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GROUPS IN WHICH SYLOW SUBGROUPS AND SUBNORMAL SUBGROUPS PERMUTE

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Abstract. We consider certain properties of finite groups in which the subnormal subgroups permute with all the Sylow subgroups. Such groups are called PST-groups. If G is such a group and H_1/K_1 and H_2/K_2 are isomorphic abelian chief factors of G such that $H_1H_2 \subseteq G'$, then they are operator isomorphic. Moreover, if all the abelian isomorphic chief factors of a PST-group G are operator isomorphic, then all the subnormal subgroups are hypercentrally embedded in G.

1. Introduction

Several authors have considered finite groups in which subnormal subgroups permute with certain classes of subgroups: see, for example, [1], [2], [3], [4], [6], and [7]. The object of this note will be to prove statements about finite groups in which the subnormal subgroups permute with the Sylow subgroups. These groups are called PST-groups. We will see that two abelian chief factors H_i/K_i are operator isomorphic if they are isomorphic and $H_1H_2 \subseteq G'$. For the proof heavy use is made of the classification of finite simple groups and the Atlas [9]. An example of Thompson in [10] shows that containment in G' is indispensable.

A subnormal subgroup S is called *hypercentrally embedded* in G if $S^G/S_G \subseteq Z_\infty(G/S_G)$, the hypercenter of G/S_G. We show that in general subnormal subgroups of PST-groups are not hypercentrally embedded. If, however, all isomorphic abelian chief factors are operator isomorphic, all subnormal subgroups are hypercentrally embedded. In [2] the authors show that if G is a soluble PST-group, then p-chief factors are operator isomorphic. Thus for soluble PST groups all the subnormal subgroups are hypercentrally embedded, a result that was observed in [4] and [7].

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2. The chief factors

Let p be a prime. A finite group G is said to be an N_p-group provided that if N is a normal subgroup of G, then all the subgroups of $O_p(G/N)$ permute with all the Sylow subgroups of G (see [7]). Robinson [10] introduced a class N_p of groups which is slightly different from ours.

A characterization of PST-groups is presented in [10] (see (ii) on p. 158), which is similar to the next proposition. As mentioned in [10], the proof is very similar to the proof of Theorem 3.1 of that paper and for that reason we simply outline a proof.

PROPOSITION 1. *A finite group G is a PST-group if and only if the non-abelian chief factors of G are simple and G satisfies N_p for all primes p.*

Proof. Assume that the non-abelian chief factors of G are simple and G satisfies N_p for all p. It is clear that all the chief factors of G are simple. Let D be the soluble residual of G. By Lemma 2.4 of [10] $D/Z(D)$ is a direct product of G-invariant non-abelian simple groups. Also by Theorem A of [7] G/D is a soluble PST-group. Let H be a subnormal subgroup of G. Argue as in the proof of Theorem 3.1 of [10] to show that H permutes with all the Sylow subgroups of G.

Conversely, assume that G is a PST-group. Since each homomorphic image of G is a PST-group, it follows that G is an N_p-group for all primes p. Thus the p-chief factors of G are simple for all p. Using a proof similar to the one used to establish Proposition 2.1 of [10] we see that the non-abelian chief factors of G are simple. □

Now we consider operator isomorphisms of the abelian chief factors of PST-groups.

THEOREM 2. *Assume that G is a finite PST-group. If H_1/K_1 and H_2/K_2 are isomorphic abelian chief factors of G and $H_1H_2 \subseteq G'$, then these factors are isomorphic as G-operator groups.*

Proof. Assume that H_1/K_1 and H_2/K_2 have order p. Let D be the soluble residual of G and recall from Proposition 1 that $Z(D)$ is the soluble radical of D. Assume that D avoids the given chief factors. Since G/D is a soluble PST-group, H_1/K_1 and H_2/K_2 are G-operator isomorphic by Theorems 6 and 8 of [2]. Now assume that D covers H_1/K_1 and H_2/K_2. Hence $Z(D)$ covers these factors. By Proposition 1 the p'-elements of G induce power automorphisms on $O_p(Z(D))$ and hence in H_1/K_1 and H_2/K_2. Thus these factors are G-isomorphic. We can now assume that $K_1 < H_1 \leq Z(D)$ and $D \leq K_2 < H_2 \leq G'$. Choose M maximal subject to $M \lhd G$ and $M \cap H_1 = K_1$.

First suppose that p divides the order of MD/D. By the previous paragraph we can assume $H_2 \subseteq MD$. But then $H_1H_2/K_1K_2 \cong H_1/K_1 \times H_2/K_2$

is a factor of $MD/M \cap D \cong M/M \cap D \times D/M \cap D$ and the p'-elements of G induce power automorphisms in this. Hence H_1/K_1 and H_2/K_2 are G-isomorphic.

Next assume that the order of MD/D is prime to p. Then we can assume that $MD \leq K_2 < H_2 \leq G$. Thus we may replace G by G/M and hence $K_1 = 1$. This means that H_1 is the unique minimal normal subgroup of G and it follows that $O_{p'}(G) = 1$ and $Z = Z(D)$ is a p-group.

Let $C = C_G(D/Z)$. By Proposition 1 and Lemma 2.6 of [10] $C = C_G(D)$ is the soluble radical of G. Note that C is a PST-group. Hence, by Theorem 1 of [1], $\gamma_\infty(C) \cap Z = 1$, which means $\gamma_\infty(C) = 1$ since $O_p{}'(G) = 1$. (Here $\gamma_\infty(C)$ is the hypercommutator or last term of the lower central series of C.) Therefore, C is a p-group.

Suppose that H_2/K_2 is G-isomorphic with a factor of $CD/D \cong C/Z$. Then we can assume that $Z \leq K_2 < H_2 \leq C$. Write $H_2 = \langle x_2, K_2 \rangle$ and $H_1 = \langle x_1 \rangle$. Then $A = \langle x_1, x_2 \rangle$ is abelian and $A \leq C$. By Proposition 1 the p'-elements of G induce power automorphisms in C and hence in A. Hence H_1 and H_2/K_2 are G-isomorphic.

Thus we can assume that $C = Z$ and H_2/K_2 is G-isomorphic with a factor of $(G/D)'$. By Proposition 1 and Lemma 2.4 of [10] we have $D/Z = Dr_{i=1}^{k}(U_i/Z)$ where U_i/Z is a non-abelian simple group. Now G/D is isomorphic with a subgroup of $Dr_{i=1}^{k}\operatorname{Out}(U_i/Z)$. Hence p divides the order of $(\operatorname{Out}(U_i/Z))'$ for some i. Also $U_i' \cap Z \neq 1$, so that p divides the order of $M(U_i/Z)$, where $M(U_i/Z)$ is the Schur multiplier of U_i/Z. Put $S_i = U_i/Z$.

At this point we appeal to the classification of finite simple groups. From the Atlas [9] we see that S_i can not be a sporadic group. Also, since p can be assumed odd, S_i is not of alternating type. Thus we are left with Chevalley and twisted Chevalley groups. Consulting Table 2 and Table 5 of [9] we see that we have to consider only

$$
\begin{aligned}
&A_n(q) = L_{n+1}(q) && \text{if } p \text{ divides } \gcd(n+1, q-1),\\
&{}^2A_n(q) = U_{n+1}(q) && \text{if } p \text{ divides } \gcd(n+1, q+1),\\
&E_6(q) && \text{if } p = 3 \text{ and } 3 \text{ divides } q-1,\\
&{}^2E_6(q) && \text{if } p = 3 \text{ and } 3 \text{ divides } q+1.
\end{aligned}
$$

In all of these cases we obtain the Sylow p-subgroup of $Z(U_i) = Z(D)$ as a subgroup of the multiplicative group of some field and the p-subgroup of $(\operatorname{Out}(U_i/Z(D))'$ as a subgroup of a normal subgroup which can be considered isomorphic to a subgroup of the multiplicative group of the same field. On both of these p-groups the same field automorphism operates, so in fact the p-chief factor belonging to $Z(D)$ and $G'C_G(U_i)/C_G(U_i)U_i$ are operator isomorphic. This completes the proof of Theorem 2. □

3. Hypercentral embedding

We begin with a counterexample.

Example. Let p be an odd prime; also let q be a prime such that $q-1$ is divisible by p but not by p^2. The group $SL(p^2, q^p)$ possesses a duality automorphism δ (which maps every matrix onto the inverse of its transpose) of order 2 and an automorphism σ of order p arising from applying the field automorphism to every matrix entry. The center of $SL(p^2, q^p)$ is of order p^2 and cyclic. Now let $H \cong SL(p^2, q^p)$ and $K = \langle H, d, s | [d, s] = d^2 = s^p = 1;\ d^{-1}hd = \delta(h);\ s^{-1}hs = \sigma(h) \text{ for all } h \in H\rangle$. We choose the subgroup $L = \langle d_1 d_2, s_1 s_2^{-1}, H_1, H_2\rangle$ of the direct product $K_1 \times K_2$ of two copies of K. We have $Z(L) = 1$, since $d_1 d_2$ inverts by conjugation all elements of $Z(H_1 H_2)$. On the other hand, if t_1, t_2 are generators of $Z(H_1), Z(H_2)$, then $[t_1 t_2, s_1 s_2^{-1}] = t_1^{kp} t_2^{-kp}$ for some k prime to p. It is easy to see that

$$\langle t_1 t_2, t_1^p t_2^{-p}\rangle = (\langle t_1, t_2\rangle)^L \subseteq Z(H_1 H_2),$$
$$\langle t_1^p t_2^p\rangle = (\langle t_1 t_2\rangle)_L,$$

and so $\langle t_1 t_2\rangle$ is subnormal and not hypercentrally embedded in L. On the other hand, L is a PST- group.

As a positive statement we obtain:

Theorem 3. *Assume that G is a finite PST-group and all abelian isomorphic chief factors of G are operator isomorphic. Then all subnormal subgroups are hypercentrally embedded in G.*

Proof. Let S be a subnormal subgroup of G. Then S/S_G is soluble. Consider a nontrivial normal p-subgroup T/S_G of S/S_G. Then $T_G = S_G$ and T^G/S_G is a normal p-subgroup of G/S_G. If T^G/S_G does not belong to the hypercenter of G/S_G, then some Sylow q-subgroups of G/S_G (where $q \neq p$) operate nontrivially on T^G/S_G. By Lemma 1 of [5], T^G/S_G is abelian and $(G/S_G)/C_{G/S_G}(T^G/S_G)$ is a direct product of a p-group and a cyclic group of order prime to p. Since p-chief factors of this quotient group would be central and those of T^G/S_G are not, we obtain that this centralizer quotient group is cyclic of order prime to p and all subgroups of T^G/S_G are normal in G/S_G; $T^G = T = T_G$, a contradiction. We obtain that the Fitting subgroup of S/S_G is contained in the hypercenter of G/S_G and so in the hypercenter of S/S_G. From this we deduce that S/S_G is nilpotent and in the hyperenter of G/S_G, as was to be shown. □

Corollary 1. *Assume that G is a finite soluble PST-group. Then all subnormal subgroups of G are hypercentrally embedded in G.*

Proof. Corollary 1 follows from Theorem 3 and Theorem H of [7]. It is also Corollary 2 of [4]. □

Carocca and Maier [8] prove the following result.

THEOREM 4. *Let G be a finite group and let S be a subnormal subgroup of G which permutes with all the Sylow subgroups of G. Then S permutes with all the pronormal subgroups of G if and only if it is hypercentrally embedded in G.*

From Theorems 3 and 4 we obtain:

COROLLARY 2. *Let G be a finite PST-group all of whose isomorphic abelian chief factors are operator isomorphic. Then the subnormal subgroups of G permute with all the pronormal subgroups of G.*

REMARK 1. Let G be a finite PST-group. By Proposition 1 and Theorem A of [6] all the subnormal subgroups of G permute with all the maximal subgroups of G. In fact, one can prove the following: Let G be a finite PST-group and let X be a locally pronormal subgroup of G. Then all the subnormal subgroups of G permute with X.

4. T-groups

It is common usage to call groups in which all subnormal subgroups are normal subgroups T-groups. Obviously the class of finite T-groups is a subclass of the class of finite PST-groups. This gives rise to a specialization of Theorem 2:

COROLLARY 3. *Two isomorphic abelian chief factors H_1/K_1 and H_2/K_2 of a T-group G are operator isomorphic whenever $H_1H_2 \subseteq G'$.*

There is another connection between these classes: we denote by G^* the nilpotent residual of G, i.e., the smallest normal subgroup K of G with nilpotent quotient group G/K. Now we can formulate our statement.

THEOREM 5. *Assume that G is a finite PST-group. Then G^* is a T-group.*

Proof. Consider a subnormal subgroup $S \subseteq G^*$ and assume $S_G \neq S$. If T/S_G is a normal p-subgroup of S/S_G, we obtain $T_G = S_G$ and T^G/S_G is a normal p-subgroup of G/S_G. If T^G/S_G is contained in the hypercenter of G/S_G, then $(G/S_G)/C_{G/S_G}(T^G/S_G)$ is nilpotent as it is a p-group. If T^G/S_G is not contained in the hypercenter of G/S_G, the quotient group $(G/S_G)/C_{G/S_G}(T^G/S_G)$ is a direct product of a cyclic group and a p-group, so it is again nilpotent. So in both cases we have $T^G/S_G \subseteq Z(G^*/S_G)$. We deduce that the Fitting subgroup of the soluble group S/S_G coincides with its centre. Now S/S_G is abelian and contained in $Z(G^*/S_G)$. Thus S/S_G is normal in G^*/S_G and S is normal in G^*. So G^* is a T-group. The proof is complete. □

Let B be a finite soluble PST-group. In the proof of Theorem H of [7] it was shown that $\mathrm{Fit}(B) = \gamma_\infty(B) \times Z_\infty(B)$. By Theorem 1 of [1] $B/Z_\infty(B)$ is a T-group so that $B'' \leq Z_\infty(B)$.

Notice also that by consulting the Atlas [9] it is noted for a non-abelian simple group S that $\mathrm{Aut}(S)/\mathrm{Inn}(S)$ is metabelian except when $S \cong D_4(q)$, where this quotient is isomorphic to S_4.

Let G be a finite PST-group. By Proposition 1 the chief factors of G are simple, and hence if $S \cong D_4(q)$ is isomorphic to some chief factor H/K of G, then $(G/K)/(H/K)C_{G/K}(H/K)$ is a subgroup of S_4 which is a PST-group and hence supersoluble. Thus it is therefore abelian, or isomorphic to S_3 or isomorphic to the dihedral group of order 8.

Let D be the soluble residual of G. Since G/D is a soluble PST-group, it follows from above that G''/D is contained in the hypercenter of G/D. Let H be the hypercenter of G. Then $H \leq C_G(D)$, the soluble radical of G. By Proposition 1, Lemma 2.4 of [10] and the above, it follows that $G'' \leq DC_G(D)$ whence $G'' \leq HD$. Now the nilpotent factors of G/H are abelian and the automorphisms induced on them by the PST-group G/H are power automorphisms. Furthermore, the abelian factors of $(G/H)' \cong G'H/H$ are central. Thus $G'H/H$ is a T-group by Theorem 4.2 of [10]. We have established:

THEOREM 6. *Let G be a finite PST-group with hypercenter H. Then $G'H/H$ is a T-group.*

REMARK 2. Let G be a finite PST-group with hypercenter H. If G is soluble, then G/H is a T-group. However, if G is the extension of $D_4(3)$ by the dihedral group of order 8, then G/H is not a T-group.

References

[1] R. K. Agrawal, *Finite groups whose subnormal subgroups permute with all Sylow subgroups.* Proc. Amer. Math. Soc. **47** (1975), 77–83.

[2] M. Alejandre, A. Ballester-Bolinches, and M. C. Pedraza-Aquilera, *Finite soluble groups with permutable subnormal subgroups*, J. Algebra **240** (2001), 705–722.

[3] A. Ballester-Bolinches and R. Esteban-Romero, *Sylow permutable subnormal subgroups of finite groups*, J. Algebra **251** (2002), 727–738.

[4] ———, *Sylow permutable subnormal subgroups of finite groups II*, Bull. Austral. Math. Soc. **64** (2001), 479–486.

[5] J. C. Beidleman and H. Heineken, *Totally permutable torsion subgroups,* J. Group Theory **2** (1999), 377–392.

[6] ———, *Pronormal and subnormal subgroups and permutability*, to appear.

[7] ———, *Finite soluble groups whose subnormal subgroups permute with certain classes of subgroups*, J. Group Theory **6** (2003), 139–158.

[8] A. Carocca and R. Maier, *Hypercentral embedding and pronormality,* Arch. Math. **7** (1998), 433–436.

[9] J. H. Conway, R. T. Curtis, S. P. Norton, R. A. Parker, and R. A. Wilson, *Atlas of finite groups*, Oxford University Press, Eynsham, 1985.

[10] D. J. S. Robinson, *The structure of finite groups in which permutability is a transitive relation*, J. Austral. Math. Soc. **70** (2001), 143–159.

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