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A REMARK ON THE QUASI-INVERSE OF A PRODUCT

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ABSTRACT. It is well known that a product ab in a ring may have an inverse without ba being invertible. However, if ab has a quasi-inverse, then so does ba. This note provides a (3-line) proof and an explanation.

It is well known that the existence of an inverse for a product ab in a ring does not imply that ba has an inverse. So it comes as a surprise that the existence of a quasi-inverse for ab implies the existence of a quasi-inverse for ba. This could be the subject of a 3-line paper, but we shall also point out the underlying reason. Reinhold Baer, who wrote about the existence of 2-sided inverses under chain conditions in [1], might have appreciated this fact.

The *quasi-inverse* of an element a in a ring is defined as an element a' such that

$$a + a' = aa' = a'a. \tag{1}$$

It occurs in the study of the Jacobson radical (see, e.g., [2], p. 191), and is related to inverses by the fact that $1 - a$ has the inverse $1 - a'$. Even for algebras without a unit element it is often convenient to adjoin a unit element and so reduce the study of quasi-inverses to that of inverses.

It is clear that if a product ab has an inverse, ba need not be invertible, e.g., ab might be 1; if ba has an inverse u say, then $uba = bau = 1$, hence $ub = ub \cdot ab = uba \cdot b = b$ and so $ba = uba = 1$. However, many examples are known where $ab = 1$ and $ba \neq 1$.

Suppose now that ab has a quasi-inverse $1-u$: thus $(1-ab)u = u(1-ab) = 1$. Then we claim that $-bua$ is a quasi-inverse for ba. The proof is a simple verification:

$$\begin{aligned}(1 - ba)(1 + bua) &= 1 - ba + (1 - ba)bua \\ &= 1 - ba + b(1 - ab)ua \\ &= 1 - ba + ba = 1,\end{aligned}$$

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and similarly $(1+bua)(1-ba) = 1$. For an explanation we consider the matrix

$$(1 - ba) \oplus 1 = \begin{pmatrix} 1 - ba & 0 \\ 0 & 1 \end{pmatrix}. \tag{2}$$

This matrix can be linearized by elementary transformations:

$$\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 - ba & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix} = \begin{pmatrix} 1 & b \\ a & 1 \end{pmatrix}. \tag{3}$$

Similarly we have

$$\begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 - ab \end{pmatrix} \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & b \\ a & 1 \end{pmatrix}. \tag{4}$$

We can write (3) and (4) more briefly as $P((1 - ba) \oplus 1)Q = T = Q(1 \oplus (1 - ab))P$, where P and Q are elementary matrices, and hence invertible. Thus

$$(1 - ba) \oplus 1 = P^{-1}Q(1 \oplus (1 - ab))PQ^{-1}.$$

Now suppose that $1 - ab$ has an inverse u, say. Then

$$(1 - ba)^{-1} \oplus 1 = QP^{-1}(1 \oplus u)Q^{-1}P; \tag{5}$$

hence $1 - ba$ has an inverse and by working out the product on the right of (5), we obtain the value $1 + bua$ for the inverse of $1 - ba$.

References

[1] R. Baer, *Inverses and zero-divisors*, Bull. Amer. Math. Soc. **48** (1942), 630–638.
[2] P. M. Cohn, *Basic algebra, groups, rings, and fields*, Springer-Verlag, London, 2002.

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