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ON BAER'S PROBLEM AND PROPERTIES OF M''-GROUPS

N. S. CHERNIKOV

On the 100th anniversary of the birth of the outstanding algebraist Reinhold Baer

Abstract. We establish many new properties of M''-groups and give a large set of new counter-examples to the well-known problems of R. Baer (1949) and S. N. Chernikov (1959) concerning socle groups, M''-groups and M'-groups. In passing we also show that for a periodic FC-group G and its locally soluble radical R the factor group $G/L(G)R$ is residually finite, where $L(G)$ is the product of all normal semisimple subgroups of G.

1. Introduction

Recall that the socle $\mathrm{Soc}(G)$ of a group G is the product of all its minimal normal subgroups, or $\mathrm{Soc}(G) = 1$, if G has no such subgroups (R. Remak). Define $\mathrm{Soc}_0(G) = 1$ and $\mathrm{Soc}_{\alpha+1}(G)/\mathrm{Soc}_\alpha(G) = \mathrm{Soc}(G/\mathrm{Soc}_\alpha(G))$.

Definition 1 (S. N. Chernikov [6, §5]). The group G is called socle if $G = \mathrm{Soc}_\gamma(G)$ for some ordinal γ.

It is obvious that a group is socle iff it has an ascending principal series. Clearly, the class of socle groups contains all hyperfinite groups and, at the same time, all locally finite-normal groups and, in particular, all periodic abelian groups. It is easy to see that an arbitrary group G satisfying the minimal condition for normal subgroups is a socle group and for each $\alpha < \gamma$, $\mathrm{Soc}_{\alpha+1}(G)/\mathrm{Soc}_\alpha(G)$ is a direct product of finitely many minimal normal subgroups of $G/\mathrm{Soc}_\alpha(G)$.

In 1949 the following natural question was raised by R. Baer.

Problem (R. Baer [2]). If the group G is socle and for each $\alpha < \gamma$, $\mathrm{Soc}_{\alpha+1}(G)/\mathrm{Soc}_\alpha(G)$ is a direct product of finitely many minimal normal subgroups of $G/\mathrm{Soc}_\alpha(G)$, does it follow that G satisfies the minimal condition for normal subgroups? (See also [6, §5] or [15, p. 151].)

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In 1959, in connection with some results of H. H. Muhammedžan [11], [12] and (indirectly) motivated by this problem, S. N. Chernikov introduced the following definitions and raised the following questions.

Definition 2 (S. N. Chernikov [6, §5]). The group G is called socle finite, if for each finite $N \underset{\neq}{\lhd} G$ $\mathrm{Soc}(G/N) \neq 1$ and $\mathrm{Soc}(G/N)$ is finite.

Definition 3 (S. N. Chernikov [6, §5]). A hyperfinite socle finite group is called an M'-group.

Definition 4 (S. N. Chernikov [6, §5]). The group G is called an M''-group if it has an ascending normal series

$$N_0 = 1 \subset \cdots \subset N_\alpha \subset \cdots \subset N_\gamma = G \qquad (1)$$

such that for $\alpha < \gamma$, $N_{\alpha+1}/N_\alpha$ is maximal among all normal subgroups of G/N_α which are direct products of finite simple groups, and N_α is finite for finite α. (If $G = 1$, then $\gamma = 0$.)

By O. J. Schmidt's Theorem (see, for instance, [15, Theorem 1.45]), M'-groups and M''-groups are locally finite.

Question 1 (S. N. Chernikov [6, §5]). Is an arbitrary M'-group Chernikov?

Question 2 (S. N. Chernikov [6, §5]). Is an arbitrary M''-group necessarily Chernikov?

The above Problem and Questions were answered in the negative by Ju. M. Mežebovskiĭ [10] and, independently, by the author [4]; see also [5]. In the present paper we continue the investigations begun in [4], [5] and, in particular, consider the Problem and Questions 1 and 2.

In what follows, $\mathbb{N}$ and $\mathbb{P}$ denote, respectively, the sets of all natural numbers and all prime numbers, and $\mathbb{Z}^+ = \mathbb{N} \cup \{0\}$. The symbols $\times$, $\leftthreetimes$, and $\wr$ denote the direct, semidirect, and wreath products, respectively. If G is a group and $\emptyset \neq H \subseteq G$, then $Z(G)$ is the centre of G, $FC(G) = \{g \in G : |G : C_G(g)| < \infty\}$ (obviously, $FC(G)$ is a subgroup of G), $H^G = \{h^g : h \in H, g \in G\}$, $H_G = \bigcap_{g \in G} H^g$, $H \,\mathrm{sn}\, G$ and $H \,\mathrm{asc}\, G$ mean as usual that H is a subnormal, resp. an ascendant subgroup of G (see, for example, [15]), and $H \,\mathrm{lasc}\, G$ means that H is a locally ascendant subgroup of G (see Definition 7). Other notations in the present paper are as in [15]. If Λ is the empty set, then we set $\prod_{\lambda \in \Lambda} G_\lambda = 1$.

Definition 5 (see [5]). The group G (finite or infinite) is called quasisimple if $G' = G$ and $G/Z(G)$ is simple. The group G is called semisimple if $G = \prod_{\lambda \in \Lambda} G_\lambda$ for some family $\{G_\lambda : \lambda \in \Lambda\}$ of its quasisimple subgroups such that $[G_\nu, G_\lambda] = 1$, $\nu \neq \lambda$.

(Obviously, $G/Z(G)$ is nonabelian.)

Definition 6 (see [5]). The subgroup $L(G)$ of the group G is defined as the product of all its normal semisimple subgroups.

Definition 7 (B. I. Plotkin; see, for instance, [13]). A subgroup H of the group G is called locally ascendant (in G), if there exists a local system of subgroups K of G such that H is ascendant in each K.

Recall also that a completely reducible group is defined as a direct product of simple groups, and the trivial group is considered to be completely reducible.

The main results of the present paper are the following theorems.

Theorem 1. *The class of M'-groups is just the class of hyperfinite M''-groups, and the class of M''-groups is a proper subclass of the class of locally finite socle finite groups. The class of locally finite-normal M'-groups coincides with the class of locally finite-normal M''-groups.*

In view of Dietzmann's Lemma (see, for instance, [15]) the class of locally finite-normal groups is just the class of periodic FC-groups.

Theorem 2. *For a group G the following statements are equivalent.*

(i) *G is an M''-group.*
(ii) *G is a periodic hyper-FC-group with Chernikov locally soluble radical R such that for every $m \in \mathbb{N}$ the set of all quasisimple subgroups $Q \lhd L(G)$ with $|Z(Q)| \leq m$ is finite or empty.*

Theorem 3. *Let G be an M''-group, H a subgroup and D the normal closure of H in G.*

(i) *If the index $|L(G) : H \cap L(G)|$ is finite, then H is an M''-group.*
(ii) *If $H \operatorname{lasc} G$, then H is an M''-group.*
(iii) *If $H \operatorname{lasc} G$ and H is almost locally soluble, then D is Chernikov.*
(iv) *If $H \operatorname{lasc} G$ and $L(H) \cap L(G)$ is Chernikov, then D is Chernikov.*
(v) *If $H \operatorname{lasc} G$ and $Z(L(H)) \cap Z(L(G))$ is finite, then D is Chernikov.*

Theorem 4. *Let G be a periodic FC-group and R its locally soluble radical. Then the factor groups $G/L(G)R$ and G/R are residually finite.*

Below, as usual, ω is the first infinite ordinal.

Theorem 5. *Let H be a countable periodic residually finite FC-group or a finite group and A be a countable abelian Chernikov group such that $A \cap H = 1$. Then there exists a non-Chernikov locally finite-normal group $G = \operatorname{Soc}_\omega(G)$ such that:*

(i) *$A, H < G = L(G) \leftthreetimes H$ and $L(G)$ is non-Chernikov.*
(ii) *The locally soluble radical R of G coincides with A, G/A is residually finite, and*

$$A = Z(G) = Z(L(G)) = C_G(L(G)).$$

(iii) *G is an M''-group and an M'-group, $\mathrm{Soc}_k(G)$ is finite for each $k \in \mathbb{N}$, and G does not satisfy the minimal condition for normal subgroups of finite index.*

Theorems 1–3 present a number of new properties of M''-groups. Theorems 4 and 5, in particular, show that the class of all locally finite-normal M''-groups is large. Theorem 5 also furnishes us with many new counter-examples to the Problem and to Questions 1 and 2 mentioned above.

The following assertion gives further information about the group G of Theorem 5.

Assertion. *For a periodic FC-group G the following statements are equivalent.*

(i) *G is an M''-group.*
(ii) *$G = \mathrm{Soc}_\omega(G)$ and for each $k \in \mathbb{N}$, $\mathrm{Soc}_k(G)$ is finite.*
(iii) *For the series (2) of G (see below) $\gamma \leq \omega$ and all G_α with finite α are finite. (In particular, G is countable or finite.)*

2. Preliminary results

The proofs of Theorems 1–5 and the Assertion will be given after a number of preliminary results.

Lemma 1. *Let $\mathfrak{X}$ be a class consisting of simple and trivial groups, and let $\mathfrak{Y}$ be the class of all direct products of $\mathfrak{X}$-groups. Let F be a group, $H \lhd F$, $G \lhd F$ and $G \supseteq H \in \mathfrak{Y}$. Then:*

(i) *There exists a subgroup $K \subseteq G$ such that $H \subseteq K$ and K is maximal in the set of all $\mathfrak{Y}$-subgroups of G that are normal in F.*
(ii) *If $\mathrm{Soc}(G) \in \mathfrak{Y}$, then for each $\mathfrak{Y}$-subgroup $B \lhd G$, $\mathrm{Soc}(G)B \in \mathfrak{Y}$. In particular, if $\mathrm{Soc}(G) \in \mathfrak{Y}$, then $\mathrm{Soc}(G) \subseteq B$ for each subgroup B that is maximal in the indicated set.*

Proof. (i) Let H_1 (respectively H_2) be the product of all nonabelian (resp. abelian) factors of the decomposition of H into the direct product of $\mathfrak{X}$-subgroups; if there are no such factors, let $H_1 = 1$ (respectively $H_2 = 1$). Let D be the subgroup generated by all nonabelian $\mathfrak{X}$-subgroups $X \,\mathrm{sn}\, G$; if there are no such X, let $D = 1$. We have, obviously, $H = H_1 \times H_2$, $H_1 \subseteq D$ and $H_2 = Z(H) \lhd F$. In view of [5, Proposition 5 and Corollary3] (for example), if $D \neq 1$, then D is the direct product of all X. It is easy to see that H_2 is contained in some subgroup S which is maximal in the set of all

F-invariant abelian $\mathfrak{Y}$-subgroups of G. Clearly, $H \subseteq DS \lhd F$ and $DS \in \mathfrak{Y}$. Suppose that $DS \subset T \subseteq G$, where $T \lhd F$ and $T \in \mathfrak{Y}$. Let R be a direct $\mathfrak{X}$-factor of T such that $R \nsubseteq DS$. Obviously, $R \subseteq Z(T)$. Clearly, $SZ(T)$ is a normal abelian $\mathfrak{Y}$-subgroup of G. By virtue of the maximality of S, we have $Z(T) \subseteq S$, which is a contradiction.

(ii) Let $\operatorname{Soc}(G) \in \mathfrak{Y}$. Obviously, for some $A \lhd G$ and $A \subseteq \operatorname{Soc}(G)$ we have $\operatorname{Soc}(G)B = A \times B$. Then $A \simeq \operatorname{Soc}(G)/\operatorname{Soc}(G) \cap B$ and, clearly, $A \in \mathfrak{Y}$. Therefore $A \times B \in \mathfrak{Y}$. □

LEMMA 2. *Let G be a group. Let $H \operatorname{lasc} G$, $K \operatorname{asc} G$ and $T \le G$, $N \lhd G$. Then $(K \cap H) \operatorname{lasc} G$, $(H \cap T) \operatorname{lasc} T$ and $HN/N \operatorname{lasc} G/N$, $KN/N \operatorname{asc} G/N$.*

The proof is obvious.

PROPOSITION 1 (B. I. Plotkin; see [13, Theorems 5.2.1.3 and 5.2.2.4]). *Let $\mathfrak{X}$ be the class of all locally nilpotent, or locally finite, or locally (finite and soluble) groups. Let R be the product of all normal $\mathfrak{X}$-subgroups of a group G. Then $R \in \mathfrak{X}$ and every $\mathfrak{X}$-subgroup $H \operatorname{lasc} G$ belongs to R.*

In what follows, just as in [15], a series in a group G is called subnormal if each of its terms is subnormal in G; given a class of groups $\mathfrak{X}$, a series in G all of whose factors belong to $\mathfrak{X}$ is called an $\mathfrak{X}$-series.

PROPOSITION 2. *Let the group G have an FC-series $\mathcal{K}$. Then the following statements hold.*

(i) *If $\mathcal{K}$ is subnormal or ascending, then G has no infinite quasisimple subgroups; in particular, each completely reducible subgroup $H \neq 1$ of G is a direct product of finite simple groups. If, in addition, $\mathcal{K}$ is abelian, then G does not possess any quasisimple subgroups.*

(ii) *If $\mathcal{K}$ is normal and ascending and G is periodic, then G has an ascending normal series*

$$G_0 = 1 \subset \cdots \subset G_\alpha \subset \cdots \subset G_\gamma = G \tag{2}$$

such that for each $\alpha < \gamma$, $G_{\alpha+1}/G_\alpha$ is a maximal normal completely reducible subgroup of G and a direct product of finite simple groups. Further, in this case, for any $N \lhd G$, $\operatorname{Soc}(G/N)$ is a direct product of finite simple groups.

(iii) *For every series (2), $\operatorname{Soc}(G/G_\alpha) \subseteq G_{\alpha+1}/G_\alpha$ and $\operatorname{Soc}_\alpha(G) \subseteq G_\alpha$, $\alpha < \gamma$.*

Proof. (i) Without loss of generality we may assume that G is quasisimple. Further, according to Baer's Theorem [1] (see, for instance, [15, Theorem 4.32(i)]), for an arbitrary FC-group X, $X/Z(X)$ is locally finite. It is easy to see that a locally finite FC-group has an ascending subnormal series with finite simple factors. Taking this into account we may also assume that each factor of $\mathcal{K}$ is either finite simple, or abelian torsion-free.

Let $\mathcal{K}$ be subnormal. Take $g \in G \backslash Z(G)$. There are neighbouring $N, K \in \mathcal{K}$ for which $g \in K \setminus N$. Obviously, $G = KZ(G)$. So $G = G' = (KZ(G))' = K'$, i.e., $G = K$. Since $G \neq NZ(G)$ and $N \operatorname{sn} G$, clearly, $N \subseteq Z(G)$. Then K/N is not abelian, so K/N is finite. Consequently $G/Z(G)$ is finite. Therefore, by Schur's Theorem (see, for instance, [15, Theorem 4.12]), $G' = G$ is finite.

Let $\mathcal{K}$ be ascending. There are neighbouring N, $K \in \mathcal{K}$ such that $N \subseteq Z(G)$ and $K \not\subseteq Z(G)$. By Lemma 2, $KZ(G)/Z(G) \operatorname{asc} G/Z(G)$. Suppose that K/N is abelian. Then $KZ(G)/Z(G)$ is abelian too. Consequently, by Proposition 1, $G/Z(G) = \langle (KZ(G)/Z(G))^{G/Z(G)} \rangle$ is locally nilpotent. But according to Malcev's Local Theorem an arbitrary nonabelian locally nilpotent group is not simple, a contradiction. Thus K/N is finite nonabelian simple. Then, by [5, Proposition 5], $G/Z(G)$ is a direct product of all distinct subgroups $(KZ(G)/Z(G))^g$, $g \in G/Z(G)$. Consequently, $G/Z(G) = KZ(G)/Z(G)$. Since $G/Z(G)$ is finite, by Schur's Theorem G is finite.

(ii) Let $G \neq 1$ and $K(\neq 1)$ be the first term of $\mathcal{K}$. In view of Baer's Theorem [1], K is locally finite. Let $g \in K \setminus \{1\}$. Then $\langle g^K \rangle$ is finite. Take a minimal normal subgroup $N \subseteq \langle g^K \rangle$ of K. Obviously, N is completely reducible and for every $g \in G$ either $[N, N^g] = 1$ or $N = N^g$. Consequently, $\langle N^G \rangle$ is the direct product of some N^g and $\langle N^G \rangle$ is completely reducible. In view of Lemma 1, for some maximal normal completely reducible subgroup G_1 of G, $\langle N^G \rangle \subseteq G_1$. By (i) G_1 is a direct product of finite simple groups. Now it is easy to show by induction that G possesses an appropriate series. The proof of the last assertion in (ii) is obvious.

(iii) By (ii) $\operatorname{Soc}(G/G_\alpha)$ is completely reducible. Therefore, in view of Lemma 1(ii), $\operatorname{Soc}(G/G_\alpha) \subseteq G_{\alpha+1}/G_\alpha$. Hence it easily follows that for each $\alpha \leq \gamma$, $\operatorname{Soc}_\alpha(G) \subseteq G_\alpha$. □

The following result immediately follows from Proposition 1.

Corollary 1. *For the series* (1) $N_{\alpha+1}/N_\alpha$ *is a maximal normal completely reducible subgroup of* G/N_α, $\operatorname{Soc}(G/N_\alpha) \subseteq N_{\alpha+1}/N_\alpha$ *and* $\operatorname{Soc}_\alpha(G) \subseteq N_\alpha$, $\alpha < \gamma$.

Lemma 3. *For the series* (1) *the union* $K = \bigcup_{i \in \mathbb{Z}^+} N_i$ *coincides with* $FC(G)$.

Proof. Obviously $K \subseteq FC(G)$. Let $g \in FC(G)$. By Dietzmann's Lemma the subgroup $T = \langle g^G \rangle$ is finite. Suppose that $g \notin K$. Then for some $t \in \mathbb{Z}^+$, $TN_t/N_t \cap N_{t+1}/N_t = 1$ and $TN_t/N_t \neq 1$. Let D be a minimal normal subgroup of G/N_t contained in TN_{t+1}/N_t. Obviously $D(N_{t+1}/N_t)$ is a direct product of finite simple groups. But then $D \subseteq N_{t+1}/N_t$, which is a contradiction. □

Lemma 4. *Let the group G have a finite maximal normal completely reducible subgroup H. Then every completely reducible subgroup $K \lhd G$ is finite.*

Proof. Indeed, $|K : C_K(H)|$ is finite and $HC_K(H)$ is normal completely reducible subgroup of G. So $C_K(H) \subseteq H$ and K is finite. □

The next proposition follows from [5, Theorem 2] and statement 2 of [5, Proposition 1].

Proposition 3. *Let G be a group. Let $\{Q_\lambda : \lambda \in \Lambda\}$ be the set of all locally ascendant quasisimple subgroups of G. Then $[Q_\lambda, Q_\nu] = 1$ for $\nu \neq \lambda$, $L(G) = L(G)' = \prod_{\lambda\in\Lambda} Q_\lambda$, $Q_\lambda \lhd L(G)$ and $Q_\lambda \lhd^2 G$, $\lambda \in \Lambda$. Also, for each $H \operatorname{lasc} G$ there exists a unique set $\Delta \subseteq \Lambda$ such that $L(H) = \prod_{\lambda\in\Delta} Q_\lambda$. In particular, $L(G)$ is semisimple, and an arbitrary set of semisimple locally ascendant subgroups of G generates a semisimple subgroup which is the product of some Q_λ's. Further, $Z(L(G)) = \prod_{\lambda\in\Lambda} Z(Q_\lambda)$, $L(G)/Z(L(G))$ is the direct product of simple subgroups $Q_\lambda Z(L(G))/Z(L(G))$, $\lambda \in \Lambda$, and $Z(L(H)) = Z(L(G)) \cap L(H)$.*

Proposition 4. *Let G be an M''-group. Then:*

(i) *For each $m \in \mathbb{N}$ all semisimple subgroups $K \operatorname{lasc} G$ satisfying $|Z(K)| \leq m$ generate a finite semisimple subgroup $H \subseteq L(G)$.*
(ii) *$L(G) = L(FC(G))$.*

Proof. (i) By Proposition 3, H is semisimple, $Z(H)$ is of finite exponent and $H/Z(H)$ is completely reducible. In view of Corollary 1 and Lemma 4, the subgroup $\langle g : Z(H) : g^p = 1 \text{ for some } p \in \mathbb{P}\rangle$ is finite. Therefore, obviously, $Z(H)$ is finite. Consequently, by Lemma 3, $Z(H)$ is contained in some finite $N_k \in (1)$. Again, by Corollary 1 and Lemma 4, HN_k/N_k is finite. So H is finite.

(ii) In view of (i) and Proposition 2(i), $L(G) \subseteq FC(G)$. Then, by Proposition 3, $L(G) \subseteq L(FC(G))$, while, on the other hand, $L(FC(G)) \subseteq L(G)$. □

Lemma 5. *Let $G/Z(G)$ be simple. Then G' is a normal quasisimple subgroup of G such that $G = G'Z(G)$ and $G'/Z(G') \simeq G/Z(G)$. Further, if $|G : Z(G)|$ is finite, then G' is finite.*

Proof. Obviously, $Z(G') = G' \cap Z(G)$ and $G = G'Z(G)$. So $G'/Z(G') \simeq G/Z(G)$ and $G'/Z(G')$ is nonabelian simple. Further, $G' = (G'Z(G))' = G''$. If $|G' : Z(G')|$ is finite, then by Schur's Theorem G'' $(= G')$ is finite. □

Proposition 5. *Let the group G satisfy the following conditions:*

(i) *There are no infinite quasisimple subgroups $Q \operatorname{lasc} G$ (equivalently, $Q \lhd L(G)$) with finite $Z(Q)$.*

(ii) *For every $m \in \mathbb{N}$ the set of all quasisimple subgroups $Q \operatorname{lasc} G$ (equivalently, $Q \lhd L(G)$) with $|Z(Q)| \leq m$ is finite or empty.*

Let the subgroup $H \operatorname{lasc} G$ have a series $H_0 = 1 \subset H_1 \subset \cdots \subset H_n = H$ with completely reducible factors. If the soluble radical R of H is Chernikov, then H is finite.

Proof. Note first that, in view of Proposition 3, $Q \operatorname{lasc} G$ iff $Q \lhd L(G)$. Let R be Chernikov. Then, obviously, R is finite and the soluble radical of H_{n-1} is finite too. Further, by Lemma 2, $H_{n-1} \operatorname{lasc} G$. Taking these properties into consideration, we may assume, of course, that H_{n-1} is already finite. Let $K/H_{n-1} = Z(H/H_{n-1})$ and $T/H_{n-1} = (H/H_{n-1})'$. Then $H/H_{n-1} = (K/H_{n-1}) \times (T/H_{n-1})$, $|K : C_K(H_{n-1})|$ and $|T : C_T(H_{n-1})|$ are finite, and $C_K(H_{n-1})$ is a normal soluble subgroup of H. Consequently, K is finite. Therefore, if $C_T(H_{n-1}) \subseteq H_{n-1}$, then H is finite. Let $C_T(H_{n-1}) \not\subseteq H_{n-1}$. It is easy to see that $C_T(H_{n-1})/Z(H_{n-1})$ is a direct product of nonabelian simple subgroups. Let $D/Z(H_{n-1})$ be a direct simple nonabelian factor of $C_T(H_{n-1})/Z(H_{n-1})$. Then, in view of Lemma 5, D' is quasisimple and $D = D'Z(H_{n-1})$. Then $|Z(D')| \leq |Z(H_{n-1})|$. Further, in view of Lemma 2, $D' \operatorname{lasc} G$. Consequently, the subgroup F generated by all D' is finite (see conditions (i), (ii) above) and $C_T(H_{n-1}) = FZ(H_{n-1})$. Therefore $C_T(H_{n-1})$ is finite. Then T and, at the same time, H are finite. □

LEMMA 6. *Let G be a group, $H \leq G$ and $|G : H|$ be finite. Let G contain quasisimple subgroups $Q \operatorname{lasc} G$ which are not contained in H. Then the subgroup K generated by all such Q is finite.*

Proof. Since the index $|G : H_G|$ is finite, we may assume, of course, that $H \lhd G$. Let Q^* be another subgroup of the type of Q. Then, by Proposition 3, $[Q, Q^*] = 1$. Since obviously QH/H is not abelian, $QH/H \neq Q^*H/H$. Thus the set of all Q is finite. Since, by statement 2 of [5, Lemma 1], $Q \cap H \subseteq Z(Q)$, $|Q : Z(Q)|$ is finite. Therefore, in view of Lemma 5, Q is finite. Consequently, K is finite. □

LEMMA 7. *Let G be a group, $H \leq G$, and assume that $|L(G) : L(G) \cap H|$ is finite. Put $K = L(G)H$ and $C = C_K(L(G))$. Then:*

(i) $L(H_K) \lhd L(K) = L(G)L(C)$.
(ii) *For any quasisimple subgroup $Q \operatorname{lasc} K$ or $Q \operatorname{lasc} H_K$, either $Q \lhd L(G)$, or $Q \lhd L(C)$.*
(iii) *There exists exactly one semisimple subgroup $B \lhd H$ such that $L(H) = L(H_K)B$ and $[L(H_K), B] = 1$. The subgroup B is necessarily finite.*
(iv) *For any quasisimple subgroup $Q \operatorname{lasc} H$, either $Q \lhd L(H_K)$ and $Q \lhd L(K)$, or $Q \lhd B$.*

Proof. Indeed, (i) and (ii) easily follow from Proposition 3. Since the index $|H : H_K|$ is finite, (iii) and (iv) easily follow from Lemma 6 and Proposition 3. □

PROPOSITION 6. *Let G be a periodic hyper FC-group with Chernikov locally soluble radical R. Then the subgroup $C = C_G(L(G))$ is Chernikov.*

Proof. In consequence of Proposition 3, $L(C) \subseteq C \cap L(G) = Z(L(G))$. Therefore, obviously, $L(C) = 1$. Let S be the locally soluble radical of C. Then $S \subseteq R$. So S is Chernikov. Further, in view of Proposition 2(ii), C has an ascending normal series with completely reducible factors. Thus, by [5, Proposition 7], C is Chernikov. □

LEMMA 8. *Let $H \lhd G$ and suppose that H possesses a subnormal abelian series. Then $[H, L(G)] = 1$.*

Proof. In view of Proposition 2(i), H has no quasisimple subgroups. Further, if $L(G) \neq 1$, then, by Proposition 3, $L(G)$ is a product of quasisimple subgroups $Q \lhd L(G)$. Consequently, in view of [5, Proposition 3], $[H, L(G)] = 1$. □

The next result is an immediate consequence of Lemma 8 and Malcev's Local Theorem.

COROLLARY 2. *Assume that $H \lhd G$ and H is locally hyperabelian. Then $[H, L(G)] = 1$.*

LEMMA 9. *Let Q and $H = \{h_1, \ldots, h_n\}$ be quasisimple and finite groups such that $Q \cap H = 1$. Then there exists a group $G = R \leftthreetimes H$ with the following properties:*

(i) $Q \lhd R = \prod_{h \in H} Q^h$.
(ii) *For arbitrary $h \in H$ and $g \in H$, $g \neq h$, $[Q^h, Q^g] = 1$ and $Q^h \cap Q^g = Z(Q)$.*
(iii) *$R/Z(Q)$ is the direct product of all $Q^h/Z(Q)$, $h \in H$.*
(iv) *$R/Z(Q)$ is the minimal normal subgroup of $G/Z(Q)$.*
(v) *$R = L(G)$ and $Z(R) = Z(Q) = Z(G)$.*
(vi) *If $N \lhd G$ and $N \not\subseteq Z(Q)$, then $N \supseteq R$.*
(vii) *If $N \lhd G$ and N is a locally soluble (more generally an SI-)group, then $N \subseteq Z(Q)$.*

Proof. Let $W = Q \wr H$. Then for some $T, V \leq W$ we have that $T \simeq Q$ and $V \simeq H$, $\langle T^W \rangle$ is the direct product of all T^h, $h \in V$, and $W = \langle T^W \rangle \leftthreetimes V$. Identify T with Q and V with H. Put $F = \langle Q^W \rangle$. Let $K = \{g_1^{h_1} \ldots g_n^{h_n} : g_k \in Z(Q), h_k \in H, k = 1, \ldots, n,$ and $g_1 \ldots g_n = 1\}$. Then $K \lhd W$, $K \cap Q = 1$ and $Z(F) = KZ(Q)$. Let G be the factor group of W by K in which in a natural manner QK/K is identified with Q and HK/K is identified with H. Put $R = F/K$. Then $G = R \leftthreetimes H$ and $N_G(Q) = R \subseteq L(G)$. Obviously, (i)–(iii) hold. By the Corollary to [15, Theorem 5.45], (iv) holds too. Further, by Proposition 3, $L(G) \subseteq N_G(Q)$. Thus $L(G) = R$.

Clearly, $Z(R) = Z(Q) \subseteq Z(G) \subseteq N_G(Q)$. So $Z(G) \subseteq R(= N_G(Q))$ and $Z(G) \supseteq Z(R)$. Therefore $Z(R) = Z(Q) = Z(G)$.

Let $R \nsubseteq N \lhd G$. Then $Q^g \nsubseteq N$ for every $g \in G$. Therefore, by [5, Proposition 3], $[N, R] = 1$. Consequently $N \cap R \subseteq Z(R) = Z(Q)$. Suppose that $N \nsubseteq Z(Q)$. Then for some $a \in R$ and $h \in H \setminus \{1\}$, $ah \in N \setminus Z(Q)$. But $Q^{ah} = Q^h \neq Q$. Thus $[ah, R] \neq 1$, which is a contradiction, and so (vi) holds.

Let N be a normal SI-subgroup of G. In view of Proposition 2(i), $Q \nsubseteq N$. Consequently, by (vi), $N \subseteq Z(Q)$ and (vii) holds. Hence the lemma is proven. □

Lemma 10. *Let $G_\lambda \lhd G$, $\lambda \in \Lambda$, and $G = \prod_{\lambda\in\Lambda} G_\lambda$. Let $\mathcal{L}$ (respectively $\mathcal{L}_\lambda$) be the set of all quasisimple subgroups $Q \operatorname{lasc} G$ (resp. $Q \lhd^2 G_\lambda$), and let $\mathcal{N}$, $\mathcal{N}^*$, $\mathcal{N}_\lambda$ be the sets of all minimal normal subgroups of G, $Z(G)$, G_λ, respectively. Then:*

(i) *$\mathcal{L} = \bigcup_{\lambda\in\Lambda} \mathcal{L}_\lambda$ and $L(G) = \prod_{\lambda\in\Lambda} L(G_\lambda)$.*

(ii) *An arbitrary noncentral minimal normal subgroup N of G lies in one of the G_λ. If for each $\nu \neq \lambda$, $[G_\lambda, G_\nu] = 1$, then $\mathcal{N} = \mathcal{N}^* \bigcup(\bigcup_{\lambda\in\Lambda} \mathcal{N}_\lambda)$ and, in particular, $\operatorname{Soc}(G) = \operatorname{Soc}(Z(G)) \prod_{\lambda\in\Lambda} \operatorname{Soc}(G_\lambda)$.*

(iii) *If for each $\nu \neq \lambda$, $[G_\lambda, G_\nu] = 1$, then $Z(G) = \prod_{\lambda\in\Lambda} Z(G_\lambda)$.*

Proof. (i) By Proposition 3, if $Q \in \mathcal{L}$, then $Q \lhd \langle Q^G \rangle$ and so $Q \lhd^2 G$. Therefore, in view of [5, Proposition 3], for some $\lambda \in \Lambda$ we have $Q \subseteq G_\lambda$. Otherwise, $[Q, G_\lambda] = 1$, $\lambda \in \Lambda$, and $Q \subseteq Z(G)$, which is a contradiction. Then $Q \in \mathcal{L}_\lambda$. On the other hand, clearly, $\bigcup_{\lambda\in\Lambda} \mathcal{L}_\lambda \subseteq \mathcal{L}$.

(ii) Indeed, for some μ, $[N, G_\mu] \neq 1$. So $N \subseteq G_\mu$. Let $[G_\lambda, G_\nu] = 1$, $\nu \neq \lambda$. Then obviously $\mathcal{N}^* \bigcup(\bigcup_{\lambda\in\Lambda} \mathcal{N}_\lambda) \subseteq \mathcal{N}$ and $N \in \mathcal{N}_\mu$. Thus $\mathcal{N} = \mathcal{N}^* \bigcup(\bigcup_{\lambda\in\Lambda} \mathcal{N}_\lambda)$.

(iii) Obviously, $\prod_{\lambda\in\Lambda} Z(G_\lambda) \subseteq Z(G)$. Let $g \in Z(G) \setminus \prod_{\lambda\in\Lambda} Z(G_\lambda) \neq \emptyset$. Then for some pairwise distinct $\lambda_1, \lambda_2, \ldots, \lambda_n \in \Lambda$ and $g_k \in G_{\lambda_k}$, $k = 1, 2, \ldots, n$, we have $g = g_1 g_2 \ldots g_n$. In that case $1 = [g, G_{\lambda_k}] = [g_k, G_{\lambda_k}]$, i.e., $g_k \in Z(G_{\lambda_k})$ and $g \in \prod_{\lambda\in\Lambda} Z(G_\lambda)$, a contradiction. □

3. Proofs of the main theorems

Proof of Theorem 1. Let $G \neq 1$ be an M'-group. Suppose that in the series (2) G_{k-1} is finite for some finite nonzero $k \leq \gamma$. Then $\operatorname{Soc}(G/G_{k-1})$ is finite. Further, by [5, Proposition 5], $G_k/G_{k-1} \subseteq \operatorname{Soc}(G/G_{k-1}) Z(G_k/G_{k-1})$. Clearly, $Z(G_k/G_{k-1})$ is completely reducible and G/G_{k-1} is an M'-group. Consequently, in view of [5, Proposition 10], $Z(G_k/G_{k-1})$ is finite. Thus

$\mathrm{Soc}(G/G_{k-1})Z(G_k/G_{k-1})$ and, at the same time, G_k are finite. So G is an M''-group.

Let $G \neq 1$ be an M''-group. Assume that $H \lhd G$, H is finite and $H \neq G$. Then some finite term $N_k \in (1)$ is not contained in H. In view of Lemma 3, H belongs to some finite term $N_l \in (1)$. Since, obviously, N_kH/H has a minimal normal subgroup of G/H, $\mathrm{Soc}(G/H) \neq 1$. Suppose further that $\mathrm{Soc}(G/H)$ is infinite. Then, obviously, $\mathrm{Soc}(G/N_l)$ is infinite too. But in accordance with Corollary 1, $\mathrm{Soc}(G/N_l)$ must be finite, a contradiction. Thus G is socle finite. Therefore, if G is hyperfinite, then G is an M'-group.

Let A be a quasicyclic group and B be an infinite locally finite p-group without nontrivial normal abelian subgroups. Then, obviously, $A \times B$ is a socle finite non-M''-group.

Finally, the last conclusion of Theorem 1 follows from the first one, because the class of locally finite-normal groups is a subclass of the class of hyperfinite groups. □

Proof of Theorem 2. Let (i) hold. Then G is a periodic hyper-FC-group. In view of Proposition 4, we only need to prove that R is Chernikov. Let H be an arbitrary nilpotent normal subgroup of G with class ≤ 2. Put $K = \langle g \in Z(H) : g^p = 1 \text{ for some } p \in \mathbb{P}\rangle$ and $T = \langle g \in H : g^p \in Z(H) \text{ for some } p \in \mathbb{P}\rangle$. Obviously, $T' \subseteq K$. By Lemmas 4 and 3 and Corollary 1, K is contained in some finite $N_k \in (1)$. Since K is finite, $Z(H)$ is Chernikov (for instance, by [7, Lemma 1.10]). Since G/N_k is an M''-group and TN_k/N_k is a normal abelian subgroup of G/N_k, TN_k/N_k is Chernikov as above. Therefore, obviously, $T/Z(H)$ is finite. Consequently, $H/Z(H)$ is Chernikov. Since $Z(H)$ and $H/Z(H)$ are Chernikov, H is Chernikov too (S. N. Chernikov; see, for instance, [7, Theorem 1.4]). Further, it is easy to see that all distinct subgroups $R \cap N_\alpha$, where $N_\alpha \in (1)$, constitute an ascending abelian series of R with G-invariant terms. Consequently, in view of Proposition 7 in [5] and Proposition 2(i), R is Chernikov.

Let (ii) hold, $G \neq 1$, and assume that for some $k \in \mathbb{N}$ and $G_k \in (2)$, G_{k-1} is finite. Since R is Chernikov, by Proposition 2(i) and Proposition 5, G_k is finite. Thus G is an M''-group. □

Proof of Theorem 3. Obviously, H is a hyper-FC-group. Let R and T be the locally soluble radicals of G and H, respectively. By Theorem 2, R is Chernikov.

(i) Let $K = L(G)H$, $C = C_K(L(G))$, let B be as in Lemma 7, and let S be the locally soluble radical of K. Since $L(G)$ is semisimple (see Proposition 3) and $L(G) \lhd K$, we have $L(G) \subseteq L(K)$. Therefore $C_K(L(K)) \subseteq C_G(L(G))$. In view of Proposition 6, $C_G(L(G))$ is Chernikov. By Corollary 2, $S \subseteq C_K(L(K))$. Thus S is Chernikov. Because $|K : H|$ is finite, T is, obviously, Chernikov. Further, by Lemma 6 for the Chernikov subgroup C, $L(C)$ is finite. In view of Lemma 7, for any quasisimple subgroup $Q \lhd L(H)$, either $Q \lhd L(G)$ or $Q \lhd L(C)$ or $Q \lhd B$. Since $L(C)$ and B are both finite, by Theorem 2 for every $m \in \mathbb{N}$ the set of all quasisimple subgroups $Q \lhd L(H)$ with $|Z(Q)| \leq m$ is finite or empty. Therefore, by Theorem 2, H is an M''-group.

Let $H \operatorname{lasc} G$. Then, by Proposition 1, $T \subseteq R$. Consequently, T is Chernikov.

(ii) Let $m \in \mathbb{N}$ and let $\mathcal{K}$ be the set of all quasisimple subgroups $Q \lhd L(G)$ with $|Z(Q)| \leq m$ and $\mathcal{N}$ be the corresponding set for $L(H)$. By Proposition 3, $\mathcal{N} \subseteq \mathcal{K}$. In view of Proposition 4, $\mathcal{K}$ is finite or empty. Thus, by Theorem 2, H is an M''-group.

(iii) Suppose that $H \neq T$. Then there is a series $H_0 = T \subset H_1 \subset \cdots \subset H_n = H$ with finite simple factors. One may assume, of course, that $H \not\subseteq K = \langle H_{n-1}^G \rangle$ and K is already Chernikov. Then HK/K is finite simple, and by Lemma 2, $HK/K \operatorname{lasc} G/K$.

Let HK/K be abelian. Then, in view of Proposition 1, D/K is locally soluble. In this case D is almost locally soluble. Therefore the index $|D : D \cap R|$ is finite and D is Chernikov.

Now suppose that HK/K is not abelian. Then, in view of [5, Proposition 5], D/K is a minimal normal subgroup of G/K and, also, is the direct product of all distinct subgroups $(HK/K)^g$, $g \in G/K$. Obviously, either $C_D(K) \subseteq K$, or $D = KC_D(K)$. By the Baer-Polovickii Theorem [3], [14] (see, for instance, [15, Theorem 3.29]), $D/C_D(K)$ is Chernikov. Therefore, in the case $C_D(K) \subseteq K$, D/K is finite and so D is Chernikov.

Next, suppose $D = KC_D(K)$. It is easy to see that for some finite nonabelian simple subgroup $B/Z(K)$, $C_D(K)/Z(K)$ is the direct product of all distinct $(B/Z(K))^g$, $g \in G/Z(K)$. Clearly, $B' \operatorname{sn} G$ and $C_D(K) = Z(K)\prod_{g\in G}(B')^g$. In view of Lemma 5, B' is finite quasisimple. Consequently, in view of Proposition 4, $\prod_{g\in G}(B')^g$ is finite. Therefore $|D : K|$ is finite and so D is Chernikov.

(iv), (v) By Proposition 3, $L(H) = \prod_{\lambda\in\Lambda} Q_\lambda$ for some quasisimple subgroups $Q_\lambda \lhd L(G)$ and, also, $Z(Q_\lambda) \subseteq Z(L(H)) \subseteq Z(L(G))$. So $|Z(Q_\lambda)| \leq |Z(L(H)) \cap Z(L(G))|$. Consequently, in case (iv), $L(H)$ is Chernikov, and in case (v), according to Proposition 4, $L(H)$ is finite. Thus in both cases $L(H)T$ is Chernikov. Therefore, by [5, Proposition 7], H is Chernikov too. Then D is Chernikov—see (iii). □

Proof of Theorem 4. Obviously, $Z(G/R) = 1$. Therefore G/R is residually finite, for instance, by [8, Proposition 2.2.9]. We will prove that $G/L(G)R$ is residually finite.

(i) Let $R = 1$ and $G \neq L(G)$. In view of [5, Proposition 8], $C_G(L(G)) = Z(L(G))$. Therefore $L(G) \neq 1$. By Proposition 3, $L(G)$ is the direct product of some nonabelian simple subgroups Q_λ, $\lambda \in \Lambda$. Obviously, $R \supseteq Z(L(G)) = 1$ and so $C_G(L(G)) = 1$. In view of Proposition 2(i), the Q_λ are finite.

Let $gL(G)$ be an arbitrary element of prime order p of $G/L(G)$, and $\Gamma = \{\lambda \in \Lambda : [g, Q_\lambda] \neq 1\}$. Obviously, for some finite $\Delta \supseteq \Gamma$, $g \in$

$N_G(\times_{\lambda\in\Delta}Q_\lambda)$ and $g^p \in \times_{\lambda\in\Delta}Q_\lambda$. Put $\langle g\rangle(\times_{\lambda\in\Delta}Q_\lambda) = H$. It is not difficult to show that $H \cap C_G(H) = 1$, $C_G(H)L(G) = C_G(H)(\times_{\lambda\in\Delta}Q_\lambda)$. Obviously, the index $|G/L(G) : C_G(H)L(G)/L(G)|$ is finite. It is easy to see that $g \notin C_G(H)(\times_{\lambda\in\Delta}Q_\lambda) = C_G(H)L(G)$. Therefore $gL(G) \notin C_G(H)L(G)/L(G)$. Thus $G/L(G)$ is residually finite.

(ii) Let $R = 1$. We next prove that for an arbitrary normal subgroup N of G with $N \subseteq L(G)$, the group G/N is residually finite. Let $aN \in G/N \setminus \{1\}$. Suppose that $a \notin L(G)$. Since by (i) $G/L(G)$ is residually finite, there exists a subgroup $K \lhd G$ such that $a \notin K \supseteq L(G)$ and $|G : K|$ is finite. Then $aN \notin K/N$ and $|G/N : K/N|$ is finite. Now let $a \in L(G)$. Clearly, the index $|G/N : C_G(\langle a^G\rangle)N/N|$ is finite. Suppose that $aN \in C_G(\langle a^G\rangle)N/N$. Then, obviously, $\langle a^G\rangle \subseteq C_G(\langle a^G\rangle)N$. Consequently, $\langle a^G\rangle' \subseteq N$. It is easy to see that $\langle a^G\rangle$ is a direct product of some Q_λ. Consequently, $\langle a^G\rangle = \langle a^G\rangle'$. Thus $a \in N$, a contradiction.

(iii) Finally, we consider the general case. The locally soluble radical of the factor group G/R coincides with 1. It is obvious that $L(G/R) \supseteq L(G)R/R \lhd G/R$. Therefore by (ii) the factor group $(G/R)/(L(G)R/R) \simeq G/L(G)R$ is residually finite, and the theorem is proved. □

Proof of Theorem 5. In view of Ph. Hall's Theorem [9], H is isomorphic to a subgroup of the direct product V of some finite groups H_k, $k \in \mathbb{N}$. We will assume that $V \cap A = 1$. Let A_k, $k \in \mathbb{N}$, be cyclic subgroups of A (not necessarily distinct in pairs) such that $2 < |A_k| \le |A_{k+1}|$ and $A = \langle A_k : k \in \mathbb{N}\rangle$. Let Q_k, $k \in \mathbb{N}$, be finite quasisimple groups such that $Z(Q_k) = A_k$ and the group A_k is a subgroup of Q_k, $Q_k \cap V = 1$ and $Q_k \cap Q_j = A_k \cap A_j$, $j \neq k$. Let for Q_k and H_k the groups G_k be as in Lemma 9 such that $G_k \cap A = A_k = Z(G_k)$ and $G_k \cap G_j = A_k \cap A_j$, $j \neq k$. (For example, for $n = |A_k|$ and an arbitrary $p \in \mathbb{P}$ such that $n|p-1$ we may take $Q_k \simeq \mathbf{SL}_n(p)$.) Further, let $G_0 = A$, D be the external direct product of the groups G_k, $k \in \mathbb{Z}^+$ (see, for instance, [16]), $N < D$,

$$\begin{aligned} N = \langle (u_k) : u_k \in G_k,\, k \in \mathbb{Z}^+,\ \text{for some } j \in \mathbb{N},\, u_j \in Z(G_j),\, u_0 = u_j^{-1} \\ \text{and } u_k = 1,\, k \in \mathbb{N} \setminus \{j\}\rangle, \\ T_j = \{(u_k) : u_k = 1 \text{ for } k \in \mathbb{Z}^+ \setminus \{j\}\},\quad j \in \mathbb{Z}^+. \end{aligned}$$

It is not difficult to show that

$$\begin{aligned} T_j \cap N = 1,\quad j \in \mathbb{Z}^+, \\ T_jN/N \cap T_0N/N = Z(T_jN/N),\quad j \in \mathbb{N}, \\ T_jN/N \cap T_kN/N = Z(T_jN/N) \cap Z(T_kN/N),\quad j,k \in \mathbb{N},\, k \neq j. \end{aligned}$$

In view of these relations we may identify in a natural way the subgroup T_jN/N of the factor group $K = D/N$ with the group G_j, for each $j \in \mathbb{Z}^+$.

Then $K = \prod_{k\in\mathbb{N}} G_k$, $[G_k, G_j] = 1$ for $j \neq k$, $\prod_{k\in\mathbb{N}} H_k$ is the direct product of subgroups H_k, $k \in \mathbb{N}$, and $A = \prod_{k\in\mathbb{N}} Z(G_k)$. We will assume that $H \leq \prod_{k\in\mathbb{N}} H_k$. In view of Lemma 10, $A = Z(K)$ and $L(K) = \prod_{k\in\mathbb{N}} L(G_k)$. Obviously, $L(K) \cap \prod_{k\in\mathbb{N}} H_k = 1$. Since $L(G_k) = \prod_{g\in H_k} Q_k^g$, and for distinct $g, h \in H_k$, $[Q_k^g, Q_k^h] = 1$, we have $L(K) = \prod_{k\in\mathbb{N}} \prod_{g\in H_k} Q_k^g$, and $[Q_k^g, Q_j^h] = 1$ if $k \neq j$ or $g \neq h$. Since $C_{G_k}(L(G_k)) = Z(G_k)$, $k \in \mathbb{N}$, it is easy to see that $C_K(L(K)) = A = Z(L(K))$.

Put $G = L(K) \leftthreetimes H$. Since $C_G(L(K)) \subseteq L(K)$, by Proposition 3, $L(G) = L(K)$. So $C_G(L(G)) = A = Z(L(G))$. By Corollary 2, $R \subseteq C_G(L(G))$. Consequently, $R = Z(L(G)) = A$ and, at the same time, R is Chernikov. Since $A \subseteq Z(G) \subseteq R$, we have $Z(G) = A$. Obviously, $L(G)$ and G are locally finite-normal and non-Chernikov.

Further, as a consequence of Proposition 3, $\mathcal{M} = \{Q_k^g : k \in \mathbb{N}, g \in H_k\}$ is the set of all quasisimple subgroups $Q \lhd L(G)$. Clearly, for each $m \in \mathbb{N}$ the set $\{Q \in \mathcal{M} : |Z(Q)| \leq m\}$ is finite or empty. Thus, by Theorem 2, G is an M''-group. Then, in view of Theorem 1, G is an M'-group. Since G is a locally finite-normal M'-group, we have $G = \bigcup_{k\in\mathbb{N}} \operatorname{Soc}_k(G) = \operatorname{Soc}_\omega(G)$ and $|\operatorname{Soc}_k(G)| < \infty$, $k \in \mathbb{N}$ (see [5, Lemma 7]).

It is easy to see that K/A, and also G/A, is infinite residually finite. Consequently, G/A, and hence G, does not satisfy the minimal condition for normal subgroups of finite index. Thus the theorem is proved. □

Remark. The group G constructed in the proof of Theorem 5 is isomorphic to a subgroup of a factor group of the direct product of groups G_k, $k \in \mathbb{N}$, by a central subgroup.

Proof of the Assertion. In view of Theorem 1, (i) holds iff G is an M'-group, and by [5, Lemma 7], G is an M'-group iff (ii) holds. Thus (i) $\Leftrightarrow$ (ii).

Let (i) hold. Then $G = \operatorname{Soc}_\omega(G)$ and by Proposition 2(iii), for the series (2) we have $\gamma \leq \omega$. Further, by virtue of Theorem 2, for each $G_\alpha \in (2)$ with finite α the locally soluble radical of G_α is Chernikov. Consequently, in view of Propositions 4(i) and 5, G_α is finite.

Obviously, (iii) $\Rightarrow$ (i). □

The next proposition may be useful when studying socle groups, M''-groups or M'-groups.

Proposition 7. *Let G be a socle group such that for each $k \in \mathbb{Z}^+$, $\operatorname{Soc}_{k+1}(G)/\operatorname{Soc}_k(G)$ is a direct product of finitely many minimal normal subgroups of group $G/\operatorname{Soc}_k(G)$. Let H be a normal subgroup of G such that $H \not\subseteq \operatorname{Soc}_k(G)$, $k = 1, 2, \ldots$. Then the intersection $H \cap \operatorname{Soc}_\omega(G)$ contains a subgroup $N \lhd G$ with the following properties:*

(i) *$N \cap \operatorname{Soc}_k(G) \neq N \cap \operatorname{Soc}_{k+1}(G)$, $k = 0, 1, 2, \ldots$.*

(ii) *Any proper G-invariant subgroup T of N is contained in one of the* $\mathrm{Soc}_k(G)$. *(In particular, T is finite if all groups* $\mathrm{Soc}_k(G)$ *are finite.)*

Proposition 7 follows at once from [5, Proposition 9], and the following lemma, which is obvious.

LEMMA 11. *Let G be a socle group. Let $N \lhd G$ and assume that $N \not\subseteq \mathrm{Soc}_\alpha(G)$ for some ordinal α. Then $N \cap \mathrm{Soc}_\alpha(G) \neq N \cap \mathrm{Soc}_{\alpha+1}(G)$.*

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DEPARTMENT OF ALGEBRA, INSTITUTE OF MATHEMATICS, NATIONAL ACADEMY OF SCIENCES OF UKRAINE, 3, TERESHCHENKIVSKA STR., 01601 KYIV-4, UKRAINE

E-mail address: chern@imath.kiev.ua