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EXTREMAL CASES OF EXACTNESS CONSTANTS AND COMPLETELY BOUNDED PROJECTION CONSTANTS

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ABSTRACT. We investigate some extremal cases of exactness constants and completely bounded projection constants. More precisely, for an n-dimensional operator space E we prove that $\lambda_{cb}(E) = \sqrt{n}$ if and only if $\mathrm{ex}(E) = \sqrt{n}$.

1. Introduction

Exactness constants and completely bounded (c.b.) projection constants are fundamental quantities in operator space theory.

For an operator space $E \subseteq B(H)$, the *c.b. projection constant* of E, $\lambda_{cb}(E)$, is defined by

$$\lambda_{cb}(E) = \inf\{\|P\|_{cb} \mid P : B(H) \to E, \text{ projection onto } E\}.$$

Let $B = B(\ell_2)$ and $\mathcal{K}$ be the ideal of all compact operators on ℓ_2, and let

$$T_E : (B \otimes_{\min} E)/(\mathcal{K} \otimes_{\min} E) \to (B/\mathcal{K}) \otimes_{\min} E$$

be the map obtained from

$$q \otimes I_E : B \otimes_{\min} E \to (B/\mathcal{K}) \otimes_{\min} E$$

by the taking quotient with respect to $\mathcal{K} \otimes_{\min} E$, where $q : B \to \mathcal{K}$ is the canonical quotient map. Then the *exactness constant* of E, $\mathrm{ex}(E)$ is defined by

$$\mathrm{ex}(E) = \left\|T_E^{-1}\right\|.$$

It is well known that the exactness constant is the same as $d_{\mathcal{SK}}(E)$, where

$$d_{\mathcal{SK}}(E) = \inf\left\{d_{cb}(E, F) : F \subseteq \mathcal{K}\right\},$$

when E is finite dimensional ([9]).

The followings are well known facts about these quantities (Chapter 7 and 17 of [12] and Section 9 of [10]):

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FACT 1. *For a finite dimensional operator space E we have*

$$\mathrm{ex}(E) = d_{\mathcal{SK}}(E) \leq \lambda_{cb}(E).$$

FACT 2. *When* $\dim(E) = n \in \mathbb{N}$, *we have*

$$\lambda_{cb}(E) \leq \sqrt{n}.$$

Thus, for an n-dimensional operator space E, $\lambda_{cb}(E)$ and $\mathrm{ex}(E)$ are both bounded by $\sqrt{n}$, and this upper bound is known to be asymptotically sharp. Indeed, we have $\mathrm{ex}(\max \ell_1^n) \geq \frac{n}{2\sqrt{n-1}}$ for $n \geq 2$ ([9]). However, it is not yet known whether there is an n-dimensional operator space E with $\lambda_{cb}(E) = \sqrt{n}$ or $\mathrm{ex}(E) = \sqrt{n}$.

In this paper we investigate the extremal cases $\lambda_{cb}(E) = \sqrt{n}$ and $\mathrm{ex}(E) = \sqrt{n}$ and prove the following theorem.

THEOREM 1. *Let $n \geq 2$ and $E \subseteq B(H)$ be an n-dimensional operator space. Then we have $\lambda_{cb}(E) = \sqrt{n}$ if and only if* $\mathrm{ex}(E) = \sqrt{n}$. *Equivalently, $\lambda_{cb}(E) < \sqrt{n}$ if and only if* $\mathrm{ex}(E) < \sqrt{n}$.

$\lambda_{cb}(E)$ is the operator space analogue of the projection constant $\lambda(X)$ of a Banach space X given by $\lambda(X) = \sup\{\lambda(X, Y) \mid X \subseteq Y\}$, where

$$\lambda(X, Y) = \inf\{\|P\| \mid P : Y \to Y \text{ projection onto } X\}.$$

See [4], [5], and [6] for more information on the Banach space case and [3] and [13] for the operator space case.

Throughout this paper, we assume that the reader is familiar with the standard results about operator spaces ([2], [10]), completely nuclear maps ([2]), and completely p-summing maps ([11]). For a linear map $T : E \to F$ between operator spaces and $1 \leq p < \infty$ we denote the completely nuclear norm and the completely p-summing norm of T by $\nu^o(T)$ and $\pi_p^o(T)$, respectively.

For an index set I, $OH(I)$ denotes the operator Hilbert space on $\ell_2(I)$, which was introduced in [10]. When $I = \{1, \ldots, n\}$ for $n \in \mathbb{N}$, we simply write OH_n. For a family of operator spaces $(E_i)_{i \in I}$ and an ultrafilter $\mathcal{U}$ on I we denote the ultraproduct of $(E_i)_{i \in I}$ with respect to $\mathcal{U}$ by $\prod_{\mathcal{U}} E_i$.

2. Proof of the main result

For the proof we need several lemmas. The first lemma is about the relationship between completely 1-summing maps and completely 2-summing maps.

LEMMA 2. *Let $v : E \to F$ be a completely 1-summing map. Then v is completely 2-summing with $\pi_2^o(v) \leq \pi_1^o(v)$.*

Proof. Let $E \subseteq B(H)$ for some Hilbert space H. Then by Remark 5.7 of [11] we have an ultrafilter $\mathcal{U}$ over an index set I and families of positive operators $(a_\alpha)_{\alpha\in I}$, $(b_\alpha)_{\alpha\in I}$, in the unit ball of $S_2(H)$ such that the following diagram commutes for some u with $\|u\|_{cb} \le \pi_1^o(v)$:

$$\begin{array}{ccc} E & \xrightarrow{v} & F \\ {\scriptstyle i}\downarrow & & \uparrow{\scriptstyle u} \\ E_\infty & \xrightarrow[\mathcal{M}]{} & E_1 \end{array} \tag{2.1}$$

where $E_\infty = i(E)$ for the complete isometry

$$i : B(H) \hookrightarrow \prod_{\mathcal{U}} B(H), x \mapsto (x)_{\alpha\in I},$$

$E_1 = \overline{Mi(E)}$ (the closure in $\prod_{\mathcal{U}} S_1(H)$) for

$$M : \prod_{\mathcal{U}} B(H) \to \prod_{\mathcal{U}} S_1(H), (x_\alpha) \mapsto (a_\alpha x_\alpha b_\alpha),$$

and $\mathcal{M} = M|_{E_\infty}$.

Next, we split M into $M = T_2T_1$, where

$$T_1 : \prod_{\mathcal{U}} B(H) \to \prod_{\mathcal{U}} S_2(H), (x_\alpha) \mapsto (a_\alpha^{1/2} x_\alpha b_\alpha^{1/2})$$

and

$$T_2 : \prod_{\mathcal{U}} S_2(H) \to \prod_{\mathcal{U}} S_1(H), (x_\alpha) \mapsto (a_\alpha^{1/2} x_\alpha b_\alpha^{1/2}).$$

Note that

$$\|T_2\|_{cb} \le \lim_{\mathcal{U}} \left\| M_\alpha : S_2(H) \to S_1(H)\ ,\ x \mapsto a_\alpha^{1/2} x b_\alpha^{1/2} \right\|_{cb} \le 1, \tag{2.2}$$

since $M_\alpha^* = N_\alpha$ for

$$N_\alpha : B(H) \to S_2(H)\ ,\ x \mapsto a_\alpha^{1/2} x b_\alpha^{1/2}$$

and $\|N_\alpha\|_{cb} \le 1$. Thus we have by Theorem 5.1 of [11] that

$$\begin{aligned} \|(vx_{ij})\|_{M_n(F)} &= \|(uT_2T_1ix_{ij})\|_{M_n(F)} \le \pi_1^o(v)\, \|(T_2T_1ix_{ij})\|_{M_n(S_1(H))} \\ &\le \pi_1^o(v)\, \|(T_1ix_{ij})\|_{M_n(S_2(H))} = \pi_1^o(v) \left\| (a_\alpha^{1/2} x_{ij} b_\alpha^{1/2}) \right\|_{M_n(S_2(H))} \end{aligned}$$

for any $n \in \mathbb{N}$ and $(x_{ij}) \in M_n(F)$, which implies $\pi_2^o(v) \le \pi_1^o(v)$. □

The second lemma is about the trace duality of completely 2-summing norms.

Lemma 3. *Let E and F be operator spaces and E be finite dimensional. Then for $v : F \to E$ we have*

$$(\pi_2^o)^*(v) := \sup\{|\mathrm{tr}(vu)| \mid \pi_2^o(u : E \to F) \le 1\} = \pi_2^o(v).$$

Proof. See Lemma 4.7 of [7]. □

The final lemma is about the relationship between the trace norm and the completely nuclear norm of a linear map on an operator space and the operator space approximation property.

LEMMA 4. *Let E be an operator space with the operator space approximation property. Then for any completely nuclear map $u : E \to E$ we can define* $\mathrm{tr}(u)$*, the trace of* u*, and we have*

$$|\mathrm{tr}(u)| \leq \nu^o(u).$$

Proof. Since E has the operator space approximation property, the canonical mapping

$$\Phi : E \widehat{\otimes} E^* \to E \otimes_{\min} E^*$$

is one-to-one by Theorem 11.2.5 of [2], where $\widehat{\otimes}$ (resp. $\otimes_{\min}$) is the projective (resp. injective) tensor product in the category of operator spaces. Thus, $\mathcal{N}^o(E)$, the set of all completely nuclear maps on E, can be identified with $E \widehat{\otimes} E^*$ with the same norm. Since we have the trace functional defined on $E \widehat{\otimes} E^*$ (7.1.12 of [2]), we can translate it to $\mathcal{N}^o(E)$, so that we have

$$|\mathrm{tr}(u)| \leq \|U\|_{E \widehat{\otimes} E^*} = \nu^o(u),$$

where $U \in E \widehat{\otimes} E^*$ is the element associated to $u \in \mathcal{N}^o(E)$. □

Let E and F be operator spaces. Then the Γ_∞-norm and the γ_∞-norm of a linear map $v : E \to F$ are defined by

$$\Gamma_\infty(v) = \inf \|\alpha\|_{cb} \|\beta\|_{cb},$$

where the infimum is taken over all Hilbert spaces H and all factorizations

$$i_F v : E \xrightarrow{\alpha} B(H) \xrightarrow{\beta} F,$$

where $i_F : F \hookrightarrow F^{**}$ is the canonical embedding, and

$$\gamma_\infty(v) = \inf \|\alpha\|_{cb} \|\beta\|_{cb},$$

where the infimum is taken over all $m \in \mathbb{N}$ and all factorizations

$$v : E \xrightarrow{\alpha} M_m \xrightarrow{\beta} F.$$

See Section 4 of [3] or [1] for the details.

Now we are ready to prove our main result. The proof follows the classical idea of [4].

Proof of Theorem 1. By Fact 1 and Fact 2 it is enough to show that the condition $\lambda_{cb}(E) = \sqrt{n}$ is inconsistent with the condition $\mathrm{ex}(E) = d_{\mathcal{SK}}(E) < \sqrt{n}$.

Step 1. $\pi_1^o(I_E) = \sqrt{n}$.

By trace duality and Lemma 4.1 and 4.2 of [3] (or see Theorem 7.6 of [1]) we have

$$\lambda_{cb}(E) = \Gamma_\infty(I_E) = \gamma_\infty(I_E) = \sup_{u \in \pi_1^o(E)} \frac{|\mathrm{tr}(u)|}{\pi_1^o(u)}.$$

Since E is finite dimensional, we can find $u \in CB(E)$ such that

$$\frac{|\mathrm{tr}(u)|}{\pi_1^o(u)} = \sqrt{n},$$

and by multiplying by a suitable constant we can also assume that $\pi_2^o(u) = \sqrt{n}$. Then, by Lemma 2, Lemma 3, and Theorem 6.13 of [11], we obtain

$$n = \sqrt{n}\pi_2^o(u) \le \sqrt{n}\pi_1^o(u) = |\mathrm{tr}(u)| \le \pi_2^o(u)\pi_2^o(I_E) = n.$$

Thus, we get

$$\pi_1^o(u) = \sqrt{n} \text{ and } |\mathrm{tr}(u)| = n.$$

Next, we show that u is actually I_E. By Proposition 6.1 of [11] we have the factorization

$$u : E \xrightarrow{A} OH_n \xrightarrow{B} E \text{ with } \pi_2^o(A) \|B\|_{cb} \le \sqrt{n}.$$

If we let $v : OH_n \to OH_n$ be defined by $v = AB$, we have $\mathrm{tr}(v) = \mathrm{tr}(v^*) = \mathrm{tr}(u)$ and

$$\begin{aligned}\|I_{OH_n} - v\|_{HS}^2 &= \mathrm{tr}\left((I_{OH_n} - v)(I_{OH_n} - v)^*\right) \\ &= \mathrm{tr}(I_{OH_n}) - 2\,\mathrm{tr}(u) + \mathrm{tr}(vv^*) \\ &= n - 2n + \|v\|_{HS}^2 = (\pi_2^o(v))^2 - n \\ &\le (\pi_2^o(A)\|B\|_{cb})^2 - n \le 0,\end{aligned}$$

which leads to the desired conclusion.

Step 2. Now we factorize I_E as in the proof of Lemma 2. Then we have an ultrafilter $\mathcal{U}$, families of positive operators $(a_\alpha)_{\alpha \in I}$, $(b_\alpha)_{\alpha \in I}$, in the unit ball of $S_2(H)$, such that the diagram (2.1) commutes for some u with

$$\|u\|_{cb} \le \pi_1^o(I_E) = \sqrt{n}.$$

Then we can find a rank n projection

$$w_1 : i(B(H)) \to i(B(H)) \text{ onto } E_\infty \text{ with } \pi_1^o(w_1) \le \sqrt{n}.$$

Consider $iu : E_1 \to i(B(H))$. Since i is a complete isometry, $i(B(H))$ is injective in the operator space sense, so that we can extend iu to

$$\tilde{u} : \prod_{\mathcal{U}} S_1(H) \to i(B(H)) \text{ with } \|\tilde{u}\|_{cb} = \|iu\|_{cb}.$$

Now consider the same factorization $M = T_2T_1$ as before. Note that

$$\pi_2^o(T_1) \le 1 \text{ and } \|T_2\|_{cb} \le 1$$

by the same calculation as the proof for (5.8) of [11] and (2.2), respectively. Then for

$$w := T_1 \tilde{u} T_2 : \prod_{\mathcal{U}} S_2(H) \to \prod_{\mathcal{U}} S_2(H)$$

we have

$$\|w\|_{HS} = \pi_2^o(w) \le \pi_2^o(T_1) \|\tilde{u}\|_{cb} \|T_2\|_{cb} \le \pi_2^o(T_1) \|u\|_{cb} \le \sqrt{n}. \tag{2.3}$$

Since $T_1 i$ is 1-1, $F := T_1 i(E)$ is n-dimensional. Furthermore, since

$$wT_1 ix = T_1 \tilde{u} T_2 T_1 ix = T_1 iu\mathcal{M}ix = T_1 ix$$

for all $x \in E$, we have $w|_F = I_F$, which means that $|\lambda_k(w)| \ge 1$ for $1 \le k \le n$, where $(\lambda_k(w))_{k\ge 1}$ is the sequence of eigenvalues of w, in non-increasing order and counted according to multiplicity. By applying Weyl's inequality (Lemma 3.5.4 of [8]) and (2.3), we get

$$n \le \sum_{k=1}^{n} |\lambda_k(w)|^2 \le \sum_{k=1}^{\infty} s_k(w)^2 = \|w\|_{HS}^2 \le n,$$

where $(s_k(w))_{k\ge 1}$ is the sequence of singular values of w. Then we have

$$|\lambda_k(w)| = \begin{cases} 1 & \text{if } 1 \le k \le n, \\ 0 & \text{if } k > n, \end{cases}$$

which implies that w has rank at most n, as does

$$w_1 := \tilde{u}\mathcal{M} = \tilde{u} T_2 T_1|_{i(B(H))} : i(B(H)) \to i(B(H)).$$

Actually, w_1 is our desired rank n projection. Indeed, we have

$$w_1 ix = \tilde{u}\mathcal{M}ix = iu\mathcal{M}ix = ix$$

for all $x \in E$, and since E_∞ is n-dimensional, w_1 maps onto E_∞. Moreover, we have

$$\pi^o(w_1) \le \|\tilde{u}\|_{cb} \, \pi_1^o(\mathcal{M}) \le \sqrt{n}$$

since $\pi_1^o(\mathcal{M}) \le 1$ ((5.7) of [11]).

Step 3. Since $d_{\mathcal{SK}}(E_\infty) = d_{\mathcal{SK}}(E) < \sqrt{n}$, we have $F \in \mathcal{K}$ and an isomorphism

$$T : E_\infty \to F \text{ with } \|T\|_{cb} \left\|T^{-1}\right\|_{cb} < \sqrt{n}.$$

By the fundamental extension theorem (Theorem 1.6 of [12]) we have extensions

$$\widetilde{T} : i(B(H)) \to B(\ell_2) \text{ and } \widetilde{T^{-1}} : B(\ell_2) \to i(B(H))$$

of T and T^{-1}, respectively, with

$$\left\|\widetilde{T}\right\|_{cb} = \|T\|_{cb} \quad \text{and} \left\|\widetilde{T^{-1}}\right\|_{cb} = \left\|T^{-1}\right\|_{cb}.$$

Let $\tilde{w_1} = \widetilde{T} w_1 \widetilde{T^{-1}} : B(\ell_2) \to B(\ell_2)$. Then clearly we have $\mathrm{ran}(\tilde{w_1}) \subseteq F$ and $\tilde{w_1}|_F = I_F$, which means that $\tilde{w_1}$ is also a rank n projection from $B(\ell_2)$ onto

F. Since $F \subseteq \mathcal{K}$ and $\mathcal{K}$ satisfies the operator space approximation property, we have by Lemma 4 and Corollary 15.5.4 of [2] that

$$\begin{aligned} n &= |\mathrm{tr}(\tilde{w}_1|_{\mathcal{K}} : \mathcal{K} \to \mathcal{K})| \leq \nu^o(\tilde{w}_1|_{\mathcal{K}} : \mathcal{K} \to \mathcal{K}) = \pi_1^o(\tilde{w}_1|_{\mathcal{K}} : \mathcal{K} \to \mathcal{K}) \\ &= \pi_1^o(\tilde{w}_1|_{\mathcal{K}} : \mathcal{K} \to B(\ell_2)) \leq \left\|\widetilde{T}\right\|_{cb} \left\|\widetilde{T^{-1}}\right\|_{cb} \pi_1^o(w_1) \\ &\leq \|T\|_{cb} \left\|T^{-1}\right\|_{cb} \sqrt{n} < n, \end{aligned}$$

This is a contradiction. □

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