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ON CAPABLE p-GROUPS OF NILPOTENCY CLASS TWO

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Dedicated to Reinhold Baer on the hundredth anniversary of his birth

Abstract. A group is called capable if it is a central factor group. Let $\mathcal{P}$ denote the class of finite p-groups of odd order and nilpotency class 2. In this paper we determine the capable 2-generator groups in $\mathcal{P}$. Using the explicit knowledge of the nonabelian tensor square of 2-generator groups in $\mathcal{P}$, we first determine the epicenter of these groups and then identify those with trivial epicenter, making use of the fact that a group has trivial epicenter if and only if it is capable. A capable group in $\mathcal{P}$ has the two generators of highest order in a minimal generating set of equal order. However, this condition is not sufficient for capability in $\mathcal{P}$. Furthermore, various homological functors, among them the exterior square, the symmetric square and the Schur multiplier, are determined for the 2-generator groups in $\mathcal{P}$.

1. Introduction

Since its publication, Philip Hall's 1940 paper [17] has pointed the way towards the classification of groups of prime power order. The paper at hand is one more contribution towards reaching the goal, the specific topic being central extensions. Here is what P. Hall himself had to say about it:

> "The question of what conditions a group G must fulfill in order that it may be the central quotient group of another group H,
> $$G \cong H/Z(H),$$
> is an interesting one. But while it is easy to write down a number of necessary conditions it is not so easy to be sure that they are sufficient."

Calling a group which is a central factor group a capable group occurred much later and is due to M. Hall and J.K. Senior [16].

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Definition 1.1. A group G is said to be capable if there exists a group H such that $G \cong H/Z(H)$, or equivalently, G is isomorphic to the inner automorphism group of a group H.

Capability of groups was first studied by R. Baer [4] and his characterization of finitely generated abelian groups which are capable has remained the only complete one within a class of groups up till now.

Theorem 1.2 ([4]). *Let A be a finitely generated abelian group written as*

$$A = \mathbb{Z}_{n_1} \oplus \mathbb{Z}_{n_2} \oplus \ldots \oplus \mathbb{Z}_{n_k}$$

such that each integer n_{i+1} is divisible by n_i, where $\mathbb{Z}_n = \mathbb{Z}$, the infinite cyclic group, if $n = 0$. Then A is capable if and only if $k \geq 2$ and $n_{k-1} = n_k$.

In 1979, Beyl, Felgner and Schmid [6] established a necessary and sufficient condition for a group to be a central quotient in terms of the epicenter defined as follows.

Definition 1.3. The epicenter $Z^*(G)$ of a group G is defined as

$$\bigcap \{\varphi Z(E); (E, \varphi) \text{ is a central extension of } G\}.$$

It can be easily seen that the epicenter is a characteristic subgroup of G contained in its center. The following criterion now characterizes capable groups.

Theorem 1.4 ([6]). *A group G is capable if and only if $Z^*(G) = 1$.*

Using this criterion, Beyl et al. give a shorter proof of Baer's result, describe finite metacyclic groups which are capable and characterize capable extra-special p-groups as follows.

Theorem 1.5 ([6]). *An extra-special p-group is capable if and only if it is dihedral of order 8 or of order p^3 and exponent p, $p > 2$.*

Though the above criterion for capability is easily formulated, its implementation is another matter. Already in [6] a remark is made about an external characterization of the epicenter, namely as the left kernel of the Ganea map [15], defined by $\gamma_G : Z(G) \times G/G' \to H_2(G)$, where $H_2(G)$ is the second homology group of G. But as in all cases before, this still requires the cumbersome process of evaluating factor sets in its implementation.

Graham Ellis [13] was able to characterize the epicenter in terms of the nonabelian tensor product as defined below. Once the nonabelian tensor product of a group is known, it is not too hard to determine its epicenter, in particular if the group has nilpotency class two. In case of an odd prime p, 2-generator p-groups of class 2 were classified and their nonabelian tensor square was determined [3]. In this paper we determine the epicenter for those groups and identify the capable ones among them. In general, we will show that Baer's criterion for capability, namely the two generators of highest order having equal order, is necessary for capability but not sufficient in the case of p-groups of class 2, p an odd prime. Along the way, we will compute various homological functors for these groups, among them the exterior and symmetric square and the Schur multiplier.

In [5], a similar investigation was done for infinite metacyclic groups. After classifying these groups and determining their nonabelian tensor squares, the various homological functors and the epicenter were computed leading to the following result about the capability of such groups.

THEOREM 1.6. *An infinite nonabelian metacyclic group is capable if and only if it is isomorphic to the infinite dihedral group.*

The nonabelian tensor square of a group is a special case of the nonabelian tensor product which has its origins in algebraic K-theory as well as in homotopy theory. For R. Brown and J.-L. Loday [9] and [10], the nonabelian tensor product is the direct outgrowth of their involvement with generalized Van Kampen Theorems. It is based on earlier work by C. Miller [22], R.K. Dennis [11], A.S.-T. Lue [21], and others. For further details we refer the reader to the website `http://www.bangor.ac.uk/~mas010/nonabtens.html`.

DEFINITION 1.7. For a group G, the nonabelian tensor square $G \otimes G$ is generated by the symbols $g \otimes h, g, h \in G$, subject to the relations

$$gg' \otimes h = ({}^g g' \otimes {}^g h)(g \otimes h), \tag{1.7.1}$$

$$g \otimes hh' = (g \otimes h)({}^h g \otimes {}^h h'), \tag{1.7.2}$$

for all $g, g', h, h' \in G$, where ${}^h g = hgh^{-1}$ denotes the conjugate of g by h.

We note here that the analogue of the bilinear map in [9] is the crossed pairing (see [8]). As a consequence of crossed pairings, the group G acts naturally on the tensor product by ${}^g(g' \otimes h) = {}^g g' \otimes {}^g h$ and there exists a homomorphism $\kappa : G \otimes G \to G$ sending $g \otimes h$ to $[g, h]$, where $ghg^{-1}h^{-1} = [g, h]$, the commutator of g and h. We write $J(G)$ for the kernel of κ. Its topological interest is the formula $J(G) \cong \pi_3 SK(G, 1)$ as given in [9] and [10], where $SK(G, 1)$ is the suspension of $K(G, 1)$. We note that $J(G)$ is a G-trivial subgroup of $G \otimes G$ contained in its center. We will use this fact throughout this paper.

Consistent with the notation and terminology in [12], we define subgroups $\triangle(G)$ and $\nabla(G)$ of $J(G)$ by the following. Let $\triangle(G)$ denote the subgroup of $J(G)$ generated by the elements $(x \otimes y)(y \otimes x)$ for $x, y \in G$. The symmetric square of G is then defined as $G \tilde{\otimes} G = {}^{(G \otimes G)}/_{\triangle(G)}$. We set $\tilde{J}(G) = {}^{J(G)}/_{\triangle(G)}$. It is shown in [10] that $\tilde{J}(G) \cong \pi_4 S^2 K(G, 1) = \pi_2^s K(G, 1)$.

Let $\nabla(G)$ denote the subgroup of $J(G)$ generated by the elements $x \otimes x$ for $x \in G$. The exterior square of G is then defined as $G \wedge G = {}^{(G \otimes G)} /_{\nabla(G)}$. We set ${}^{J(G)}/_{\nabla(G)} = M(G)$, which is otherwise known as the Schur multiplier of G. It has been shown in [22] that $M(G) \cong H_2(G)$, the second homology group of G.

We summarize these results in the following theorem, which (modulo different notation and terminology) can already be found in [10].

THEOREM 1.8. *Let G be a group. Then the rows are exact in the following commutative diagram*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & J(G) & \longrightarrow & G \otimes G & \xrightarrow{\kappa} & G' & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & \tilde{J}(G) & \longrightarrow & G \tilde{\otimes} G & \xrightarrow{\kappa''} & G' & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & M(G) & \longrightarrow & G \wedge G & \xrightarrow{\kappa'} & G' & \longrightarrow & 1
\end{array}$$

where $\kappa(g \otimes h) = \kappa''(g \tilde{\otimes} h) = \kappa'(g \wedge h) = [g, h]$.

In Section 3, we will compute $G \tilde{\otimes} G$, $G \wedge G$ as well as $J(G)$, $\tilde{J}(G)$ and $M(G)$ for the 2-generator p-groups of class 2, p an odd prime, as classified in [3].

Similar to the tensor center $Z^{\otimes}(G) = \{a \in G \mid a \otimes g = 1_{\otimes},\ \forall g \in G\}$ [12], we can define the exterior center of a group as

$$Z^{\wedge}(G) = \{a \in G \mid a \wedge g = 1_{\wedge},\ \forall g \in G\}.$$

Here $1_{\otimes}$ and $1_{\wedge}$ denote the identities in $G \otimes G$ and $G \wedge G$, respectively. It can be easily shown that $Z^{\otimes}(G)$ and $Z^{\wedge}(G)$ are characteristic and central subgroups of G with $Z^{\otimes}(G) \subseteq Z^{\wedge}(G)$. In [13], Ellis obtains the desired external characterization of the epicenter as follows.

THEOREM 1.9. *For any group G, the epicenter coincides with the exterior center, i.e., $Z^*(G) = Z^{\wedge}(G)$.*

As we will see in the last section, using the above theorem and Theorem 1.4, we will be able to determine the epicenters of all 2-generator p-groups of class 2 with p an odd prime and will identify the capable groups among them.

2. Some preparatory results

This section contains some results to be used throughout the rest of the paper. We start with some tensor expansion formulas for groups of nilpotency class two which can be found in Proposition 3.2 and Lemma 3.4 of [3].

LEMMA 2.1. *Let G be a group of nilpotency class 2 with $g, g', h, h' \in G$ and m, n integers. Then*

$$gg' \otimes h = (g \otimes h)(g' \otimes h)([g, g'] \otimes h)(g' \otimes [g, h]), \tag{2.1.1}$$

$$g \otimes hh' = (g \otimes h)(g \otimes h')(g \otimes [h, h'])([h, g] \otimes h'), \tag{2.1.2}$$

$$g^n \otimes h^m = (g \otimes h)^{nm}(h \otimes [g, h])^{n\binom{m}{2}}(g \otimes [g, h])^{m\binom{n}{2}}. \tag{2.1.3}$$

For orders of products, commutators, and tensors we have the following estimates.

LEMMA 2.2. *Let G be a group of nilpotency class 2 and $g, h \in G$ of odd order $o(g)$ and $o(h)$, respectively. Then*

$$o(gh) \leq \max\{o(g), o(h)\}, \tag{2.2.1}$$

$$o([g, h]) \leq \min\{o(g), o(h)\}, \tag{2.2.2}$$

$$o(g \otimes h) \leq \min\{o(g), o(h)\}. \tag{2.2.3}$$

Proof. Since $\langle g, h\rangle$ is regular, (2.2.1) and (2.2.2) follow immediately. To show (2.2.3), we observe that for any $x, y \in G$ and an integer n we can expand linearly and so (2.1.3) yields

$$(x \otimes [x, y])^n = x^n \otimes [x, y] = x \otimes [x, y]^n.$$

Setting $k = \min\{o(g), o(h)\}$, the above together with (2.2.2) yields

$$(g \otimes [g, h])^k = (h \otimes [h, g])^k = 1_\otimes. \tag{2.2.4}$$

Expanding by (2.1.3) and using (2.2.4), we arrive at $g^k \otimes h = (g \otimes h)^k = g \otimes h^k$. Since $k = \min\{o(g), o(h)\}$, it follows that $o(g \otimes h) \leq \min\{o(g), o(h)\}$, the desired result. □

PROPOSITION 2.3. *For any group G, $J(G) \supseteq \nabla(G) \supseteq \triangle(G)$.*

Proof. The containment $J(G) \supseteq \nabla(G)$ is clear. To prove $\nabla(G) \supseteq \triangle(G)$, we expand $gh \otimes gh$ by (1.1.1) and (1.1.2) and observe that the G-action on $\nabla(G)$ is trivial, arriving at

$$gh \otimes gh = {}^g(h \otimes g)(h \otimes h)(g \otimes g) \cdot {}^g(g \otimes h).$$

The fact that $\nabla(G)$ is central in $G \otimes G$ leads to

$$gh \otimes gh = (h \otimes h)(g \otimes g) \cdot {}^g((h \otimes g)(g \otimes h)).$$

Observing that G acts trivially on $\triangle(G)$ and rearranging the terms in the above yield

$$(h \otimes g)(g \otimes h) = (g \otimes g)^{-1}(h \otimes h)^{-1}(gh \otimes gh). \tag{2.3.1}$$

Hence any generator of $\triangle(G)$ is contained in $\nabla(G)$ and our claim follows. □

In case of a group of class 2, the generators for $G \otimes G$, $\nabla(G)$ and $\triangle(G)$ can be expressed in terms of tensors involving only generators of G.

Proposition 2.4. *Given a group G of nilpotency class 2 and a generating set X for G, we have*

$$G \otimes G = \langle u \otimes v, u \otimes [v,w] \mid u,v,w \in X \rangle, \tag{2.4.1}$$

$$\nabla(G) = \langle u \otimes u, (u \otimes v)(v \otimes u) \mid u,v \in X \rangle, \tag{2.4.2}$$

$$\triangle(G) = \langle (u \otimes u)^2, (u \otimes v)(v \otimes u) \mid u,v \in X \rangle. \tag{2.4.3}$$

Proof. Since G is nilpotent of class two, (2.4.1) follows by Lemma 2.5 in [2]. Containment in one direction is clear for (2.4.2) and (2.4.3) using the definitions of $\nabla(G)$ and $\triangle(G)$.

To show $\nabla(G) \subseteq \langle u \otimes u, (u \otimes v)(v \otimes u) | u,v \in X \rangle$, we observe that every $g \in G$ can be written as $g = \prod_{i=1}^{n} u_i^{m_i} \cdot \prod_{1 \leq i < j \leq n} [u_i, u_j]^{l_{ij}}$ with $u_i, u_j \in X$ and integers m_i, l_{ij}, involving only finitely many u_i, u_j in X. By Theorem 3.1 in [2], the generator $g \otimes g$ of $\nabla(G)$ can be written as a finite product of elements of the form $u_i \otimes u_j$ and $u_i \otimes [u_j, u_k]$. Denoting with $\exp(x \otimes y)$ the exponent of the term $x \otimes y$, inspection of the exponents in the expansion of $g \otimes g$ by Theorem 3.1 in [2] reveals that

$$\exp(u_i \otimes u_i) = m_i^2, \tag{2.4.4}$$

$$\exp(u_i \otimes u_j) = \exp(u_j \otimes u_i), \tag{2.4.5}$$

$$\exp(u_i \otimes [u_j, u_k]) = 0. \tag{2.4.6}$$

By (2.4.5) and since $G \otimes G$ is abelian, the term $u_i \otimes u_j$ and $u_j \otimes u_i$ can be combined and we arrive at $\nabla(G) \subseteq \langle u \otimes u, (u \otimes v)(v \otimes u) \mid u,v \in X \rangle$. Thus (2.4.2) follows.

To prove the other inclusion in (2.4.3), we observe that $(g \otimes h)(h \otimes g) = (gh \otimes gh)(g \otimes g)^{-1}(h \otimes h)^{-1}$ by (2.3.1). Next, we apply the expansion formula of Theorem 3.1 in [2] to $gh \otimes gh, (g \otimes g)^{-1}$ and $(h \otimes h)^{-1}$, form their product and collect terms. Note that if $u \in X$ occurs in the presentation of g and h with $\exp_g(u) = m$ and $\exp_h(u) = n$, respectively, then $\exp_{gh}(u) = m + n$. Thus, by (2.4.4), we obtain $\exp_{(g \otimes h)(h \otimes g)}(u \otimes u) = (m+n)^2 - m^2 - n^2 = 2mn$ and arrive at $\triangle(G) \subseteq \langle (u \otimes u)^2, (u \otimes v)(v \otimes u) | u,v \in X \rangle$. Hence (2.4.3) follows. $\square$

In [7], the tensor centralizer of an element $g \in G$ was introduced as $C^{\otimes}(g) = \{a \in G \mid a \otimes g = 1_{\otimes}\}$. It was shown that $C^{\otimes}(g)$ is a subgroup of G contained in the centralizer of g and $\bigcap_{g \in G} C^{\otimes}(g) = Z^{\otimes}(G)$, the tensor center of G. To effectively compute the epicenter of a group, we introduce the epicentralizer of an element g in G as $C^*(g) = \{a \in G \mid a \wedge g = 1_{\wedge}\}$. The following proposition establishes a connection between the tensor centralizer, epicentralizer, and centralizer of an element.

Proposition 2.5. *Let G be a group with $g \in G$. Then $C^*(g)$ is a subgroup of G with $C^{\otimes}(g) \subseteq C^*(g) \subseteq C_G(g)$ and $\bigcap_{g\in G} C^*(g) = Z^*(G)$.*

Proof. The containments and the fact that the intersection of the epicentralizers equals the epicenter are obvious. To show $C^*(g)$ is a subgroup, observe that both $1, g \in C^*(g)$ so $C^*(g)$ is nonempty. It follows from Proposition 7 of [8] and (1.7.1), that $v^{-1}u \wedge g = {}^{v^{-1}}((u \wedge g) \cdot (v \wedge g)^{-1}) = 1_\wedge$, provided $u, v \in C^*(g)$. Hence $v^{-1}u \in C^*(g)$ and the proposition is established. □

We will make use of the following classification of 2-generator p-groups of nilpotency class 2, p an odd prime.

Theorem 2.6 ([3]). *Let p be an odd prime and G a 2-generator p-group of nilpotency class two. Then G is isomorphic to exactly one group of the following three types:*

$$G \cong (\langle c\rangle \times \langle a\rangle) \rtimes \langle b\rangle,\ \text{where } [a,b] = c, [a,c] = [b,c] = 1,$$
$$o(a) = p^\alpha, o(b) = p^\beta, o(c) = p^\gamma,$$
$$\alpha, \beta, \gamma \in \mathbb{N} \text{ with } \alpha \geq \beta \geq \gamma \geq 1. \tag{2.6.1}$$

$$G \cong \langle a\rangle \rtimes \langle b\rangle,\ \text{where } [a,b] = a^{p^{\alpha-\gamma}},\ o(a) = p^\alpha, o(b) = p^\beta,$$
$$\alpha, \beta, \gamma \in \mathbb{N} \text{ with } \alpha \geq 2\gamma,\ \beta \geq \gamma \geq 1. \tag{2.6.2}$$

$$G \cong (\langle c\rangle \times \langle a\rangle) \rtimes \langle b\rangle,\ \text{where } [a,b] = a^{p^{\alpha-\gamma}} \cdot c,$$
$$[c,b] = a^{-p^{2(\alpha-\gamma)}} c^{-p^{\alpha-\gamma}},$$
$$o(a) = p^\alpha, o(b) = p^\beta, o(c) = p^\sigma,$$
$$\alpha, \beta, \gamma, \sigma \in \mathbb{N} \text{ with } \gamma > \sigma \geq 1, \alpha + \sigma \geq 2\gamma, \beta \geq \gamma. \tag{2.6.3}$$

The nonabelian tensor squares of the above groups were computed in [3]. However for the groups of type (2.6.2) the extraneous condition of $\alpha \geq \beta$ was given resulting in an incorrect order for the generator $b \otimes a$ of the tensor square. In order to correct this error, the following lemma, a proof of which can be found in [1], is used.

Lemma 2.7. *Let p be an odd prime and α, β, γ positive integers such that $2\gamma \leq \alpha$, and denote with $[n]_p$ the exponent of p in the prime factorization of n. Then*
$$\left[\sum_{k=0}^{p^{\beta-1}} (p^\alpha - p^{\alpha-\gamma} + 1)^k\right]_p \begin{cases} = \beta & \text{if } \beta \leq \alpha, \\ \geq \alpha & \text{if } \beta > \alpha. \end{cases}$$

3. Homological functors

This section determines the various homological functors as outlined in the introduction for the groups described in Theorem 2.6. It begins with a general result to facilitate determining these functors, but which is also of interest in its own right.

THEOREM 3.1. *Let G be a torsion group with no elements of even order. Then:*

$$\triangle(G) = \nabla(G); \tag{3.1.1}$$

$$G\tilde{\otimes}G = G \wedge G; \tag{3.1.2}$$

$$\tilde{J}(G) = M(G). \tag{3.1.3}$$

Proof. Observe that $\triangle(G) \subseteq \nabla(G)$ by Proposition 2.3. To establish (3.1.1), it remains to be shown that $\nabla(G) \subseteq \triangle(G)$. Let $g \in G$ and $g \otimes g$ be a generator of $\nabla(G)$. By (2.2.3) we have $1_\otimes = (g \otimes g)^{o(g)}$. Moreover $(g \otimes g)^2 = (g \otimes g)(g \otimes g) \in \triangle(G)$. Since $o(g)$ is odd, we have $\gcd(o(g), 2) = 1$ and there exist integers j, k such that $1 = j \cdot o(g) + 2k$ and $g \otimes g = (g \otimes g)^{j \cdot o(g)}(g \otimes g)^{2k}$. So it follows that $g \otimes g \in \triangle(G)$ and $\nabla(G) \subseteq \triangle(G)$. The remaining assertions follow from the definitions of the functors and (3.1.1). □

The following curious result is of topological interest concerning the Isomorphism Theorem of Hurewicz [23] and is an immediate corollary of the preceding theorem.

COROLLARY 3.2. *Let G be a torsion group with no elements of even order. Then*

$$H_2(G) \cong \pi_4 S^2 K(G, 1).$$

Proof. By [10], $\tilde{J}(G) = \pi_4 S^2 K(G, 1)$, and since $M(G) = H_2(G)$, the claim follows by (3.1.3). □

We are now in a position to compute the homological functors for 2-generator p-groups of class 2 with p an odd prime.

PROPOSITION 3.3. *Let G be a group of type (2.6.1), that is, G is isomorphic to $(\langle c \rangle \times \langle a \rangle) \rtimes \langle b \rangle$, where $[a, b] = c, [a, c] = [b, c] = 1, o(a) = p^\alpha, o(b) = p^\beta, o(c) = p^\gamma, \alpha, \beta, \gamma$ are integers with $\alpha \geq \beta \geq \gamma \geq 1$, and p is an odd prime. Then:*

$$G \otimes G \cong \mathbb{Z}_{p^\alpha} \times \mathbb{Z}_{p^\beta}^3 \times \mathbb{Z}_{p^\gamma}^2; \tag{3.3.1}$$

$$G\tilde{\otimes}G = G \wedge G \cong \mathbb{Z}_{p^\beta} \times \mathbb{Z}_{p^\gamma}^2; \tag{3.3.2}$$

$$\nabla(G) = \langle a \otimes a \rangle \times \langle b \otimes b \rangle \times \langle (a \otimes b)(b \otimes a) \rangle; \tag{3.3.3}$$

$$J(G) = \nabla(G) \times \langle a \otimes [a, b] \rangle \times \langle b \otimes [a, b] \rangle \times \langle (b \otimes a)^{p^\gamma} \rangle; \tag{3.3.4}$$

$$\tilde{J}(G) = H_2(G) \cong \mathbb{Z}_{p^{\beta-\gamma}} \times \mathbb{Z}_{p^\gamma}^2. \tag{3.3.5}$$

Proof. We observe that (3.3.1) follows by Theorem 4.3 of [3]. Since $o(a \otimes b) = o(b \otimes a) = p^{\beta}$, a direct decomposition for the tensor square of G is

$$G \otimes G = \langle a \otimes a\rangle \times \langle b \otimes b\rangle \times \langle (a \otimes b)(b \otimes a)\rangle \\ \times \langle a \otimes [a,b]\rangle \times \langle b \otimes [a,b]\rangle \times \langle b \otimes a\rangle.$$

Using (2.4.1) and the above, we obtain (3.3.3). Since $o([a,b]) = p^{\gamma}$, it follows that $(b \otimes a)^{p^{\gamma}} \in J(G)$, so (3.3.4) holds. Using the above direct product decomposition for $G \otimes G$, (3.3.3) and Theorem 3.1, we arrive at (3.3.2). Similarly, $\tilde{J}(G) = J(G)/\nabla(G) \cong \mathbb{Z}_{p^{\beta-\gamma}} \times \mathbb{Z}^2_{p^{\gamma}}$ and hence we obtain (3.3.5) by another application of Theorem 3.1. □

Proposition 3.4. *Let G be a group of type* (2.6.2)*, that is, G is isomorphic to $\langle a\rangle \rtimes \langle b\rangle$, where $[a,b] = a^{p^{\alpha-\gamma}}, o(a) = p^{\alpha}, o(b) = p^{\beta}, o([a,b]) = p^{\gamma}, \alpha, \beta, \gamma$ are integers with $\alpha \geq 2\gamma, \beta \geq \gamma \geq 1$, and p is an odd prime. Then for $\delta = \min\{\alpha-\gamma, \beta\}$ and $\tau = \min\{\alpha, \beta\}$ we have:*

(3.4.1) $$G \otimes G \cong \mathbb{Z}_{p^{\alpha-\gamma}} \times \mathbb{Z}_{p^{\beta}} \times \mathbb{Z}_{p^{\tau}} \times \mathbb{Z}_{p^{\delta}};$$

(3.4.2) $$G \tilde{\otimes} G = G \wedge G \cong \mathbb{Z}_{p^{\tau}};$$

(3.4.3) $$\nabla(G) = \langle a \otimes a\rangle \times \langle b \otimes b\rangle \times \langle (a \otimes b)(b \otimes a)\rangle;$$

(3.4.4) $$J(G) = \nabla(G) \times \langle (a \otimes b)^{p^{\gamma}}\rangle;$$

(3.4.5) $$\tilde{J}(G) = H_2(G) \cong \mathbb{Z}_{p^{\tau-\gamma}}.$$

Proof. By Theorem 4.2 of [3], a direct decomposition of the tensor square of G is

$$G \otimes G = \langle a \otimes a\rangle \times \langle b \otimes b\rangle \times \langle (a \otimes b)(b \otimes a)\rangle \times \langle b \otimes a\rangle,$$

where $o(a \otimes a) = p^{\alpha-\gamma}, o(b \otimes b) = p^{\beta}, o((a \otimes b)(b \otimes a)) = p^{\delta}$ with $\delta = \min\{\alpha-\gamma, \beta\}$ and $\tau = \min\{\alpha, [n]_p\}$ for $n = \sum_{k=0}^{p^{\beta}-1}(p^{\alpha} - p^{\alpha-\gamma} + 1)^k$. By Lemma 2.7, $[n]_p \geq \alpha$ if $\beta > \alpha$ and $[n]_p = \beta$ if $\beta \leq \alpha$. It follows that $\tau = \min\{\alpha, \beta\}$, thus proving (3.4.1). The rest of the proof is similar to the one for Proposition 3.3 and will be omitted here. □

Proposition 3.5. *Let G be a group of type* (2.6.3)*, that is, G is isomorphic to $(\langle c\rangle \times \langle a\rangle) \rtimes \langle b\rangle$, where $[a,b] = a^{p^{\alpha-\gamma}}c, [c,b] = a^{-p^{2(\alpha-\gamma)}}c^{-p^{\alpha-\gamma}}, o(a) = p^{\alpha}, o(b) = p^{\beta}, o(c) = p^{\sigma}, \alpha, \beta, \gamma, \sigma \in \mathbb{N}$ with $\gamma > \sigma \geq 1, \alpha + \sigma \geq 2\gamma, \beta \geq \gamma$. Then:*

(3.5.1) $$G \otimes G \cong \mathbb{Z}^2_{p^{\alpha-\gamma+\sigma}} \times \mathbb{Z}_{p^{\beta}} \times \mathbb{Z}_{p^{\min\{\alpha,\beta\}}} \times \mathbb{Z}^2_{p^{\sigma}};$$

(3.5.2) $$G \tilde{\otimes} G = G \wedge G = \mathbb{Z}_{p^{\min\{\alpha,\beta\}}} \times \mathbb{Z}^2_{p^{\sigma}};$$

(3.5.3) $$\nabla(G) = \langle a \otimes a\rangle \times \langle b \otimes b\rangle \times \langle (a \otimes b)(b \otimes a)\rangle;$$

(3.5.4) $$J(G) = \nabla(G) \times \langle a \otimes [a,b]\rangle \times \langle b \otimes [a,b]\rangle \times \langle (a \otimes b)^{p^{\gamma}}\rangle;$$

$$\tilde{J}(G) = H_2(G) \cong \mathbb{Z}_{p^{\min\{\alpha,\beta\}-\gamma}} \times \mathbb{Z}_{p^\sigma}^2. \tag{3.5.5}$$

Proof. As mentioned in [20], the group given in Theorem 4.4 of [3] as the tensor square corresponding to a group of type (2.6.3) is only a homomorphic image of the actual tensor square which in [20] is listed as the group given in (3.5.1). From the presentation given in [20], we derive the following direct decomposition:

$$\begin{aligned} G \otimes G = \langle a \otimes a\rangle \times \langle b \otimes b\rangle \times \langle (a \otimes b)(b \otimes a)\rangle \\ \times \langle a \otimes [a,b]\rangle \times \langle b \otimes [a,b]\rangle \times \langle a \otimes b\rangle, \end{aligned}$$

where $o(a \otimes a) = o((a \otimes b)(b \otimes a)\rangle = p^{\alpha-\gamma+\sigma}$, $o(a \otimes [a,b]) = o(b \otimes [a,b]) = p^\sigma, o(b \otimes b) = p^\beta$, and $o(a \otimes b) = p^{\min\{\alpha,\beta\}}$. Again, the rest of the proof is similar to the one for Proposition 3.3 and will be omitted here. □

4. Epicenters and capability

This section details the computation of the epicenters for 2-generator p-groups of class 2 with p an odd prime and determines which are the capable among them. It will be shown that Baer's necessary and sufficient condition for finitely generated abelian groups to be capable, namely having two factors of maximal order in a minimal generating set of equal order, is only necessary for p-groups of class 2, but not sufficient.

Our first lemma will help identify the elements in the epicentralizer of a group element.

LEMMA 4.1. *Let there be given an odd prime p and G a p-group of nilpotency class 2. If $u, v \in G$ with $o(u) = p^j, o(v) = p^k$, and $j \geq k$, then $u^{p^k} \in C^{\otimes}(v)$ and $u^{p^k} \in C^*(u^n v)$ for any integer n.*

Proof. By (2.2.3), $1_\otimes = (u \otimes v)^{p^k}$, and by (2.1.3) and (2.2.2) we have

$$u^{p^k} \otimes v = (u \otimes v)^{p^k} (u \otimes [u,v]^{\binom{p^k}{2}}) = (u \otimes v)^{p^k},$$

so it follows that $u^{p^k} \in C^{\otimes}(v)$. The second claim is a consequence of the first, since

$$u^{p^k} \otimes u^n v = (u^{p^k} \otimes u^n) \cdot {}^{u^n}(u^{p^k} \otimes v) = (u \otimes u)^{np^k},$$

and hence $u^{p^k} \in C^*(u^n v)$. □

THEOREM 4.2. *Let $G = \langle a, b\rangle$ be a 2-generator p-group of nilpotency class 2, p an odd prime. Then*

$$Z^*(G) = \langle a^{p^\beta}, b^{p^\alpha}\rangle, \text{ if } G \text{ is of type } (2.6.1) \text{ or } (2.6.2), \text{ and}$$

$$Z^*(G) = \langle a^{p^\beta}, b^{p^\alpha}, [a,b]^{p^\sigma}\rangle, \text{ if } G \text{ is of type } (2.6.3)\ .$$

Proof. Let G be a group of type (2.6.1), (2.6.2), or (2.6.3). Consider a^{p^β} and $b^{p^\alpha} \in G$, where one or both of these elements are equal to the identity. If one is not, consider $k = \min\{\alpha, \beta\}$, call $u = a$ and $v = b$ if $k = \beta$, and $u = b$ and $v = a$ if $k = \alpha$. We observe that $u^{p^k} \neq 1$ for this choice of u. If $g \in G$, then $g = u^m h$ with $h = v^n[u,v]^l$ for integers m, n and l. By Lemma 2.2, it follows $o([u,v]) \leq o(v) \leq o(u)$ so $o(h) \leq o(u)$. Hence Lemma 4.1 implies $u^{p^k} \in C^*(g)$. Thus, for all three types of groups, Proposition 2.5 yields $\langle a^{p^\beta}, b^{p^\alpha} \rangle \subseteq Z^*(G)$. Now let G be of type (2.6.3). Then, by Proposition 3.5, we have $o(a \otimes [a,b]) = o(b \otimes [a,b]) = p^\sigma$. Hence $a \otimes [a,b]^{p^\sigma} = (a \otimes [a,b])^{p^\sigma} = 1_\otimes$, and similarly $b \otimes [a,b]^{p^\sigma} = 1_\otimes$. We conclude $[a,b]^{p^\sigma} \in Z^\otimes(G)$ and so $[a,b]^{p^\sigma} \in Z^*(G)$ since $Z^\otimes(G) \subseteq Z^*(G)$. Hence $\langle a^{p^\beta}, b^{p^\alpha}, [a,b]^{p^\sigma} \rangle \subseteq Z^*(G)$ for a group of type (2.6.3).

To prove the other inclusion, we first observe that $Z(G) = \langle a^{p^\gamma}, b^{p^\gamma}, [a,b] \rangle$ for any of the groups given in Theorem 2.6. Let $g \in Z^*(G)$ with $g = a^m b^n [a,b]^l$ for integers m, n and l. Then p^γ divides m and n, since $g \in Z(G)$. For $x = a$ or b we can expand linearly and obtain

$$g \otimes x = (a^m \otimes x)(b^n \otimes x)([a,b]^l \otimes x).$$

Using (2.3.1) and the above remark, we arrive at

$$\begin{aligned} g \otimes b &= (a \otimes b)^m (b \otimes b)^n ([a,b] \otimes b)^l \in \nabla(G), \\ g \otimes a &= (a \otimes a)^m (b \otimes a)^n ([a,b] \otimes a)^l \in \nabla(G). \end{aligned}$$

By (2.3.1), we have $a \otimes b \equiv (b \otimes a)^{-1} \bmod \nabla(G)$. Since $\nabla(G)$ and $\langle a \otimes b \rangle$ are distinct direct summands of $G \otimes G$ by Propositions 3.3, 3.4 and 3.5, it follows that $\nabla(G) \cap \langle a \otimes b \rangle = \nabla(G) \cap \langle b \otimes a \rangle = \{1_\otimes\}$. We conclude $m \equiv n \equiv 0 \bmod \min\ \{p^\alpha, p^\beta\}$.

If G is of type (2.6.2), we can assume $l = 0$. It follows that $g \in \langle a^{p^\beta}, b^{p^\alpha} \rangle$. Now assume G is of type (2.6.1) or (2.6.3). Then, similarly as above, we obtain $(\nabla(G) \times \langle a \otimes b \rangle) \cap \langle [a,b] \otimes a \rangle = (\nabla(G) \times \langle a \otimes b \rangle) \cap \langle [a,b] \otimes b \rangle = \{1_\otimes\}$. From Proposition 3.3 it follows that $l \equiv 0 \bmod p^\gamma$ if G is of type (2.6.1) and we conclude $Z^*(G) \subseteq \langle a^{p^\beta}, b^{p^\alpha} \rangle$. Similarly, if G is of type (2.6.3), Proposition 3.5 yields $l \equiv 0 \mod p^\sigma$. It follows that $Z^*(G) \subseteq \langle a^{p^\beta}, b^{p^\alpha}, [a,b]^{p^\sigma} \rangle$. This concludes the proof of Theorem 4.2. $\square$

We mention here that Theorem 4.2 shows that $Z^\otimes(G) = \langle [a,b]^{p^\sigma} \rangle$ for any group of type (2.6.3). Since $1 \leq \sigma < \gamma$ and $Z^\otimes(G) \subseteq Z^*(G)$, such a group is never capable. Groups of type (2.6.1) or (2.6.2) have trivial tensor center which does not allow us to draw any conclusions about the capability of such groups. However, for a group of type (2.6.1) or (2.6.2) we have $Z^*(G) = \langle a^{p^\beta}, b^{p^\alpha} \rangle$ which is trivial if and only if $\alpha = \beta$. The following corollary is now immediate.

Corollary 4.3. *Let G be a 2-generator p-group of nilpotency class 2, p an odd prime. Then G is capable if and only if G is of type* (2.6.1) *or* (2.6.2) *with $\alpha = \beta$.*

In our last theorem we will show that Baer's criterion, namely that the two generators of highest order have equal order, which is necessary and sufficient for finitely generated abelian groups, is only necessary but not sufficient for finite p-groups of odd order and nilpotency class 2.

Theorem 4.4. *Let p be an odd prime and $G = \langle a_1, \ldots, a_k \rangle$ a finite p-group, nilpotent of class 2, with $o(a_i) = p^{\alpha_i}, \alpha_i \geq \alpha_{i+1}, i = 1, \ldots, k-1$, and $\{a_1, \ldots, a_k\}$ a minimal generating set of G. If G is capable, then $\alpha_1 = \alpha_2$. However, for any odd prime p, there exists a group with $\alpha_1 = \alpha_2$ that is not capable.*

Proof. We first observe that the groups in the above theorem are regular. Hence there exists a generating system as specified in the assumptions. Moreover, the $\alpha_i,\ i = 1, \ldots, k$, are unique.

Suppose now that G is capable. For any element $g \in G$ there exist non-negative integers $m_1, \ldots, m_k$ and $c \in G'$ such that $g = a_1^{m_1} a_2^{m_2} \ldots a_k^{m_k} c$. Setting $h = a_2^{m_2} \ldots a_k^{m_k} c$, we have $g = a_1^{m_1} h$ with $o(h) \leq p^{\alpha_2}$. By Lemma 4.1, $a_1^{p^{\alpha_2}}$ epicentralizes $a_1^{m_1} h = g$ and, since g was arbitrary, $a_1^{p^{\alpha_2}} \in Z^*(G)$. Since G is capable, the epicenter is trivial, thus $a_1^{p^{\alpha_2}} = 1$, and $\alpha_1 = \alpha_2$, proving the first assertion.

To show that the condition is not sufficient, consider a group G of type (2.6.3) with $\alpha = \beta$. By Theorem 4.2 we have $Z^*(G) = \langle [a,b]^{p^\sigma} \rangle$. Since $1 \leq \sigma < \gamma$, the epicenter of G is nontrivial. Hence G is not capable by Theorem 1.4. $\square$

The question arises whether a characterization of the capable p-groups of nilpotency class 2 can be given, at least in the case of an odd prime p. As things stand now, that does not appear to be an attainable goal in view of the characterization given here for the 2-generator groups in this class. To apply our methods, very explicit knowledge of the groups was needed. However for another subclass of these groups, namely the ones of exponent p, some sufficient conditions for capability are known, and some necessary ones.

Ellis [13] shows that a p-group of class 2 and exponent p is capable, provided that for a subset $\{x_1, \ldots, x_k\}$ of G corresponding to a basis of $G/Z(G)$ the nontrivial commutators of the form $[x_i, x_j]$ with $1 \leq i < j \leq k$ are distinct and constitute a basis for the vector space G'. On the other hand, Heineken [18] shows that $p^2 < |G/Z(G)| < p^6$ for a capable p-group of class 2 with an elementary abelian commutator subgroup of rank 2. In [19] Heineken and Nikolova generalize this result in the case the group G has exponent p and $G' = Z(G)$. They show if such a group is capable with $Z(G)$ of rank k, then the rank of G/G' is at most $2k+\binom{k}{2}$. In conclusion, it seems to be a reasonable question to ask whether one can narrow the gap between the necessary and the sufficient conditions, leading to a characterization of capable p-groups of exponent p and nilpotency class 2 for odd primes p.

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