Illinois Journal of Mathematics
Volume 51, Number 4, Winter 2007, Pages 1439–1454
S 0019-2082

ACYCLICITY OVER LOCAL RINGS WITH RADICAL CUBE ZERO

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Abstract. This paper studies infinite acyclic complexes of finitely generated free modules over a commutative noetherian local ring $(R,\mathfrak{m})$ with $\mathfrak{m}^3 = 0$. Conclusive results are obtained on the growth of the ranks of the modules in acyclic complexes, and new sufficient conditions are given for total acyclicity. Results are also obtained on the structure of rings that are not Gorenstein and admit acyclic complexes; part of this structure is exhibited by every ring R that admits a non-free finitely generated module M with $\operatorname{Ext}_R^n(M,R) = 0$ for a few $n > 0$.

Introduction

In this paper, R is a commutative noetherian local ring with maximal ideal $\mathfrak{m}$. Throughout, *module* means finitely generated module.

A chain complex of R-modules

$$\boldsymbol{A} \;=\; \cdots \longrightarrow A_{i+1} \xrightarrow{\partial_{i+1}} A_i \xrightarrow{\partial_i} A_{i-1} \xrightarrow{\partial_{i-1}} A_{i-2} \longrightarrow \cdots$$

is *acyclic* if $\mathrm{H}(\boldsymbol{A}) = 0$. The focus of this paper is on complexes of free modules, so we adopt the convention that an *acyclic complex* consists of free modules. Such a complex $\boldsymbol{A}$ is said to be *totally acyclic* if also the dual complex $\boldsymbol{A}^* = \operatorname{Hom}_R(\boldsymbol{A}, R)$ is acyclic.

Over a Gorenstein ring, every acyclic complex is totally acyclic [6, (4.1.3)]. Moreover, a module over such a ring is maximal Cohen–Macaulay if and only if it is the cokernel of some differential in an acyclic complex [6, Thm. (1.4.8) and (1.4.9)]. Thus, acyclic complexes abound over Gorenstein rings.

Over a ring that is not Gorenstein, a non-trivial acyclic complex need not even exist. Indeed, this is the case for rings that are Golod and not Gorenstein [5, Exa. 3.5(2)]. Yet, examples of acyclic complexes over non-Gorenstein rings

Received May 21, 2006; received in final form November 23, 2006.
2000 *Mathematics Subject Classification.* 13D02, 13D25.
Key words and phrases. Totally acyclic complexes, complete resolutions, infinite syzygy, infinite syzygies, totally reflexive modules, Betti numbers, minimal free resolutions.
L.W.C. was partly supported by grants from the Danish Natural Science Research Council and the Carlsberg Foundation.

do exist, and the ones given in [4], [6], [17], [18] are, in fact, examples of totally acyclic complexes. It has proved harder to come by acyclic complexes that are not totally acyclic. However, in [12] Jorgensen and Şega construct an acyclic, but not totally acyclic, complex over a local ring with $\mathfrak{m}^3 = 0$.

This paper started from the observation that the ring considered in [12] has a specific structure, described by Yoshino [18] in a related context. To explain this we introduce some notation:

Let k denote the residue field $R/\mathfrak{m}$. Two principal invariants of R are the embedding dimension and the socle dimension:

$$e = \operatorname{rank}_k \mathfrak{m}/\mathfrak{m}^2 \qquad \text{and} \qquad r = \operatorname{rank}_k(0 : \mathfrak{m}).$$

The ith Bass number of R is $\mu_R^i = \operatorname{rank}_k \operatorname{Ext}_R^i(k, R)$; note that $\mu_R^0 = r$, as $(0 : \mathfrak{m}) \cong \operatorname{Hom}_R(k, R)$. For an R-module M, the ith Betti number is $\beta_i^R(M) = \operatorname{rank}_k \operatorname{Ext}_R^i(M, k)$. The formal power series

$$\mathrm{I}_R(t) = \sum_{i=0}^{\infty} \mu_R^i t^i \qquad \text{and} \qquad \mathrm{P}_k^R(t) = \sum_{i=0}^{\infty} \beta_i^R(k) t^i$$

are, respectively, the Bass series of R and the Poincaré series of k.

A complex of free R-modules $\boldsymbol{A}$ is *minimal* if $\partial(\boldsymbol{A}) \subseteq \mathfrak{m}\boldsymbol{A}$. In particular, if $\boldsymbol{A}$ is minimal and acyclic, then either $A_i \neq 0$ for all $i \in \mathbb{Z}$ or $\boldsymbol{A}$ is the zero complex. Every acyclic complex contains a minimal one as a direct summand with contractible complement.

In [18] Yoshino proves that when a non-Gorenstein local ring R with $\mathfrak{m}^3 = 0$ admits a non-zero minimal totally acyclic complex, either of the two numbers e and r completely determines the homological invariants $\mathrm{I}_R(t)$ and $\mathrm{P}_k^R(t)$. The same holds for the ring considered by Jorgensen and Şega in [12].

Let $(R, \mathfrak{m})$ be a local ring with $\mathfrak{m}^3 = 0 \neq \mathfrak{m}^2$. Suppose R is not Gorenstein and admits a non-zero minimal acyclic complex $\boldsymbol{A}$. This paper considers the following questions:

A. Does the existence of $\boldsymbol{A}$ impose conditions on the structure of R?
B. What is the asymptotic behavior of the sequences

$$\{\operatorname{rank}_R A_i\}_{i \geqslant 0} \quad \text{and} \quad \{\operatorname{rank}_R A_i\}_{i \leqslant 0}?$$

C. When is $\boldsymbol{A}$ totally acyclic?

Accordingly, the main results are collected in three theorems.

THEOREM A. *Let $(R, \mathfrak{m}, k)$ be a local ring that is not Gorenstein and has $\mathfrak{m}^3 = 0 \neq \mathfrak{m}^2$. If there exists a non-zero minimal acyclic complex $\boldsymbol{A}$ of finitely generated free R-modules, then the ring has the following properties:*

(a) $(0 : \mathfrak{m}) = \mathfrak{m}^2$.
(b) $e = r + 1$*; in particular,* length $R = 2e$.

(c) $\mathrm{P}_k^R(t) = \dfrac{1}{(1-t)(1-rt)}$.

(c′) *The graded ring* $\mathrm{gr}_{\mathfrak{m}}(R)$ *is Koszul.*[1]

If, in addition, $\mathrm{H}^n(\boldsymbol{A}^*) = 0$ *for some integer* n*, then*

(d) $\mathrm{I}_R(t) = \dfrac{r-t}{1-rt}$.

Yoshino proved in [18, Thm. 3.1] that R has this structure if it is standard graded and $\boldsymbol{A}$ is totally acyclic; see also Observation 3.3.

For the acyclic complex constructed in [12], the sequence $\{\mathrm{rank}_R A_i\}_{i\geqslant 0}$ is strictly increasing and $\{\mathrm{rank}_R A_i\}_{i\leqslant 0}$ is constant. A natural question, posed in [12], is whether the opposite behavior, namely $\{\mathrm{rank}_R A_i\}_{i\geqslant 0}$ constant and $\{\mathrm{rank}_R A_i\}_{i\leqslant 0}$ strictly increasing, is possible. For rings with $\mathfrak{m}^3 = 0$ the answer is negative:

Theorem B. *Let* $(R, \mathfrak{m}, k)$ *be a local ring that is not Gorenstein and has* $\mathfrak{m}^3 = 0 \neq \mathfrak{m}^2$*. If* $\boldsymbol{A}$ *is a non-zero minimal acyclic complex of finitely generated free* R*-modules, then one of the following holds:*

(I) *The residue field* k *is not a direct summand of* $\mathrm{Coker}\,\partial_i$ *for any* $i \in \mathbb{Z}$*, and there is a positive integer* a *such that*

$$a = \mathrm{rank}_R A_i \quad \textit{for all } i \in \mathbb{Z}.$$

Moreover, $\mathrm{length}_R \mathrm{Coker}\,\partial_i = ae$ *for all* $i \in \mathbb{Z}$*.*

(II) *There is an integer* $\varkappa$*, such that* k *is a direct summand of* $\mathrm{Coker}\,\partial_{\varkappa+2}$ *and not of* $\mathrm{Coker}\,\partial_{i+2}$ *for any* $i < \varkappa$*, and a positive integer* a *such that*

$$a = \mathrm{rank}_R A_i \quad \textit{for all integers} \quad i \leqslant \varkappa \quad \textit{and}$$
$$\mathrm{rank}_R A_{i+1} > \mathrm{rank}_R A_i \quad \textit{for all integers} \quad i \geqslant \varkappa.$$

Moreover, $\mathrm{length}_R \mathrm{Coker}\,\partial_{i+2} = ae$ *for all integers* $i \leqslant \varkappa$*.*

In case (II) the sequence $\{\mathrm{rank}_R A_i\}_{i\geqslant\varkappa}$ has exponential growth by work of Lescot [13, Thm. B]. More precise statements about the growth of this sequence are obtained by Gasharov and Peeva in [7, Cor. 2.3(ii)] and [15, Prop. 3].

The totally acyclic complex constructed in [7, Prop. 3.4] is of type (I), and the acyclic complex from [12, Lem. 1.4] is of type (II) with $\varkappa = 0$.

If R is Gorenstein and $\mathfrak{m}^3 = 0 \neq \mathfrak{m}^2$, then any acyclic complex $\boldsymbol{A}$ is totally acyclic, and the sequences $\{\mathrm{rank}_R A_i\}_{i\geqslant 0}$ and $\{\mathrm{rank}_R A_i\}_{i\leqslant 0}$ have the same

[1] R is itself standard graded, i.e., $R \cong \mathrm{gr}_{\mathfrak{m}}(R)$, if and only if k is a retract of R as a ring (see [14, Prop. 1.1]), and by Cohen's Structure Theorem this happens if and only if R is equicharacteristic.

growth, either exponential or polynomial of the same degree. This follows from work of Sjödin [16], Lescot [13], and Avramov and Buchweitz [2]; see 3.1 for a summary. If R is not Gorenstein, the first implication in Theorem C contains the result from [18, Thm. 3.1] that all modules in a totally acyclic complex have the same rank.

Theorem C. *Let $(R, \mathfrak{m})$ be a local ring that is not Gorenstein and has $\mathfrak{m}^3 = 0 \neq \mathfrak{m}^2$. Suppose $\boldsymbol{A}$ is a non-zero minimal acyclic complex of finitely generated free R-modules. Set*

$$\mathcal{H} = \{i \in \mathbb{Z} \mid \mathrm{H}^i(\boldsymbol{A}^*) = 0\}$$

and consider the conditions:

(i) *The set $\mathcal{H}$ contains infinitely many positive integers.*
(ii) *All the free modules A_i have the same rank.*
(iii) *If $l-1$ and $l+1$ are in $\mathcal{H}$, then so is l.*

The following implications hold:

$$\text{(i)} \implies \text{(ii)} \implies \text{(iii)}.$$

In particular, if two out of every three consecutive integers belong to $\mathcal{H}$, then $\mathcal{H} = \mathbb{Z}$, i.e., $\boldsymbol{A}$ is totally acyclic.

This theorem compares to [12, Prop. 2.1], which holds for standard graded artinian rings: An acyclic complex $\boldsymbol{A}$ is totally acyclic if $\mathbb{Z} \setminus \mathcal{H}$ is a finite set of integers of the same parity.

1. Modules over local rings with $\mathfrak{m}^3 = 0$

In the rest of this paper, the local ring $(R, \mathfrak{m}, k)$ is assumed to have $\mathfrak{m}^3 = 0 \neq \mathfrak{m}^2$. Resolutions of modules over such rings were first studied by Sjödin [16] and Lescot [13]; we open this section with a collection of results from [13].

In the sequel, the socle $(0 : \mathfrak{m})$ is denoted $\operatorname{Soc} R$. It is clear that $\mathfrak{m}^2 \subseteq \operatorname{Soc} R$; most of the results from [13] require $\operatorname{Soc} R = \mathfrak{m}^2$, which is equivalent to assuming k is not a direct summand of $\mathfrak{m}$; cf. [13, Lem. 3.2]. This condition is fulfilled automatically for the rings we are interested in; see Theorem A. In fact, it is not too restrictive either: It is not hard to check that if k is a retract of R as a ring, then $\operatorname{Soc} R = \mathfrak{m}^2$ or R is a trivial extension of a ring with that property. That is, $R = Q \ltimes V$, were $(Q, \mathfrak{n})$ is a local ring with $\mathfrak{n}^3 = 0$ and $\operatorname{Soc} Q = \mathfrak{n}^2$, and V is a k-vector space.

1.1. From [13, Prop. 3.9(2), Thm. B, and Lem. 3.5] one has:

(a) If $\operatorname{Soc} R \neq \mathfrak{m}^2$, then for every non-free R-module M the sequence $\{\beta_i^R(M)\}_{i \geqslant 1}$ is strictly increasing.

(b) If $\operatorname{Soc} R = \mathfrak{m}^2$, then for every R-module M the sequence $\{\beta_i^R(M)\}_{i\geqslant 1}$ is eventually constant or has exponential growth. In the latter case there is an integer j such that the sequence $\{\beta_i^R(M)\}_{i\geqslant j}$ is strictly increasing.

(c) If $\operatorname{Soc} R = \mathfrak{m}^2$ and M is an R-module with $\mathfrak{m}^2 M = 0$, then

$$\beta_i^R(M) \geqslant e\beta_{i-1}^R(M) - r\beta_{i-2}^R(M) \quad \text{for all } i \geqslant 2.$$

The next lemma complements 1.1(b) and contains a special case of a result by Gasharov and Peeva [7, Cor. 2.3].

1.2. LEMMA. *Assume* $\operatorname{Soc} R = \mathfrak{m}^2$ *and* $e \geqslant 1 + r$. *For a non-zero* R*-module* M *with* $\mathfrak{m}^2 M = 0$ *there exist integers* $m \geqslant n \geqslant 0$, *where possibly* $m = \infty$, *such that*

$$\cdots > \beta_{m+1}^R(M) > \beta_m^R(M) = \cdots = \beta_n^R(M) < \beta_{n-1}^R(M) < \cdots < \beta_0^R(M).$$

Moreover, if $e > 1 + r$, *then* $m = n$ *or* $m = n + 1$.

Proof. There exists a least $n \geqslant 0$ such that $\beta_{n+1}^R(M) \geqslant \beta_n^R(M)$. The first inequality below is by 1.1(c),

$$\begin{aligned} \beta_{n+2}^R(M) &\geqslant e\beta_{n+1}^R(M) - r\beta_n^R(M) \\ &\geqslant e\beta_{n+1}^R(M) - r\beta_{n+1}^R(M) \\ &= (e - r)\beta_{n+1}^R(M) \\ &\geqslant \beta_{n+1}^R(M). \end{aligned} \tag{1}$$

By iteration, one has $\beta_{i+1}^R(M) \geqslant \beta_i^R(M)$ for all $i \geqslant n$, and it is immediate that $\beta_{j+1}^R(M) > \beta_j^R(M)$ implies $\beta_{i+1}^R(M) > \beta_i^R(M)$ for all $i \geqslant j$. With

$$m = \inf\{i \in \mathbb{Z} \mid \beta_{i+1}^R(M) > \beta_i^R(M)\} \geqslant n$$

one has $m = \infty$ or $\beta_{i+1}^R(M) > \beta_i^R(M)$ for all $i \geqslant m$.

Finally, if $e > r + 1$ then (1) yields

$$\beta_{n+2}^R(M) > \beta_{n+1}^R(M),$$

which forces $n + 1 \geqslant m \geqslant n$. □

For an R-module M, let M_i denote the ith syzygy of M.

1.3. Assume $\operatorname{Soc} R = \mathfrak{m}^2$. Let M be a non-zero R-module with $\mathfrak{m}^2 M = 0$, and let h be a positive integer. Following [13, Def. 3.1], M is said to be h-*exceptional* if k is not a direct summand of the syzygies M_i for $1 \leqslant i \leqslant h$. If M is h-exceptional for every $h \geqslant 1$, then M is said to be *exceptional*.

Let $h \geqslant 1$. By the proof of [13, Lem. 3.3] an R-module M is h-exceptional if and only if the Betti numbers satisfy:

$$\begin{aligned} \beta_1^R(M) &= e\beta_0^R(M) - \operatorname{rank}_k \mathfrak{m}M \quad \text{and} \\ \beta_i^R(M) &= e\beta_{i-1}^R(M) - r\beta_{i-2}^R(M) \quad \text{for all} \quad 2 \leqslant i \leqslant h. \end{aligned} \tag{1.3.1}$$

1.4. Assume $\operatorname{Soc} R = \mathfrak{m}^2$ and let h be a positive integer. If R admits an h-exceptional module, then k is h-exceptional. This is [13, Lem. 3.6].

The equalities (1.3.1) can be rewritten as an equality of polynomials [13, Lem. 3.3]: k is h-exceptional if and only if

$$\left[\mathrm{P}_k^R(t)\right]_{\leqslant h} = \left[\frac{1}{1-et+rt^2}\right]_{\leqslant h}, \tag{1.4.1}$$

where $[-]_{\leqslant h}$ denotes the terms of degree at most h. In particular, k is exceptional if and only if

$$\mathrm{P}_k^R(t) = \frac{1}{1-et+rt^2}. \tag{1.4.2}$$

1.5. Lemma. *Assume R is not Gorenstein. If there exists a syzygy module $N \neq 0$ with $\operatorname{Ext}_R^h(N,R) = 0$ for some $h \geqslant 2$, then $\operatorname{Soc} R = \mathfrak{m}^2$.*

Moreover, the following hold for an R-module $M \neq 0$ with $\mathfrak{m}^2 M = 0$:

(a) *If $\operatorname{Ext}_R^2(M,R) = 0$ and M is a syzygy, then M is 1-exceptional.*
(b) *If $\operatorname{Ext}_R^{h+1}(M,R) = 0$ for some $h \geqslant 2$, then M is h-exceptional.*
(c) *If $\operatorname{Ext}_R^h(M,R) = 0$ for infinitely many $h \geqslant 1$, then M is exceptional.*

Proof. If $\operatorname{Soc} R \neq \mathfrak{m}^2$, then k is a direct summand of N_1 because it is a second syzygy; see [13, Lem. 3.2 and proof of Lem. 3.3]. Therefore,

$$0 = \operatorname{Ext}_R^h(N,R) = \operatorname{Ext}_R^{h-1}(N_1,R) = \operatorname{Ext}_R^{h-1}(k,R) \oplus \operatorname{Ext}_R^{h-1}(N_1',R)$$

for some module N_1'. In particular, $\operatorname{Ext}_R^{h-1}(k,R) = 0$ and that contradicts the assumption that R is not Gorenstein.

Note that by this argument, the hypotheses in parts (a)–(c) ensure that $\operatorname{Soc} R = \mathfrak{m}^2$, so it makes sense to speak about exceptionality.

(a): Applied to $N = M$ and $h = 2$ the argument above shows that M is 1-exceptional.

(b): Suppose M is not h-exceptional. Then $M_i = k \oplus M_i'$ for some $1 \leqslant i \leqslant h$ and some R-module M_i'. Now there are equalities

$$\begin{aligned} \operatorname{Ext}_R^{h+1-i}(k,R) \oplus \operatorname{Ext}_R^{h+1-i}(M_i',R) &= \operatorname{Ext}_R^{h+1-i}(M_i,R) \\ &= \operatorname{Ext}_R^{h+1}(M,R) \\ &= 0. \end{aligned}$$

Again, this contradicts the assumption on R. Whence, M is h-exceptional.

(c): In view of (b), it is sufficient to remark that if M is h-exceptional for some $h \geqslant 1$, then M is i-exceptional for all $1 \leqslant i \leqslant h$. $\square$

The next result is extracted from the proof of [13, Lem. 3.3].

1.6. Assume $\operatorname{Soc} R = \mathfrak{m}^2$. If M is a 1-exceptional module, then

$$\operatorname{rank}_k \mathfrak{m} M_1 = r\beta_0^R(M).$$

For an R-module M, set $M^* = \operatorname{Hom}_R(M,R)$ and let $\ell(M)$ denote the length of M. The following equalities are proved already in [18], ostensibly under stronger hypotheses. For completeness, we include a proof.

1.7. Lemma. *For an R-module M with $\mathfrak{m}^2 M = 0$ one has:*

(a) $\ell(M) = \operatorname{rank}_k \mathfrak{m} M + \beta_0^R(M)$.

(b) *If* $\operatorname{Ext}_R^1(M,R) = 0$, *then* $\ell(M^*) = r\ell(M) - \beta_0^R(M)\mu_R^1$.

Proof. We can assume $M \neq 0$. Set

$$s = \operatorname{rank}_k \mathfrak{m} M \qquad \text{and} \qquad b = \beta_0^R(M).$$

(a): The exact sequence $0 \to \mathfrak{m} M \to M \to M/\mathfrak{m} M \to 0$ is isomorphic to

$$0 \to k^s \to M \to k^b \to 0. \tag{1}$$

In particular, we have $\ell(M) = s + b$.

(b): Dualizing (1), we obtain the exact sequence

$$0 \to \operatorname{Hom}_R(k,R)^b \to M^* \to \operatorname{Hom}_R(k,R)^s \to \operatorname{Ext}_R^1(k,R)^b \to 0,$$

which by part (a) and additivity of length yields

$$\ell(M^*) = br + sr - b\mu_R^1 = r\ell(M) - b\mu_R^1. \qquad \square$$

1.8. Lemma. *Assume R is not Gorenstein. If there exists a non-free R-module M such that $\operatorname{Ext}_R^{n+1}(M,R) = 0$ for some $n \geqslant 2$, then the Bass series of R satisfies:*

$$[\mathrm{I}_R(t)]_{\leqslant n} = \left[\frac{r - et + t^2}{1 - et + rt^2}\right]_{\leqslant n}.$$

Proof. It follows from Lemma 1.5 that $\operatorname{Soc} R = \mathfrak{m}^2$. Let $E = \mathrm{E}(k)$ be the injective envelope of k. Since the module M is not free and

$$\operatorname{Tor}_{n+1-i}^R(M, E_i) = \operatorname{Tor}_{n+1}^R(M,E) = \operatorname{Hom}_R(\operatorname{Ext}_R^{n+1}(M,R),E) = 0$$

for $0 < i \leqslant n$, the syzygy module E_1 is $n-1$ exceptional and does not contain k as a direct summand. Now [10, 2.8(3&4)] yield

$$\beta_0^R(E_1) = e(r-1) \quad \text{and} \quad \ell(E_1) = (r-1)(1+e+r), \tag{1}$$

and from Lemma 1.7(a) we get

$$\operatorname{rank}_k \mathfrak{m}E_1 = (r-1)(1+e+r) - e(r-1) = r^2 - 1. \tag{2}$$

The Betti numbers of the module E are the Bass numbers of R; that is $\mu_R^i = \beta_i^R(E) = \beta_{i-1}^R(E_1)$. Rewriting the equations (1.3.1) for the module E_1 as an equality of polynomials gives the first equality below. The second follows by (1) and (2).

$$\begin{aligned}
[\mathrm{I}_R(t)]_{\leqslant n} &= r + t\left[\frac{\beta_0^R(E_1) - (\operatorname{rank}_k \mathfrak{m}E_1)t}{1-et+rt^2}\right]_{\leqslant n-1} \\
&= r + t\left[\frac{e(r-1) - (r^2-1)t}{1-et+rt^2}\right]_{\leqslant n-1} \\
&= \left[\frac{r(1-et+rt^2) + e(r-1)t - (r^2-1)t^2}{1-et+rt^2}\right]_{\leqslant n} \\
&= \left[\frac{r-et+t^2}{1-et+rt^2}\right]_{\leqslant n}.
\end{aligned}$$

□

2. Proofs of Theorems A-C

In this section we prove the three main theorems, stated in the Introduction.

2.1. Let $\boldsymbol{A}$ be a minimal acyclic complex; throughout this section we use the following notation:

$$b_i = \operatorname{rank}_R A_i, \quad \mathrm{C}_i(\boldsymbol{A}) = \operatorname{Coker}\partial_{i+1} \cong \operatorname{Ker}\partial_{i-1}, \quad \text{and} \quad s_i = \operatorname{rank}_k \mathfrak{m}\mathrm{C}_i(\boldsymbol{A})$$

for $i \in \mathbb{Z}$. Note that $\beta_0^R(\mathrm{C}_i(\boldsymbol{A})) = b_i$.

2.2. Remark. Assume R is not a hypersurface ring. The Betti numbers of k are then strictly increasing; see [1, rem. 8.1.1(3)]. Let $\boldsymbol{A}$ be an acyclic R-complex; note that for any $j \in \mathbb{Z}$ the inequalities

$$\beta_{b_j}^R(k) > \beta_{b_j-1}^R(k) > \cdots > \beta_0^R(k) = 1$$

show that $\beta_{b_j}^R(k) > b_j$, so k cannot be a direct summand of $\mathrm{C}_i(\boldsymbol{A})$ for any $i \leqslant j - b_j$. In particular, k is not a direct summand of $\mathrm{C}_i(\boldsymbol{A})$ for any $i \leqslant -b_0$.

For a minimal acyclic complex $\boldsymbol{A}$ set

$$\varkappa = \inf\{i \mid k \text{ is a direct summand of } \mathrm{C}_{i+1}(\boldsymbol{A})\}$$

and note that

$$\infty \geqslant \varkappa \geqslant -b_0 > -\infty.$$

If $\operatorname{Soc} R = \mathfrak{m}^2$, then (1.3.1) yields

$$b_i = eb_{i-1} - rb_{i-2} \quad \text{for all} \quad i \leqslant \varkappa, \quad \text{and} \tag{2.2.1}$$

$$b_{\varkappa+1} > eb_{\varkappa} - rb_{\varkappa-1}. \tag{2.2.2}$$

2.3. *Proof of Theorem A.* We may, after a shift, assume k is not a direct summand of $\mathrm{C}_{-i}(\boldsymbol{A})$ for any $i \geqslant 0$; cf. Remark 2.2.

(a): Suppose $\operatorname{Soc} R \neq \mathfrak{m}^2$; by 1.1(a) one gets

$$b_0 > b_{-1} > b_{-2} > \cdots > b_{-b_0} > 0,$$

which is absurd. Therefore, $\operatorname{Soc} R = \mathfrak{m}^2$.

(b): Set $a_i = b_{-i}$. Then (2.2.1) translates to

$$a_i = e a_{i+1} - r a_{i+2} \quad \text{for all} \quad i \geqslant 0.$$

By Proposition A.1(b) it follows that $e = r + 1$.

(c): For $h \geqslant 1$ our assumption on the complex $\boldsymbol{A}$ implies that $\mathrm{C}_{-h}(\boldsymbol{A})$ is an h-exceptional R-module. By 1.4, k is then h-exceptional for all $h \geqslant 1$ and hence exceptional. The expression for the Poincaré series of k now follows from (1.4.2) and part (b).

(c′): It follows from (b) and (c) that the Hilbert series of $\operatorname{gr}_{\mathfrak{m}}(R)$ is

$$1 + et + rt^2 = (1+t)(1+rt) = \frac{1}{\mathrm{P}_k^R(-t)}.$$

As $\mathrm{P}_k^R(t) = \mathrm{P}_k^{\operatorname{gr}_{\mathfrak{m}}(R)}(t)$ by [14, Thm. 2.3], it follows that $\operatorname{gr}_{\mathfrak{m}}(R)$ is a Koszul algebra by [14, Thm. 1.2].

(d): After a shift, we may assume $n = -1$; then $\mathrm{H}^n(\boldsymbol{A}^*) = 0$ translates to

$$\operatorname{Ext}_R^{i+1}(\mathrm{C}_{-i}(\boldsymbol{A}), R) = 0 \quad \text{for all} \quad i \geqslant 0.$$

Lemma 1.8 now applies to the modules $\mathrm{C}_{-i}(\boldsymbol{A})$ for all $i > 0$ and yields

$$\mathrm{I}_R(t) = \frac{r - et + t^2}{1 - et + rt^2}.$$

Using part (b) we obtain, after simplification, the desired equality. □

2.4. *Proof of Theorem B.* Note that $\operatorname{Soc} R = \mathfrak{m}^2$ and $e = r + 1$ by Theorem A. It was already remarked in 2.2 that $\infty \geqslant \varkappa > -\infty$. Thus, either $\varkappa = \infty$ or we may, after a shift, assume $\varkappa = 0$. The first case corresponds to (I) and the second to (II). Set $a_i = b_{-i}$; in either case (2.2.1) translates to

$$a_i = e a_{i+1} - r a_{i+2} \quad \text{for all} \quad i \geqslant 0.$$

By Proposition A.1(a) there is a positive integer a, such that

$$b_i = a \quad \text{for all} \quad i \leqslant 0.$$

In case $\varkappa = 0$, the inequality (2.2.2) becomes

$$b_1 - b_0 > r(b_0 - b_{-1}) = 0.$$

Thus, $b_1 > b_0$ and Lemma 1.2 applied to $\mathrm{C}_0(\boldsymbol{A})$ yields the desired conclusion.

In case $\varkappa = \infty$, the equality (2.2.1) translates to

$$b_i - b_{i-1} = r(b_{i-1} - b_{i-2}) \quad \text{for all} \quad i \in \mathbb{Z}.$$

Since $b_0 = a = b_{-1}$, it follows by recursion that $b_i = a$ also for $i > 0$.

For $i \leqslant \varkappa$, the residue field k is not a direct summand of $\mathrm{C}_i(\boldsymbol{A})$. Therefore, one has $\operatorname{rank}_k \mathfrak{m}\mathrm{C}_i(\boldsymbol{A}) = ar$ by 1.6, and Lemma 1.7(a) yields the desired

$$\ell(\mathrm{C}_i(\boldsymbol{A})) = ar + a = ae. \qquad \square$$

2.5. *Proof of Theorem C.* First note that $\operatorname{Soc} R = \mathfrak{m}^2$ and $e = r + 1$ by Theorem A.

(i) $\Longrightarrow$ (ii): Let C be any cokernel $\mathrm{C}_i(\boldsymbol{A})$. By assumption, $\operatorname{Ext}_R^h(C, R) = 0$ for infinitely many $h > 0$, so C is exceptional by Lemma 1.5(c). Thus, k is not a direct summand of any cokernel $\mathrm{C}_i(\boldsymbol{A})$, and it follows by Theorem B that all the modules A_i have the same rank.

(ii) $\Longrightarrow$ (iii): After a shift we may assume $l = 1$, so 0 and 2 are in $\mathcal{H}$. Consider the dual complex

$$\boldsymbol{A}^* = \cdots \to A^*_{-2} \xrightarrow{\partial^*_{-1}} A^*_{-1} \xrightarrow{\partial^*_0} A^*_0 \xrightarrow{\partial^*_1} A^*_1 \xrightarrow{\partial^*_2} A^*_2 \to \cdots .$$

By definition, $\mathrm{H}^1(\boldsymbol{A}^*) = \operatorname{Ker} \partial^*_2 / \operatorname{Im} \partial^*_1$. We will show that $\operatorname{Ker} \partial^*_2$ and $\operatorname{Im} \partial^*_1$ have the same length, ae, where a is the common rank of the modules A_i.

First note that for all $i \in \mathbb{Z}$ we have $\operatorname{Ker} \partial^*_i = (\operatorname{Coker} \partial_i)^* = \mathrm{C}_{i-1}(\boldsymbol{A})^*$, by left-exactness of $\operatorname{Hom}_R(-, R)$. By assumption $\mathrm{H}^0(\boldsymbol{A}^*) = 0 = \mathrm{H}^2(\boldsymbol{A}^*)$; this means that $\operatorname{Ext}^1_R(\mathrm{C}_{-1}(\boldsymbol{A}), R) = 0$ and $\operatorname{Ext}^1_R(\mathrm{C}_1(\boldsymbol{A}), R) = 0$. For $i = \pm 1$ it follows by Lemma 1.7(b), Theorem B, and Theorem A(b,d) that

$$\begin{aligned}
\ell(\mathrm{C}_i(\boldsymbol{A})^*) &= r\ell(\mathrm{C}_i(\boldsymbol{A})) - \beta^R_0(\mathrm{C}_i(\boldsymbol{A}))\mu^1_R \\
&= ra(r+1) - a(r^2 - 1) \\
&= a(r+1) \\
&= ae.
\end{aligned}$$

Thus we have

$$\ell(\operatorname{Ker} \partial^*_2) = ae = \ell(\operatorname{Ker} \partial^*_0).$$

For all $i \in \mathbb{Z}$ we have $\ell(\operatorname{Ker} \partial^*_i) + \ell(\operatorname{Im} \partial^*_i) = a\ell(R)$; moreover, since $\mathrm{H}^0(\boldsymbol{A}^*) = 0$ we have $\ell(\operatorname{Ker} \partial^*_1) = \ell(\operatorname{Im} \partial^*_0)$. Combining these equations we find

$$\begin{aligned}
\ell(\operatorname{Im} \partial^*_1) &= a\ell(R) - \ell(\operatorname{Ker} \partial^*_1) \\
&= a\ell(R) - \ell(\operatorname{Im} \partial^*_0) \\
&= \ell(\operatorname{Ker} \partial^*_0) \\
&= ae. \qquad \square
\end{aligned}$$

3. Concluding remarks and questions

In this section we sum up the state of the three questions raised in the Introduction. The assumption that $(R, \mathfrak{m}, k)$ is local with $\mathfrak{m}^3 = 0 \neq \mathfrak{m}^2$ is still in force.

A. Structure of a non-Gorenstein ring admitting an acyclic complex. One answer to this question is given by Theorem A. It remains open whether the additional assumption, in Theorem A(d), that some cohomology module vanishes, is fulfilled automatically. See also Question 3.4 below.

It also remains open whether every non-Gorenstein ring R with the structure described in Theorem A admits a non-zero minimal acyclic complex. For a construction of totally acyclic complexes over certain rings, see [4, Thm. (3.1)].

If one allows for non-finitely generated modules, an acyclic R-complex can always be constructed by copying part of the argument for [11, Prop. 6.1(3)]: Let $\boldsymbol{P}$ be a projective resolution of the injective hull of k, then the mapping cone of the homothety morphism $R \to \operatorname{Hom}_R(\boldsymbol{P}, \boldsymbol{P})$ is an acyclic complex of flat R-modules, and flat modules are free, as R is artinian.

B. Asymptotic behavior of ranks. For a minimal acyclic R-complex $\boldsymbol{A}$, the possible asymptotic behaviors of $\{\operatorname{rank}_R A_i\}_{i \geqslant 0}$ and $\{\operatorname{rank}_R A_i\}_{i \leqslant 0}$ are now completely understood. For non-Gorenstein rings it is explained by Theorem B. For Gorenstein rings we collect the results in:

3.1. Summary. Let R be Gorenstein. Then $\operatorname{Soc} R = \mathfrak{m}^2$. For a minimal acyclic complex $\boldsymbol{A}$, the sequences $\{\operatorname{rank}_R A_i\}_{i \geqslant 0}$ and $\{\operatorname{rank}_R A_i\}_{i \leqslant 0}$ have the same growth, either exponential or polynomial of the same degree. We show this below by arguing on the embedding dimension of R, but first we make an observation: Because R is artinian and Gorenstein, every non-free R-module is a cokernel in a minimal acyclic complex, which is determined uniquely up to isomorphism. By the Krull–Schmidt theorem every R-module decomposes uniquely as a sum of indecomposable modules, and it follows that every minimal acyclic complex is isomorphic to a sum of acyclic complexes whose cokernels are indecomposable.

If $e = 1$, then all modules in a minimal acyclic R-complex have the same rank. Indeed, by Cohen's Structure Theorem, R is isomorphic to D/t^3D, where D is a discrete valuation domain with maximal ideal tD. Up to isomorphism there are, therefore, three indecomposable R-modules: R, $R/\mathfrak{m}$, and $R/\mathfrak{m}^2$. The two non-free ones have constant Betti numbers equal to 1.

If $e = 2$, then R is a complete intersection ring. Let C be a cokernel in $\boldsymbol{A}$. In view of the isomorphism $\operatorname{Ext}^i_R(C, C) \cong \operatorname{Ext}^i_R(C^*, C^*)$ it follows by [2, Cor. 5.7] that $\{\operatorname{rank}_R A_i\}_{i \geqslant 0}$ and $\{\operatorname{rank}_R A_i\}_{i \leqslant 0}$ have polynomial growth of the same degree. By Lemma 1.2 it follows that either all the modules in $\boldsymbol{A}$ have the same rank or, after a shift, one has

$$\cdots > \operatorname{rank}_R A_{m+1} > \operatorname{rank}_R A_m = \cdots = \operatorname{rank}_R A_0 < \operatorname{rank}_R A_{-1} < \cdots,$$

where $\infty > m \geqslant 0$.

The indecomposable modules over the ring $R = k[X,Y]/(X^2,Y^2)$ are classified in [8]. For every $l > 0$ an indecomposable module of even length $2l$ determines an acyclic complex $\boldsymbol{A}$ with $\operatorname{rank}_R A_n = l$ for all $n \in \mathbb{Z}$; this follows from [8, Prop. 5] and Lemma 1.7(a). The indecomposable modules of odd length are exactly the syzygies and cosyzygies of k; see also [3, 4.2.3]. After a shift, these modules all determine the same acyclic complex $\boldsymbol{A}$, for which $\{\operatorname{rank}_R A_i\}_{i\in\mathbb{Z}}$ is the sequence

$$\cdots > n+1 > n > \cdots > 2 > 1 = 1 < 2 < \cdots < n < n+1 < \cdots .$$

If $e \geqslant 3$ and $\boldsymbol{A}$ is non-zero, then $\{\operatorname{rank}_R A_i\}_{i\geqslant 0}$ and $\{\operatorname{rank}_R A_i\}_{i\leqslant 0}$ have exponential growth by [13, Thm. B]. Moreover, after a shift one has

$$\cdots > \operatorname{rank}_R A_2 > \operatorname{rank}_R A_1 \geqslant \operatorname{rank}_R A_0 < \operatorname{rank}_R A_{-1} < \cdots$$

by Lemma 1.2. An example is $R = k[X,Y,Z]/(X^2-Y^2, Y^2-Z^2, XY, YZ, XZ)$.

C. Acyclicity and total acyclicity. All acyclic complexes known to the authors, including the one from [12], can be obtained by a standard technique:

3.2. Construction. Let M be an R-module; take minimal free resolutions

$$\cdots \longrightarrow P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{p} M \longrightarrow 0$$
$$\cdots \longrightarrow Q_2 \xrightarrow{\partial_2} Q_1 \xrightarrow{\partial_1} Q_0 \xrightarrow{\pi} M^* \longrightarrow 0$$

and form the complex

$$\boldsymbol{A} = \cdots \to Q_2 \xrightarrow{\partial_2} Q_1 \xrightarrow{\partial_1} Q_0 \xrightarrow{p^*\circ\pi} P_0^* \xrightarrow{d_1^*} P_1^* \xrightarrow{d_2^*} P_2^* \to \cdots \tag{3.2.1}$$

with P_0^* in degree 0. If $\operatorname{Ext}_R^i(M,R) = 0$ for all $i > 0$, then $\boldsymbol{A}$ is acyclic and

$$M = \operatorname{Coker}\partial_1^{\boldsymbol{A}^*} \quad \text{and} \quad \mathrm{H}^i(\boldsymbol{A}^*) = 0 \quad \text{for all} \quad i < 0.$$

Moreover, if M^* is without non-zero free direct summands, then $\boldsymbol{A}$ is minimal.

On the other hand, if $\boldsymbol{A}$ is some acyclic complex with $\mathrm{H}^i(\boldsymbol{A}^*) = 0$ for all $i < 0$, then the module $M = \operatorname{Coker}\partial_1^{\boldsymbol{A}^*}$ has $\operatorname{Ext}_R^i(M,R) = 0$ for all $i > 0$.

3.3. Observation. In combination, [10, Prop. 2.9][2] and Lemma 1.8 can be reformulated as follows:

Let $(R,\mathfrak{m},k)$ be a local ring that is not Gorenstein and has $\mathfrak{m}^3 = 0 \neq \mathfrak{m}^2$. If there exists an R-module $M \neq 0$ and an integer $n \geqslant 3$ such that $\mathfrak{m}^2 M = 0$ and

$$\operatorname{Ext}_R^{n-1}(M,R) = \operatorname{Ext}_R^{n}(M,R) = \operatorname{Ext}_R^{n+1}(M,R),$$

then there are equalities $\beta_n^R(M) = \cdots = \beta_1^R(M) = \beta_0^R(M)$, and the ring has properties (a) *and* (b) *from Theorem A. Moreover, the ring satisfies:*

[2]Which contains a typo: the equalities of Betti numbers should be $b_0(M) = b_1(M) = \cdots = b_{j+1}(M)$.

(c) $\left[\mathrm{P}_k^R(t)\right]_{\leqslant n} = \left[\dfrac{1}{(1-t)(1-rt)}\right]_{\leqslant n}$.

(d) $\left[\mathrm{I}_R(t)\right]_{\leqslant n} = \left[\dfrac{r-t}{1-rt}\right]_{\leqslant n}$.

If M is a module with $\operatorname{Ext}_R^i(M,R) = 0$ for all $i > 0$, and $\boldsymbol{A}$ is the corresponding acyclic complex, cf. 3.2, then $\boldsymbol{A}$ satisfies the hypothesis of Theorem A. The module M satisfies the hypothesis of the result above for all integers $n \geqslant 3$, so it yields the same conclusion as Theorem A.

This naturally raises the following

3.4. Question. Is every minimal acyclic R-complex $\boldsymbol{A}$ obtainable from a module M with $\operatorname{Ext}_R^i(M,R) = 0$ for all $i > 0$ and thus of the form (3.2.1)?

The authors are not aware of any example of an acyclic, but not totally acyclic, complex in which all the modules have the same rank. Hence one may even ask

3.5. Question. For an acyclic R-complex $\boldsymbol{A}$ set

$$\varkappa = \inf\{i \mid k \text{ is a direct summand of } \mathrm{C}_{i+1}(\boldsymbol{A})\}$$

as in Remark 2.2. Does one always have $\mathrm{H}^i(\boldsymbol{A}^*) = 0$ for all $i < \varkappa$?

Appendix

Here we prove a technical result on sequences of positive integers that satisfy a certain second order linear recursion formula.

A.1. Proposition. *Let $e > 0$ and $r > 1$ be integers. If there exists a sequence of positive integers $\{a_i\}_{i\geqslant 0}$ such that*

$$a_i = ea_{i+1} - ra_{i+2} \quad \textit{for all} \quad i \geqslant 0,$$

then the following hold:

(a) *The sequence $\{a_i\}_{i\geqslant 0}$ is constant.*
(b) $e = r + 1$.

Proof. Set $q_i = a_i/a_{i+1}$. From the recursion formula one gets for each $i \geqslant 0$:

$$(1)_i \qquad q_i = e - \frac{r}{q_{i+1}}.$$

Subtract $(1)_{i+1}$ from $(1)_i$ to get

$$(2)_i \qquad q_i - q_{i+1} = r\frac{q_{i+1} - q_{i+2}}{q_{i+1}q_{i+2}}.$$

Next we show that

$$q_i = q_0 \quad \text{for all} \quad i \geqslant 0. \tag{3}$$

Let $i \geqslant 0$; multiplying the equations $(2)_0, \dots, (2)_i$ one gets

$$q_0 - q_1 = r^{i+1} \frac{q_{i+1} - q_{i+2}}{q_1 q_2^2 \cdot \dots \cdot q_{i+1}^2 q_{i+2}}.$$

Rewrite this equality in terms of the a_is and simplify as follows

$$\begin{aligned}
\frac{a_0}{a_1} - \frac{a_1}{a_2} &= r^{i+1} \frac{\dfrac{a_{i+1}}{a_{i+2}} - \dfrac{a_{i+2}}{a_{i+3}}}{\dfrac{a_1}{a_2} \cdot \dfrac{a_2^2}{a_3^2} \cdot \dots \cdot \dfrac{a_{i+1}^2}{a_{i+2}^2} \cdot \dfrac{a_{i+2}}{a_{i+3}}} \\
&= r^{i+1} \frac{\dfrac{a_{i+1}a_{i+3} - a_{i+2}^2}{a_{i+2}a_{i+3}}}{\dfrac{a_1 a_2}{a_{i+2}a_{i+3}}} \\
&= r^{i+1} \frac{a_{i+1}a_{i+3} - a_{i+2}^2}{a_1 a_2}.
\end{aligned}$$

Multiplication by $a_1 a_2$ yields

$$a_0 a_2 - a_1^2 = r^{i+1}(a_{i+1}a_{i+3} - a_{i+2}^2) \quad \text{for all} \quad i \geqslant 0. \tag{4}$$

Thus, r^{i+1} divides $a_0 a_2 - a_1^2$ for all $i \geqslant 0$, which forces $a_0 a_2 - a_1^2 = 0$ as $r > 1$. By (4) we now have $a_i a_{i+2} - a_{i+1}^2 = 0$ for all $i \geqslant 0$; that is, $q_i = q_{i+1}$ and therefore $q_i = q_0$ for all $i \geqslant 0$.

The recursion formula may now be rewritten

$$q_0^2 - e q_0 + r = 0. \tag{5}$$

Since q_0 is rational, it follows by the Rational Root Test that q_0 is an integer and divides r. If $q_0 > 1$, then

$$a_0 > a_1 > a_2 > \cdots > 0,$$

which is impossible. Thus, $q_0 = 1$ and then (3) implies part (a) while (b) follows from (5). □

Acknowledgments. The authors thank Luchezar Avramov and Răzvan Veliche for interesting discussions related to this material. Thanks are also due to Srikanth Iyengar, Greg Piepmeyer, and Diana White for useful comments on the exposition.

Note added in proof. After the acceptance of this paper, Hughes, Jorgensen, and Şega [9] have shown that both questions have negative answers.

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