, x, y]^a = [[a, x]^r, y][y, a] = [a, x, y]^r.$$

Also, by (21), (24) and Lemma 2.8, there is a generator b of B such that

$$(26) \quad [a, x, y] = b[a, u],$$

where $u \in G^2$. Let $[a, u]^a = [a, u]^s$. By Lemma 3.2, r and s can be chosen so that $r \equiv s$ modulo 2^{n-1} . But $[a, u] \in [A, G^2] = [A^2, G]$, and so $|[a, u]| \leq 2^{n-1}$, by Lemma 2.3(iii). Therefore $[a, u]^a = [a, u]^r$. Then by (25) and (26), $b[a, u]^a = (b[a, u])^r$, i.e., $b[a, u]^r = b^r[a, u]^r$ and so

$$(27) \quad r \equiv 1 \pmod{|B|}.$$

We distinguish 2 cases.

Suppose that $|B| < 2^n$. By Lemma 2.7, each element of $[A, G]$ has the form $c[a, g]$, where $c \in B$. Let $[a, g]^a = [a, g]^t$. Since r is unique modulo 2^n ($= |[a, x]|$), Lemma 3.2 shows that we can choose $t \equiv r \pmod{2^{n-1}}$ and so $c^t = c^r = c$, by (27). Therefore $(c[a, g])^a = c[a, g]^t = (c[a, g])^t$. Thus a

conjugates each element of $[A, G]$ to a power and so acts as a universal power automorphism on $[A, G]$, as required.

Suppose that $|B| = 2^n$. In this case it follows from (27) that $[a, x]^a = [a, x]$. We must show that a also centralises $[a, y]$. Again we distinguish 2 cases.

(i) Suppose that $w = [x, y, a] \notin K$. By the Three Subgroup Lemma (see [10, 5.1.10]), $w \in [A, G, G] \leq B[A^4, G]$ (by (21)) and so $w = b'v$, where $B = \langle b' \rangle$ and $v \in [A^4, G]$. But $[x, y] \in G^2$ and therefore

$$w \in [A, G^2] = [A^2, G],$$

(by Lemma 2.5) of exponent 2^{n-1} . Also $|v| \leq 2^{n-2}$ and hence $|b'| \leq 2^{n-1}$, a contradiction.

(ii) Suppose that $w \in K$. By (24) and the Three Subgroup Lemma, we must have $[a, y, x] \notin K$. Thus $[Y, G]$ is not contained in K and we may repeat the argument from (23) with x and y interchanged. Then we deduce that a centralises $[a, y]$ as required. This completes the proof of Theorem 2.2. \square

Now we can prove our main structure theorem.

Proof of Theorem 2.1. We have $A = \langle a \rangle$, a cyclic permutable subgroup of a finite 2-group G satisfying $(*)$, and $N = [A, G]'$. We must show that $N \leq \Omega(A)$ and that a acts on $[A, G]/N$ as a power automorphism.

Let $[A, G]$ have exponent 2^n . If $n \leq 2$, then A centralises $[A, G]$, by Lemma 2.3(iv). So A^G does the same. Therefore $N = 1$ and the theorem is true. Thus we may suppose that $n \geq 3$.

Let $W = [A, G]^{2^{n-1}}$. By Lemma 3.1, $H = [A, G]$ satisfies the hypotheses of Lemma 3.4. Therefore by Lemma 2.5(i), $W = [A^{2^{n-1}}, G]$. By Lemma 3.2, there is an integer $r \equiv 1 \pmod{4}$ such that a conjugates each commutator $[a, g]$ to its r th power modulo W . But H/W has class at most 2 and derived subgroup elementary abelian, by Lemma 3.1, and hence a conjugates each element of H/W to its r th power. In the usual way we deduce that H/W has all its subgroups normal and is therefore abelian, as in Lemma 2.9. So we have proved that

$$(28) \quad H/W \text{ is abelian with } a \text{ conjugating each element to its } r\text{th power.}$$

Suppose that $A \cap W = 1$. In this case, $|A \cap H| = 2^m$ with $m \leq n - 1$. By (28), $r \equiv 1 \pmod{2^m}$. Let $g \in G$ and $[a, g]^a = [a, g]^s$. By Lemma 3.2, we can choose $s \equiv r \pmod{2^{n-1}}$ (since r is unique modulo 2^{n-1}) and hence $s \equiv 1 \pmod{2^m}$. Therefore a conjugates each element of the form $a^i[a, g]$ in H to its s th power. Then by Lemma 2.7, A normalises every subgroup of H and we deduce as usual that

$$(29) \quad [A, G] \text{ is abelian with } a \text{ acting on it as a power automorphism.}$$

To complete the proof we distinguish 2 cases.

(i) *Suppose that $A_G = 1$.* Since $[a, g]^{2^n} = 1$ for all $g \in G$, we have $[a^{2^n}, g] = 1$ for all g , by Lemma 2.3(iii), and so $a^{2^n} = 1$. Thus by Lemma 2.11, $|A| = 2^n$. But by Lemma 2.6(iii) applied to $A^{2^{n-1}}$, $W \leq Z(G)$ and so $A \cap W = 1$. Then (29) holds and the theorem is true.

(ii) *Suppose that $A_G \neq 1$.* If $A \cap H = 1$, then $A \cap W = 1$ and again (29) follows. Therefore with $|A \cap H| = 2^m$, we may assume that $m \geq 1$. Clearly $m \leq n$. Since a centralises $A \cap H$, we see from (28) that $r \equiv 1 \pmod{2^{m-1}}$. Choose $g \in G$. Then as above, $[a, g]^a = [a, g]^s$ with $s \equiv r \pmod{2^{n-1}}$. Hence $s \equiv 1 \pmod{2^{m-1}}$. Thus, modulo $\Omega(A)$, a conjugates each element of the form $a^i[a, g]$ in H to its s th power. Now in the usual way, using Lemma 2.7, we see that

$H/\Omega(A)$ is abelian with a acting on it as a power automorphism.

Therefore $N = H' \leq \Omega(A)$ as required. If $N = \Omega(A)$, then we are finished. On the other hand, if $N = 1$, then the theorem follows from Theorem 2.2. \square

It is clear that the complications in our work arise when $B = A \cap [A, G] \neq 1$. In many cases, including metacyclic groups and the first two examples of Section 5, it turns out that $B \triangleleft G$. Were this always true, then it would surely be of some significance. However, we shall find in Example 5.3 that it is not the case in general. But B always lies in $Z([A, G])$, as we now see.

COROLLARY 4.1. *Let A be a cyclic permutable subgroup of a finite group G satisfying (*). Then $A \cap [A, G] \leq Z([A, G])$.*

Proof. Using standard arguments (see [3, Theorem 1.1]) we easily reduce to the case where G is a 2-group. Let $H = [A, G]$ and $B = A \cap H$. Then we must show that $[H, B] = 1$. By the Three Subgroup Lemma, this will follow from

$$(30) \quad [B, G, A] = 1.$$

Let H have exponent 2^n . By Theorem 2.1, we may assume that

$$N = H' = \Omega(A) \leq H^{2^{n-1}},$$

the last inclusion following from Lemma 3.2 (which forces $H/H^{2^{n-1}}$ to be abelian). Let $|B| = 2^m$ and $A = \langle a \rangle$. By Theorem 2.1, a acts on H/N as a power automorphism, raising each element to the r th power, say, where $r \equiv 1 \pmod{4}$ (Lemma 2.3(iv)). Also $r \equiv 1 \pmod{2^{m-1}}$, since a centralises B/N . Put $B = \langle b \rangle$. Then since $\Omega(B) = \Omega(A) \leq Z(G)$, it follows from Lemma 2.3(iii) that $|[b, g]| \leq 2^{m-1}$ for all g belonging to G . Therefore

$$(31) \quad [b, g]^r = [b, g].$$

Certainly $b \in \langle a^2 \rangle$. So $[b, g] \in \langle [a, g]^2 \rangle$, by Lemma 2.3(iii). However,

$$[a, g]^a \equiv [a, g]^r \pmod{N},$$

so $[a, g]^{2a} = [a, g]^{2r}$ and thus $[b, g]^a = [b, g]^r = [b, g]$, by (31). Therefore (30) follows. \square

5. Examples and counterexamples

We showed in [3] that when G is a p -group, for an odd prime p , with a cyclic permutable subgroup $A = \langle a \rangle$, then every element of $[A, G]$ has the form $[a, g]$ for some element g in G . Were this always the case when $p = 2$, then we could simply have omitted the requirement that A has odd order in [3, Theorem 1.1]. However, we know that it is not the case, simply by looking at the dihedral group of order 16. One might have conjectured the weaker statement each element of $[A, G]$ has the form $[a^i, g]$ for some integer i , and again that would have implied that $[A, G]$ is abelian with a acting as a power automorphism. But in fact this weaker statement also fails, even when (*) is satisfied. Our first example shows this.

EXAMPLE 5.1. There is a group G of order 2^{10} , with a cyclic permutable subgroup $A = \langle a \rangle$ of order 8 satisfying (*), such that not every element of $[A, G]$ has the form $[a^i, g]$ with i an integer and g an element of G .

Construction. Let $X = \langle x \rangle$, $Y = \langle y \rangle$ and $Z = \langle z \rangle$ be cyclic groups of order 8 and let H be the split extension of $Y \times Z$ by X where x acts as follows:

$$y^x = yz, \quad z^x = z.$$

Then x acts faithfully, H has order 2^9 and $H' = Z(H) = Z$. Now H has an automorphism of order 2 defined by

$$(32) \quad x \mapsto x^5, \quad y \mapsto y^5.$$

For, $H = \langle x, y \mid x^8 = y^8 = [x, y, x] = [x, y, y] = 1 \rangle$ is a presentation of H and the map (32) preserves the relations and is surjective. Since z is fixed, there is an extension G of H by a group of order 2 defined as follows:

$$G = \langle H, a \mid a^2 = z^2, [x, a] = x^4, [y, a] = y^4 \rangle$$

(see [13, 9.7.1(ii)]).

Let $A = \langle a \rangle$ of order 8. We claim that

$$(33) \quad A \text{ is permutable in } G.$$

For, $a^2 \in Z(G)$, so we can factor G by A^2 and assume that $a^2 = 1$. Now $z^2 = 1$ and G has class 2, with $G' = \langle x^4, y^4, z \rangle \cong C_2 \times C_2 \times C_2$. Also a conjugates x and y to their 5th powers and hence a conjugates every element of G to its 5th power. Thus (33) is true.

Next we claim that (*) is satisfied. For, let $W = \langle x^4, y^4, z^4 \rangle$. Then W is elementary abelian of order 8 and normal in G . Since G/W has class 2, it

has exponent 4 and it follows that G has exponent 8. Also $[A, G] = W$. Let $K = \langle k \rangle$ be a cyclic subgroup of G . We must show that

$$(34) \quad K \text{ is permutable in } AK.$$

If $|K : A \cap K| \leq 4$, then $[A, K] \leq A^2 K^4 = A^2$, by (33). (Recall that $A^2 \leq Z(G)$.) Thus K normalises A and so $a^k = a^r$ with $r \equiv 1 \pmod{4}$, since $[A, G]$ is elementary. Then (34) is true. Therefore we may suppose that $A \cap K = 1$ and $|K| = 8$. By (1), AK is metacyclic and so there is a cyclic normal subgroup N of order 8 in AK such that AK/N is cyclic of order 8. If $AK = AN$, then again AK must have a modular subgroup lattice. Therefore we may assume that $AK = KN = S$, say. Thus $S' = [K, N] = [A, K]$ of order ≤ 2 and again (34) follows. This proves that (*) is satisfied.

Finally, since $a^2 \in Z(G)$, it suffices to show that $a^4 (= z^4 \in [A, G])$ cannot be expressed in the form $[a, g]$ for any element g . Thus let $g = a^i x^j y^k z^\ell$. Then

$$[a, g] = [a, x^j y^k] = [a, y^k][a, x^j]^{y^k} = y^{4k}(x^{4j})^{y^k}.$$

If k is odd, then $[a, g] = y^4 x^{4j} z^{4j} = y^4$ or $y^4 x^4 z^4$. If k is even, then $[a, g] = x^{4j} = x^4$ or 1. Thus in no case do we get a^4 for $[a, g]$. \square

It follows that there was no hope in the present work of trying to emulate the approach in [3]. It appears that Lemma 2.8 here is, in some sense, the best possible result in this direction.

Our second example shows that $[A, G]$ in Theorems A and 2.1 is *not* abelian in general.

EXAMPLE 5.2. There is a group G of order 2^{17} with a cyclic permutable subgroup A of order 2^7 satisfying (*) such that $[A, G]$ is not abelian. By Theorem A, this implies that $[A, G]' = \Omega(A)$ and A does not act on $[A, G]$ as a group of power automorphisms.

Construction. Let $C = \langle c \rangle$ be a cyclic group of order 2^7 . Then C has an automorphism of order 2^5 defined by $c \mapsto c^{21}$. Thus there is a group G_1 of order 2^{12} which is an extension of C by a cyclic group of order 2^5 , presented by

$$\langle a, c \mid a^{128} = 1, a^{32} = c^{32}, [c, a] = c^{20} \rangle$$

(see [13, 9.7.1(ii)]). Let $A = \langle a \rangle$ of order 2^7 . Since $21 \equiv 1 \pmod{4}$, $G_1 = AC$ has a modular subgroup lattice.

Next we check easily that the map $a \mapsto a^{65}$, $c \mapsto a^{64}c (= c^{65})$ defines an automorphism of G_1 of order 2. Since $a^{32} \in Z(G_1)$, the Scott reference above shows that there is a group

$$(35) \quad G_2 = \langle a, b_1, c \mid a^{128} = 1, a^{32} = b_1^2 = c^{32}, \\ [c, a] = c^{20}, [a, b_1] = [c, b_1] = a^{64} \rangle,$$

which is an extension of G_1 by a group of order 2. So $|G_2| = 2^{13}$. Also $G_2 = AC\langle b_1 \rangle = A\langle b_1 \rangle C$.

By similar routine calculations we see that G_2 has an automorphism α defined by

$$a \mapsto ab_1^{-1}, \quad b_1 \mapsto b_1, \quad c \mapsto a^{80}c.$$

This map preserves the relations of G_2 and is surjective. Also

$$\alpha^4 : a \mapsto a^{65}, \quad b_1 \mapsto b_1, \quad c \mapsto a^{64}c,$$

agreeing with conjugation in G_2 by b_1 . Thus a third application of [13, 9.7.1(ii)] shows that there is a group

$$G_3 = \langle a, b_2, c \mid a^{128} = 1, a^{32} = b_2^8 = c^{32}, [c, a] = c^{20}, [a, b_2] = b_2^{-4}, [c, b_2] = a^{16} \rangle,$$

which is an extension of G_2 by a cyclic group of order 4. Here $b_2^4 = b_1$ and the last two relations of (35) are consequences of the last two relations of the above presentation of G_3 . Thus $|G_3| = 2^{15}$. Also $G_3 = A\langle b_2 \rangle C$.

Finally we extend G_3 by a cyclic group of order 4 to get our group G as follows. Routine calculations show that G_3 has an automorphism β defined by

$$a \mapsto ab_2^{-1}, \quad b_2 \mapsto b_2, \quad c \mapsto a^{52}b_2^2c.$$

Also β^4 maps $a \mapsto ab_2^{-4}, b_2 \mapsto b_2, c \mapsto ca^{16}$, agreeing with conjugation in G_3 by b_2 . Therefore putting $b^4 = b_2$, we get a group

$$(36) \quad G = \langle a, b, c \mid a^{128} = 1, a^{32} = b^{32} = c^{32}, b^a = b^5, c^a = c^{21}, c^b = a^{52}b^8c \rangle,$$

which is an extension of G_3 by a cyclic group of order 4. The last relation in the presentation of G_3 is a consequence of the relations in (36). The group G has order 2^{17} and this has been confirmed using MAGMA by M. F. Newman, to whom we are most grateful. MAGMA also showed that G has class 4. Putting $B = \langle b \rangle$, we have $G = BA\langle b_2 \rangle C = ABC$. In fact, G is the product of A, B and C in any order.

We claim that

$$(37) \quad A \text{ is a permutable subgroup of } G.$$

For, AB has a modular subgroup lattice and so, by Lemma 2.3(iii),

$$\langle [a^8, b] \rangle = \langle [a, b]^8 \rangle = \langle b^{32} \rangle = \langle a^{32} \rangle.$$

Similarly $\langle [a^8, c] \rangle = \langle c^{32} \rangle = \langle a^{32} \rangle$. Therefore $A^8 \triangleleft G$ and in order to establish (37) we may factor G by A^8 and include $a^8 = 1$ as a relation. Then

$$(38) \quad c^{b^2} = (a^4b^8c)^b = (ab^{-4})^4b^8a^4b^8c = a^8c = c,$$

since $[b^4, a] = b^{16}$ has order 2 and lies in $Z(AB)$. We claim that

$$(39) \quad a \text{ normalises each cyclic subgroup } \langle b^i c^j \rangle \text{ with power action } \equiv 1 \pmod{4}.$$

To see this, we first calculate $(b^i c^j)^2$. By (38), if i is even, then b^i and c^j commute; while if i is odd, then

$$(c^j)^{b^i} = (a^4 b^8 c)^j = a^{4j} b^{8j} c^{j+16\varepsilon},$$

where $\varepsilon = \binom{j}{2}$. (Note that $[a^4, c] = c^{16}$.) Therefore if i is even, then $(b^i c^j)^2 = b^{2i} c^{2j}$; while if i is odd, then

$$(b^i c^j)^2 = a^{4j} b^{2i+8j} c^{2j+16\varepsilon}.$$

We distinguish 4 possibilities.

Case 1. i and j are both odd. Then we have $(b^i c^j)^2 = a^4 b^{2i+8j} c^{2j+16\varepsilon}$ and so $(b^i c^j)^4 = b^{4i+16} c^{4j}$. Then $(b^i c^j)^{20} = b^{20i+16} c^{20j} = b^{4i} c^{20j} = [b^i c^j, a]$. Therefore

$$(40) \quad (b^i c^j)^a = (b^i c^j)^{21}.$$

Case 2. i is odd and j is even. Here $(b^i c^j)^2 = b^{2i+8j} c^{2j+16\varepsilon}$ and $(b^i c^j)^4 = b^{4i} c^{4j} = b^{4i} c^{20j}$, since $c^{16j} = 1$. So

$$(41) \quad (b^i c^j)^a = (b^i c^j)^5.$$

Case 3. i is even and j is odd. Now $(b^i c^j)^{20} = b^{20i} c^{20j} = b^{4i} c^{20j}$ and we have (40).

Case 4. i and j are both even. Then $(b^i c^j)^4 = b^{4i} c^{4j} = b^{4i} c^{20j}$ and we have (41).

Thus (39) is true, i.e., modulo A^8 , $A\langle b^i c^j \rangle$ is a subgroup with a modular subgroup lattice and so all its subgroups are permutable. It follows that, for any integer k ,

$$A\langle a^k b^i c^j \rangle \equiv A\langle b^i c^j \rangle \pmod{A^8}.$$

Now passing back to the group G as presented by (36), in which A^8 is a normal subgroup, we see that $A\langle a^k b^i c^j \rangle$ is a subgroup and so (37) is true.

Finally we show that (*) is satisfied, i.e., for all cyclic subgroups X of G ,

$$(42) \quad AX \text{ has a modular subgroup lattice.}$$

Certainly $AX = K$ (say) is metacyclic by (1), so there is a cyclic normal subgroup N of K with K/N cyclic. By (3) we may assume that $K = NX$. Thus $K' \leq N^2$ and so if $N^2 \leq X$, then $X \triangleleft K$ and again K has a modular subgroup lattice. Therefore suppose that N^2 is not contained in X . In this case, since we may assume that X is not in N , we have $K/X^2 N^4$ of order 8 and so $A^8 \leq X^2 N^4$. But in proving that A is permutable above, we saw that K/A^8 has a modular subgroup lattice and therefore the same is true of $K/X^2 N^4$. Thus this quotient is not dihedral. Since it has 2 minimal normal subgroups, it must be abelian and so $K' \leq X^2 N^4 \cap N = N^4$. Therefore (42) is true and the construction of Example 5.2 is complete. \square

We observe that $[A, G]$ in the above example is not abelian, since b^4 and c^4 both belong to $[A, G]$, while $[c^4, b^4] = c^{-4}(ca^{16})^4 = a^{64} \neq 1$. We list other facts about this group G .

- (i) By Theorem 2.1 we have $[A, G]' = A^{64}$ of order 2.
- (ii) We must have $[A, G] \neq \{[a^i, g] \mid i \text{ an integer, } g \in G\}$, otherwise $[A, G]$ would be abelian.
- (iii) The element a does *not* act on $[A, G]$ as a power automorphism, by Theorem 2.2.

In this example $A \cap [A, G] = B$ (say) $= A^{16} \triangleleft G$. Similarly in Example 5.1, with the same notation, B has order 2 and lies in $Z(G)$. In fact, all examples to which we have so far made reference have $B \triangleleft G$. To conclude, we show that this is not always the case.

EXAMPLE 5.3. There is a group G of order 2^{10} with a cyclic permutable subgroup A of order 8 satisfying $(*)$, such that $A \cap [A, G]$ is not normal in G .

Construction. Let $X = \langle x \rangle$ and $A = \langle a \rangle$ be cyclic groups of order 32 and 8, respectively, and let H be the split extension of X by A defined as follows:

$$H = \langle a, x \mid a^8 = x^{32} = 1, x^a = x^5 \rangle.$$

Then H has an automorphism θ defined by $a \mapsto a^5, x \mapsto a^2x^5$. We see that θ^4 fixes a and maps x to x^{17} and this coincides with conjugation in H by a^4 . Therefore by [13, 9.7.1(ii)], there is a group

$$G = \langle a, x, y \mid a^8 = x^{32} = 1, x^a = x^5, [a, y] = y^4, x^y = a^2x^5, a^4 = y^4 \rangle.$$

Here G is an extension of H by a cyclic group of order 4 and $|G| = 2^{10}$. Also X^4 is centralised by y and so $X^4 \triangleleft G$. Thus $A^4X^4 \triangleleft G$ and we must have $[A, G] = A^4X^4$. Put $B = A \cap [A, G]$. Then $B = A^4$. But $[a^4, x] = x^{16} \neq 1$ and so B is not normal in G .

We claim that

$$A \text{ is permutable in } G \text{ and } (*) \text{ is satisfied.}$$

First we note that a typical element of G has the form

$$(43) \quad g = a^k y^i x^j,$$

for suitable integers i, j, k . By routine calculations one shows, using induction on i , that $x^{y^i} = a^{2i}x^{4i+1}$. Then we find that

$$(44) \quad g^4 = a^q y^r x^s,$$

where $q = 4(k + ij), r = 2i(5^k + 1), s = (8i + 2)j5^k + 16i \binom{j5^k}{2} + 2j + 16(k + ij)j$; and $g^8 = x^t$, where $t = (16i + 4)j5^k + 4j$. To see that $A\langle g \rangle$ is a subgroup, we distinguish 3 cases.

Suppose that j is odd. Then $\langle g^8 \rangle = X^8 \triangleleft G$ and modulo X^8 we have A^2 central in G . Also modulo A^2X^8 , the derived subgroup of G is X^4 and

therefore G/A^2X^8 has class 2. Thus $g^a \equiv a^k y^i x^{5j} \equiv g^5 \pmod{A^2X^8}$ and so $A\langle g \rangle$ is a subgroup.

Suppose that $j \equiv 2 \pmod{4}$. Now $g^8 = x^{8j}$ generating X^{16} ($\triangleleft G$). Also modulo X^{16} , a^2 is centralised by x^2 and by y and therefore by g . The quotient $\langle a, x^2, y \rangle / A^2X^{16}$ has derived subgroup generated by X^8 and thus has class 2. Then

$$g^a \equiv a^k y^i x^{5j} \equiv g^5 \pmod{A^2X^{16}}$$

and again $A\langle g \rangle$ is a subgroup.

Suppose that $j \equiv 0 \pmod{4}$. Let $K = \langle a, x^4, y \rangle$, so $g \in K$. We have $K' = \langle a^4, x^{16} \rangle$, the 4-group, lying in $Z(K)$. Also $a^2 \in Z(K)$. Thus $g^a \equiv a^k y^i x^{5j} \equiv g^5 \pmod{A^2}$ and again $A\langle g \rangle$ is a subgroup.

Therefore A is a permutable subgroup of G . It remains to show that $(*)$ is satisfied. Let g be given by (43). By (1), $A\langle g \rangle = L$, say, is metacyclic and so there is a cyclic normal subgroup N of L with L/N cyclic. As in Example 5.2, we may assume that $L = N\langle g \rangle$. We must show that $L' \leq N^4$. Therefore suppose that this is not the case. Then $L' = N^2 \neq 1$. Put $M = \langle g \rangle$. By (3), we may assume that M does not contain N^2 (otherwise $M \triangleleft L$).

Now $M^2N^4 \triangleleft L$ and $L/M^2N^4 \cong D_8$, the dihedral group of order 8. So

$$(45) \quad L/L^4 \text{ has } D_8 \text{ as an epimorphic image.}$$

Observe that $a^4 = y^4 \in L^4$. Thus

$$(46) \quad [a, g] = [a, y^i x^j] = [a, x^j][a, y^i][a, y^i, x^j] \equiv x^{-4j+16ij} \pmod{L^4}.$$

Since $y^i x^j \in L$, putting $k = 0$ in (44) gives $x^s \in L^4$, where

$$s = (8i + 4)j + 16i \binom{j}{2} + 16ij.$$

Thus if j is odd, then $x^4 \in L^4$ and so L/L^4 is abelian, by (46), a contradiction. If $j \equiv 2 \pmod{4}$, then (44) gives $\langle g^4 \rangle \equiv \langle x^8 \rangle \equiv \langle [a, g] \rangle \pmod{L^4}$, by (46). So again L/L^4 is abelian, contradicting (45). Similarly if $j \equiv 4 \pmod{8}$, then $\langle g^4 \rangle \equiv \langle x^{16} \rangle \equiv \langle [a, g] \rangle \pmod{L^4}$; while if $j \equiv 0 \pmod{8}$, then $[a, g] \in L^4$. Thus L/L^4 is abelian in every case and we conclude that $(*)$ is satisfied. \square

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