REFINED ANALYTIC TORSION: COMPARISON THEOREMS AND EXAMPLES

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ABSTRACT. Braverman and Kappeler introduced a refinement of the Ray-Singer analytic torsion associated to a flat vector bundle over a closed odd-dimensional manifold. We study this notion and improve the Braverman-Kappeler theorem comparing the refined analytic torsion with the Farber-Turaev refinement of the combinatorial torsion. Using this result we establish, modulo sign, the Burghelea-Haller conjecture, comparing their complex analytic torsion with the Farber-Turaev torsion in the case when the flat connection can be deformed in the space of flat connections to a Hermitian connection. We then compute the refined analytic torsion of lens spaces and answer some of the questions posed in [5, Remark 14.9].

1. Introduction

Let M be a closed oriented odd dimensional manifold and let E be a complex vector bundle over M endowed with a flat connection ∇. In a series of papers [5], [4], [6], [7], M. Braverman and T. Kappeler defined and studied a nonzero element

$$\rho_{\mathrm{an}}(\nabla) \in \mathrm{Det}\left(H^\bullet(M, E)\right)$$

of the complex determinant line $\mathrm{Det}\left(H^\bullet(M, E)\right)$ of the cohomology $H^\bullet(M, E)$ of M with coefficients in the complex vector bundle E. They called this element *refined analytic torsion*. It can be viewed as an analytic analogue of the refinement of the Reidemeister torsion due to Turaev [21], [22] and, more generally, to Farber and Turaev [14], [15]. Recall that the Farber-Turaev torsion $\rho_{\varepsilon,o}(\nabla)$ depends on the Euler structure ε, the cohomology orientation o, and the connection ∇.

The following extension of the Cheeger-Müller theorem [11], [18] was proven in [7, Theorem 5.11]: For each connected component C of the space $\mathrm{Flat}(E)$

of flat connections on E, there exists a constant $\theta^C \in \mathbb{R}$, such that

$$(1.1) \qquad\qquad \frac{\rho_{\mathrm{an}}(\nabla)}{\rho_{\varepsilon,o}(\nabla)} = e^{i\theta^C} \cdot f_{\varepsilon,o}(\nabla),$$

where $f_{\varepsilon,o}(\nabla)$ is a holomorphic function of $\nabla \in \mathrm{Flat}(E)$, given by an explicit local expression. Equality (1.1) does not give us any information about the constant θ^C and its dependence on C, ε, and o. In particular, Braverman and Kappeler posed the following two questions in [5, Remark 14.9],

QUESTION 1. *Does the constant θ^C depend on the connected component C of the space $\mathrm{Flat}(E)$ of flat connections on E?*

QUESTION 2. *For which connections ∇ one can find an Euler structure ε and the cohomological orientation o such that $\rho_{\mathrm{an}}(\nabla) = \rho_{\varepsilon,o}(\nabla)$?*

In Section 3 we compute the constant θ^C for any connected component C of $\mathrm{Flat}(E)$ which contains a Hermitian connection. In Section 5 and Section 6 of this paper we compute the refined analytic torsion of lens spaces and study its relationship with the cohomological Turaev torsion of lens spaces. Our explicit calculation for the three-dimensional lens space $L(5; 1, 1)$ shows that, in general, *the constant θ^C does depend on the connected component C of* $\mathrm{Flat}(E)$. This result provides a positive answer to Question 1 above. (Note that, in the case of lens spaces, $\mathrm{Flat}(E)$ is discrete and coincides with the space of acyclic Hermitian connections. Hence, connected components C of $\mathrm{Flat}(E)$ are one-element subsets).

We then compute the quotient of the refined analytic torsion and cohomological Turaev torsion of the five-dimensional lens space $L(3; 1, 1, 1)$. In this case we show that *for all connections, all Euler structures and all cohomological orientations, the cohomological Turaev torsion and the refined analytic torsion are not equal*. This provides a partial answer to Question 2 above.

In [9], [10] Burghelea and Haller defined a complex valued quadratic form, referred to as *complex Ray-Singer torsion*. This torsion is defined for a complex flat vector bundle over a closed manifold of arbitrary dimension, provided that the complex vector bundle admits a non-degenerate complex valued symmetric bilinear form b. Burghelea and Haller ([10, Conjecture 5.1]; see also Conjecture 4.1 below) conjectured that the complex Ray-Singer torsion is roughly speaking equal to the square of the Farber-Turaev torsion and established the conjecture in some non-trivial situations. Braverman and Kappeler, [8], expressed the Burghelea-Haller complex Ray-Singer torsion in terms of the square of the refined analytic torsion $\rho_{\mathrm{an}}(\nabla)$ and the eta invariant $\eta(\nabla)$. In particular, they proved a weak version of the Burghelea-Haller conjecture. In Section 4 we improve this result for the case when ∇ belongs to a connected

component of the space of flat connections on the associated complex vector bundle E which contains a Hermitian connection. Our result establishes, modulo sign, the Burghelea-Haller conjecture for this case.

This paper is organized as follows. In Section 2 we recall the definitions and properties of refined analytic torsion from [5], [4], [6]. In Section 3 we study the comparison theorem of the refined analytic torsion and the cohomological Farber-Turaev torsion from [7, Theorem 5.11] and present the formula of the constant θ^C. In Section 4 we present our result about the Burghelea-Haller conjecture. In Section 5 we compute the refined analytic torsion of lens spaces. In Section 6 we compute the Turaev torsion of lens spaces. At the end of Section 6 we calculate the constant θ^C in the case of the three-dimensional lens space $L(5; 1, 1)$ and the quotient of the refined analytic torsion and cohomological Turaev torsion of the five-dimensional lens space $L(3; 1, 1, 1)$ and explain how our computation gives answers to Question 1 and Question 2 above.

2. Refined analytic torsion

Throughout this paper we will assume that M is a closed oriented manifold of odd dimension $d = 2n - 1$ and E is a complex vector bundle over M endowed with a flat connection ∇. Fix a Riemannian metric g^M on M. In [6], Braverman and Kappeler defined a non-zero element (cf. [6, Section 7]),

$$(2.1) \qquad \rho(\nabla, g^M) \in \mathrm{Det}\left(H^\bullet(M, E)\right).$$

In general, $\rho(\nabla, g^M)$ might depend on the Riemannian metric g^M. Hence they introduced *the refined analytic torsion* $\rho_{\mathrm{an}}(\nabla)$ (cf. Definition 2.3), which is a slight modification of $\rho(\nabla, g^M)$ and is independent of g^M.

2.1. The odd signature operator. The refined analytic torsion is defined in terms of the odd signature operator which was introduced by Atiyah, Patodi, and Singer, [1, p. 44], [2, p. 405], and, in a more general setting, by Gilkey, [16, pp. 64–65]. Hence, let us begin by recalling the definition of this operator.

Let $\Omega^\bullet(M, E)$ denote the space of smooth differential forms on M with values in E. Fix a Riemannian metric g^M on M and let $* : \Omega^\bullet(M, E) \to \Omega^{d-\bullet}(M, E)$ denote the Hodge $*$-operator. Define the *chirality operator* $\Gamma = \Gamma(g^M) : \Omega^\bullet(M, E) \to \Omega^\bullet(M, E)$ by the formula

$$(2.2) \qquad \Gamma\omega := i^n(-1)^{\frac{k(k+1)}{2}} * \omega, \qquad \omega \in \Omega^k(M, E),$$

where n is given as above by $n = \frac{d+1}{2}$. Note that $\Gamma^2 = 1$.

DEFINITION 2.1.   The (even part of the) *odd signature operator* $B_{\text{even}} = B_{\text{even}}(\nabla, g^M)$ acting on an even form $\omega \in \Omega^{2p}(M, E)$ is defined by the formula

$$B_{\text{even}}\,\omega := (\Gamma\nabla + \nabla\Gamma)\omega \ \in \ \Omega^{d-2p-1}(\,M, E\,) \oplus \Omega^{d-2p+1}(\,M, E\,).$$

The operator B_{even} is an elliptic differential operator, whose leading symbol is symmetric with respect to any Hermitian metric h^E on E.

2.2. The η-invariant. Let θ be an Agmon angle for B_{even}; see [5, Definition 3.4] or [6, Definition 6.3] for the choice of this angle. The η-function of B_{even} is defined by the formula

$$\eta_\theta(s, B_{\text{even}}) = \sum_{\text{Re } \lambda_k > 0} m_k(\lambda_k)^{-s} - \sum_{\text{Re } \lambda_k < 0} m_k(-\lambda_k)^{-s},$$

where λ_k is the eigenvalue of B_{even} and m_k is the algebraic multiplicity of λ_k. It is known, [16], that $\eta_\theta(s, B_{\text{even}})$ has a meromorphic extension to the whole complex plane $\mathbb{C}$ with isolated simple poles, and that it is regular at 0.

Let m_+ (respectively, m_-) denote the number of eigenvalues (counted with their algebraic multiplicities) of B_{even} on the positive (respectively, negative) part of the imaginary axis. Let m_0 denote the algebraic multiplicity of 0 as an eigenvalue of B_{even}.

DEFINITION 2.2.   The η-invariant $\eta(\nabla)$ of B_{even} is defined by the formula

$$\eta(\nabla) = \frac{\eta_\theta(\,0, B_{\text{even}}\,) + m_+ - m_- + m_0}{2}.$$

Note that $\eta(\nabla)$ is independent of the angle θ (cf. [5, Subsection 3.10]).

2.3. The refined analytic torsion. Let $B_{\text{trivial}} = \Gamma d + d\Gamma \ : \ \Omega^\bullet(M) \to \Omega^\bullet(M)$. Define

$$\eta_{\text{trivial}} = \eta_{\text{trivial}}(g^M) = \frac{\eta_\theta(\,0, B_{\text{trivial}}\,)}{2}$$

to be the η-invariant corresponding to the trivial line bundle $M \times \mathbb{C} \to M$ over M.

Recall that the element $\rho(\nabla, g^M) \in \text{Det}\left(H^\bullet(M, E)\right)$ of the determinant line of the cohomology of M with coefficients in E was defined in [6, Section 7]. If the bundle E is acyclic, $\text{Det}\left(H^\bullet(M, E)\right)$ is canonically isomorphic to $\mathbb{C}$. In this case, $\rho(\nabla, g^M)$ can be viewed as a complex number, which is equal to the *graded determinant* of the operator B_{even} (cf. [5, Section 6]).

DEFINITION 2.3.   Let (E, ∇) be a flat vector bundle on M. The *refined analytic torsion* is the element

$$(2.3) \ \ \rho_{\text{an}}(\nabla) := \rho(\nabla, g^M) \cdot \exp\left(i\pi \cdot \text{rank } E \cdot \eta_{\text{trivial}}(g^M) \right) \ \in \ \text{Det}\left(H^\bullet(M, E),\right.$$

where g^M is any Riemannian metric on M.

It is shown in [6, Theorem 9.6] that $\rho_{\mathrm{an}}(\nabla)$ is independent of the choice of the metric g^M. Note that when $\dim M \equiv 1 \, (\mathrm{mod} \, 4)$, $\eta_{\mathrm{trivial}} = 0$, and, hence, $\rho_{\mathrm{an}}(\nabla) = \rho(\nabla, g^M)$.

3. Comparison between the refined analytic torsion and the Farber-Turaev torsion

We now recall the definition of the canonical involution on the complex determinant line $\mathrm{Det}\left(H^\bullet(M, E)\right)$ of the cohomology $H^\bullet(M, E)$ of M with coefficients in E. We then derive the formula of the phase of the refined analytic torsion and recall the formula of the phase of the Farber-Turaev torsion. Then we compute the constant θ^C and improve the Braverman-Kappeler theorem comparing the refined analytic torsion and the Farber-Turaev torsion.

3.1. Involution on the determinant line. In this subsection we recall the definition of the canonical involution on the complex line $\mathrm{Det}\left(H^\bullet(M, E)\right)$ from [6, Subsection 10.1].

Let M be a closed oriented manifold of odd dimension $d = 2n - 1$ and let E be a flat complex vector bundle over M admitting a flat Hermitian metric h^E and endowed with a flat connection ∇. Then

$$dh^E(u, v) = h^E(\nabla u, v) + h^E(u, \nabla v), \qquad u, v \in C^\infty(M, E),$$

and h^E can be extended canonically to a sesquilinear map

$$h^E : \Omega^\bullet(M, E) \times \Omega^\bullet(M, E) \to \Omega^\bullet(M, \mathbb{C}).$$

For $\omega_1, \omega_2 \in \Omega^\bullet(M, E)$ and for each $j = 0, \dots, d$, we then obtain a sesquilinear pairing

$$(3.1) \quad h^E : \Omega^j(M, E) \times \Omega^{d-j}(M, E) \to \mathbb{C}, \quad (\omega_1, \omega_2) \mapsto \int_M h^E(\omega_1 \wedge \omega_2).$$

The pairing (3.1) induces a non-degenerate sesquilinear pairing

$$(3.2) \qquad H^j(M, E) \otimes H^{d-j}(M, E) \longrightarrow \mathbb{C}, \qquad j = 0, \dots, d,$$

and allows us to identify $H^j(M, E)$ with the dual space of $H^{d-j}(M, E)$. Using the construction of Subsection 3.4 of [6] we thus obtain a canonical involution

$$(3.3) \qquad D : \mathrm{Det}\left(H^\bullet(M, E)\right) \longrightarrow \mathrm{Det}\left(H^\bullet(M, E)\right).$$

Note that if the flat bundle E is acyclic, then the complex determinant line $\mathrm{Det}\left(H^\bullet(M, E)\right)$ is canonically isomorphic to $\mathbb{C}$ and under this isomorphism the involution (3.3) coincides with the complex conjugation.

If $h \in \mathrm{Det}\left(H^\bullet(M, E)\right)$ and $D(h) = h$, then the element h will be called *real*. The real elements of $\mathrm{Det}\left(H^\bullet(M, E)\right)$ form a real line.

If $h \in \mathrm{Det}\left(H^\bullet(M, E)\right)$ can be represented in the form $h = h_0 e^{i\phi}$, where h_0 is real, then $\phi \in \mathbb{R}$ will be called the *phase* of h. It is defined up to an integral multiple of π and we will denote it by $\mathrm{Ph}(h)$.

3.2. On sign conventions. The definition of the canonical involution D in [6, Subsection 10.1] is different from the definition of the canonical involution in [14, Subsection 2.1] by a factor $(-1)^\nu$, $\nu \in \mathbb{Z}$. Hence, if we denote by $\widetilde{\mathrm{Ph}}(h)$ the phase of $h \in \mathrm{Det}\left(H^\bullet(M, E)\right)$ as it is defined in [14], then

$$(3.4) \qquad \mathrm{Ph}(h) = \widetilde{\mathrm{Ph}}(h) + \frac{\pi\nu}{2} \qquad \mathrm{mod}\, \pi\mathbb{Z}.$$

Note, however, that if the bundle E is acyclic, then both involutions coincide with the complex conjugation (cf. Subsection 3.1 and [14, Lemma 2.2]). Hence, for the acyclic case, $\nu = 0$ and $\mathrm{Ph}(h) = \widetilde{\mathrm{Ph}}(h)$.

3.3. Phase of the refined analytic torsion. In this subsection we derive the formula of the phase of the refined analytic torsion $\rho_{\mathrm{an}}(\nabla)$. We have the following proposition.

PROPOSITION 3.1. *Let M be a closed oriented manifold of odd dimension $d = 2n - 1$ and let E be a flat complex vector bundle over M admitting a flat Hermitian metric and endowed with a flat connection ∇. Then the phase of the refined torsion $\rho_{\mathrm{an}}(\nabla)$ is given by the following formula:*

$$(3.5) \qquad \mathrm{Ph}\left(\rho_{\mathrm{an}}(\nabla)\right) = -\pi\left(\eta(\nabla) - \mathrm{rank}\, E \cdot \eta_{\mathrm{trivial}}\right) \qquad \mathrm{mod}\, \pi\mathbb{Z}.$$

Proof. From [6, Theorem 10.3], we have

$$(3.6) \qquad D\left(\rho_{\mathrm{an}}(\nabla)\right) = \rho_{\mathrm{an}}(\nabla) \cdot e^{2i\pi\left(\eta(\nabla) - \mathrm{rank}\, E \cdot \eta_{\mathrm{trivial}}\right)}.$$

If $\rho_{\mathrm{an}}(\nabla) = \rho_0 e^{i\phi}$, where ρ_0 is real with respect to the canonical involution (3.3), then we obtain $D\left(\rho_{\mathrm{an}}(\nabla)\right)/\rho_{\mathrm{an}}(\nabla) = e^{-2i\phi}$. Therefore,

$$(3.7) \qquad \mathrm{Ph}\left(\rho_{\mathrm{an}}(\nabla)\right) = -\pi\left(\eta(\nabla) - \mathrm{rank}\, E \cdot \eta_{\mathrm{trivial}}\right) \qquad \mathrm{mod}\, \pi\mathbb{Z}. \qquad \square$$

3.4. Phase of the Farber-Turaev torsion. The homological version of the formula of the phase of the Farber-Turaev torsion was computed in [14]. Similarly, we can compute the cohomological version of the formula of the phase of the Farber-Turaev combinatorial torsion $\rho_{\varepsilon,o}(\nabla)$.

Following Farber [13], we denote by Arg_∇ the unique cohomology class $\mathrm{Arg}_\nabla \in H^1(M, \mathbb{C}/\mathbb{Z})$ such that for every closed curve $\gamma \in M$ we have

$$(3.8) \qquad \det\left(\mathrm{Mon}_\nabla(\gamma)\right) = \exp\left(2\pi i\langle \mathrm{Arg}_\nabla, [\gamma]\rangle\right),$$

where $\mathrm{Mon}_\nabla(\gamma)$ denotes the monodromy of the flat connection ∇ along the curve γ and $\langle \cdot, \cdot \rangle$ denotes the natural pairing

$$H^1(M, \mathbb{C}/\mathbb{Z}) \times H_1(M, \mathbb{Z}) \longrightarrow \mathbb{C}/\mathbb{Z}.$$

Note that when ∇ is a Hermitian connection, $\mathrm{Mon}_\nabla(\gamma)$ is unitary and $\mathrm{Arg}_\nabla \in H^1(M, \mathbb{R}/\mathbb{Z})$ (cf. [13]).

Let $c(\varepsilon) \in H_1(M, \mathbb{Z})$ denote the characteristic class of the Euler structure ε (cf. [15, Subsection 5.2] or [22]). Then we have the following proposition ([14, Theorem 2.3]).

PROPOSITION 3.2. *Let M be a closed oriented manifold of odd dimension $d = 2n - 1$ and let E be a flat complex vector bundle over M admitting a flat Hermitian metric and endowed with a flat connection ∇. Then the phase of the Farber-Turaev torsion $\rho_{\varepsilon,o}(\nabla)$ is given by the following formula:*

$$(3.9) \qquad \mathrm{Ph}\left(\rho_{\varepsilon,o}(\nabla)\right) = \pi\langle\, \mathrm{Arg}_\nabla, c(\varepsilon)\,\rangle + \frac{\pi\nu}{2} \qquad \mathrm{mod}\,\pi\mathbb{Z},$$

where $\nu \in \mathbb{Z}$. If, moreover, the bundle E is acyclic, then $\nu = 0$ (cf. Subsection 3.2).

3.5. Comparison between the Farber-Turaev torsion and the refined analytic torsion. In [5], [7], Braverman and Kappeler computed the ratio

$$R = R(\nabla, \varepsilon, o) := \frac{\rho_{\mathrm{an}}(\nabla)}{\rho_{\varepsilon,o}(\nabla)}.$$

We now briefly recall their result. First, we need to introduce some additional notations.

Let us denote by $\widehat{L}(p) \in H_\bullet(M, \mathbb{Z})$ the Poincaré dual of the cohomology class $[L(p)]$, where $L(p) = L_M(p)$ is the Hirzebruch L-polynomial in the Pontrjagin forms of the Riemannian metric g^M. Let $\widehat{L}_1 \in H_1(M, \mathbb{Z})$ denote the component of $\widehat{L}(p)$ in $H_1(M, \mathbb{Z})$. Then

$$\langle\, [L(p)] \cup \mathrm{Arg}_\nabla, [M]\,\rangle = \langle\, \mathrm{Arg}_\nabla, \widehat{L}_1\,\rangle \ \in \ \mathbb{C}/\mathbb{Z}.$$

Note that when $\dim M \equiv 3 \ (\mathrm{mod}\,4)$, $\widehat{L}_1 = 0$.

The following Braverman-Kappeler theorem comparing the refined analytic torsion with the Farber-Turaev torsion was proven in [7, Theorem 5.11]. We will restrict to the case when the connected component C of $\mathrm{Flat}(E)$ contains a Hermitian connection.

THEOREM 3.3. *Suppose that M is a closed oriented odd dimensional manifold. Let E be a flat complex vector bundle over M admitting a Hermitian metric and endowed with a flat connection ∇. Let ε be an Euler structure on M and let o be a cohomological orientation of M. Then, for each connected component C of the set $\mathrm{Flat}(E)$ that contains a Hermitian connection, there exists a constant $\theta^C = \theta_o^C \in \mathbb{R}/2\pi\mathbb{Z}$, depending on o (but not on ε), such that, for any connection $\nabla \in C$,*

$$(3.10) \qquad\qquad \frac{\rho_{\mathrm{an}}(\nabla)}{\rho_{\varepsilon,o}(\nabla)} = \pm e^{i\theta_o^C} \cdot e^{-\pi i\langle\, \mathrm{Arg}_\nabla, c(\varepsilon) + \widehat{L}_1\,\rangle}.$$

Now we compute the constant θ^C which appears in the quotient of the refined analytic torsion and the cohomological Farber-Turaev torsion of M (cf. (3.10)). We have the following theorem.

THEOREM 3.4.   *Suppose that M is a closed oriented odd dimensional manifold. Let E be a flat complex vector bundle over M admitting a Hermitian metric and endowed with a flat connection ∇. Let ε be an Euler structure on M and let o be a cohomological orientation of M. If the connected component C of $\mathrm{Flat}(E)$ contains a Hermitian connection, then, for some $\nu \in \mathbb{Z}$:*

(1) *If $\dim M \equiv 1 \,(\mathrm{mod}\,4)$, then*

$$(3.11) \qquad \theta^C = -\pi\big(\,\mathrm{Re}\,\eta(\nabla) - \mathrm{Re}\langle\,\mathrm{Arg}_\nabla, \widehat{L}_1\,\rangle\,\big) + \frac{\pi\nu}{2} \qquad \mathrm{mod} \quad \pi\mathbb{Z}.$$

(2) *If $\dim M \equiv 3 \,(\mathrm{mod}\,4)$, then*

$$(3.12) \qquad \theta^C = -\pi\big(\,\mathrm{Re}\,\eta(\nabla) - \mathrm{rank}\,E \cdot \eta_{\mathrm{trivial}}\,\big) + \frac{\pi\nu}{2} \qquad \mathrm{mod} \quad \pi\mathbb{Z}.$$

If, moreover, the bundle E is acyclic, then $\nu = 0$ (cf. Subsection 3.2).

Proof. If $\nabla \in C$ is a Hermitian connection, then the theorem follows by combining Proposition 3.1 and Proposition 3.2 with Theorem 3.3.

Suppose that $\nabla_t (t \in [0,1])$ is a smooth family of connections in C such that $\nabla_0 = \nabla$ is Hermitian. From Theorem 12.3 and Lemma 12.6 of [5] we conclude that

$$(3.13) \qquad\qquad \frac{d}{dt}\eta(\nabla_t) = \frac{d}{dt}\langle\mathrm{Arg}_{\nabla_t}, \widehat{L}_1\rangle.$$

Lemma 5.5 of [7] shows that

$$(3.14) \qquad\qquad \exp\big(\,\pi\,\mathrm{Im}\langle\,\mathrm{Arg}_\nabla, \widehat{L}_1\,\rangle\,\big) = \exp\big(\,\pi\,\mathrm{Im}\,\eta(\nabla)\,\big).$$

Hence by combining (3.13), (3.14) with the case that $\nabla \in C$ is a Hermitian connection, the theorem follows. $\square$

From Theorem 3.3 and Theorem 3.4, we have the following theorem which improves the Braverman-Kappeler theorem (Theorem 3.3).

THEOREM 3.5.   *Suppose that M is a closed oriented odd dimensional manifold. Let E be a flat complex vector bundle over M admitting a Hermitian metric and endowed with a flat connection ∇. Let ε be an Euler structure on M and let o be a cohomological orientation of M. If the connected component C of $\mathrm{Flat}(E)$ contains a Hermitian connection, then*

$$(3.15) \qquad\qquad \frac{\rho_{\mathrm{an}}(\nabla)}{\rho_{\varepsilon,o}(\nabla)} = \pm i^\nu \cdot e^{-\pi i\langle\,\mathrm{Arg}_\nabla, c(\varepsilon)\,\rangle} \cdot e^{-i\pi(\eta(\nabla) - \mathrm{rank}\,E \cdot \eta_{\mathrm{trivial}})},$$

where $\nu \in \mathbb{Z}$. If, moreover, the bundle E is acyclic, then $\nu = 0$ (cf. Subsection 3.2).

4. Comparison between the Farber-Turaev torsion and the Burghelea-Haller torsion

In [9], [10] Burghelea and Haller introduced a refinement of the square of the Ray-Singer torsion for a closed manifold of arbitrary dimension, provided that the complex vector bundle E admits a non-degenerate complex valued symmetric bilinear form b. They defined a complex valued quadratic form $\tau_{b,\nabla}$ on the determinant line $\mathrm{Det}\left(H^\bullet(M, E)\right)$. Then they defined a complex valued quadratic form, referred to as *complex Ray-Singer torsion*. For the closed oriented odd dimensional manifold M and the complex vector bundle E over M endowed with a flat connection ∇, it is given by

$$(4.1) \qquad \tau_{b,\gamma,\nabla}^{\mathrm{BH}} := \tau_{b,\nabla} \cdot e^{-2\int_M \omega_{\nabla,b}\wedge\gamma},$$

where $\gamma \in \Omega^{d-1}(M)$ is an arbitrary closed $(d-1)$-form and $\omega_{\nabla,b} \in \Omega^1(M)$ is the Kamber-Tondeur form (cf. [10, Section 2]).

Burghelea and Haller conjectured, [10, Conjecture 5.1], that for a suitable choice of γ the form $\tau_{b,\gamma,\nabla}^{\mathrm{BH}}$ is roughly speaking equal to the square of the Farber-Turaev torsion and established the conjecture in some non-trivial situations. Though the conjecture is for manifolds of arbitrary dimensions, we restrict to the odd dimensional case and adopt the following formulation from [8, Conjecture 1.9].

CONJECTURE 4.1 (Burghelea-Haller). *Let M be a closed oriented manifold of odd dimension $d = 2n - 1$ and let E be a flat complex vector bundle over M endowed with a flat connection ∇. Let b be a non-degenerate symmetric bilinear form on E. Let ε be an Euler structure on M represented by a non-vanishing vector field X and let o be a cohomological orientation of M. Fix a Riemannian metric g^M on M and let $\Psi(g^M) \in \Omega^{d-1}(TM\backslash\{0\})$ denote the Mathai-Quillen form, [3, pp. 40–44], [17, Section 7]. Set*

$$(4.2) \qquad \gamma_\varepsilon = \gamma_\varepsilon(g^M) := X^*\Psi(g^M).$$

Then

$$(4.3) \qquad \tau_{b,\gamma_\varepsilon,\nabla}^{\mathrm{BH}}(\rho_{\varepsilon,o}(\nabla)) = 1.$$

In [8] Braverman and Kappeler expressed the Burghelea-Haller complex Ray-Singer torsion in terms of the square of the refined analytic torsion $\rho_{\mathrm{an}}(\nabla)$ and the eta invariant $\eta(\nabla)$. In particular, they proved the following weak version of the Burghelea-Haller conjecture: $\tau_{b,\gamma_\varepsilon,\nabla}^{\mathrm{BH}}(\rho_{\varepsilon,o}(\nabla))$ is locally constant in ∇ and

$$|\tau_{b,\gamma_\varepsilon,\nabla}^{\mathrm{BH}}(\rho_{\varepsilon,o}(\nabla))| = 1.$$

In the following theorem we improve this result for the case when ∇ belongs to a connected component of the space of flat connections on the associated complex vector bundle E which contains a Hermitian connection. More precisely, we have the following theorem:

THEOREM 4.2. *Suppose the assumptions of Conjecture 4.1 hold, and assume in addition that the connected component C of the set $\mathrm{Flat}(E)$ of flat connections on E contains a Hermitian connection. Then*

$$(4.4) \qquad \tau^{\mathrm{BH}}_{b,\gamma_\varepsilon,\nabla}(\rho_{\varepsilon,o}(\nabla)) = \pm 1, \qquad for\ all \quad \nabla \in C.$$

Proof. In Theorem 1.10 of [8], Braverman and Kappeler proved that $\tau^{\mathrm{BH}}_{b,\gamma_\varepsilon,\nabla}(\rho_{\varepsilon,o}(\nabla))$ is constant on C with absolute value 1. Hence it is enough to prove the equality (4.4) in the case when $\nabla \in C$ is a Hermitian connection.

From (1.1), (1.2) and Theorem 1.4 of [8] we have

$$(4.5) \qquad \tau^{\mathrm{BH}}_{b,\gamma_\varepsilon,\nabla}(\rho_{\mathrm{an}}(\nabla)) = \pm e^{-2\pi i \left(\eta(\nabla) - \mathrm{rank}\, E \cdot \eta_{\mathrm{trivial}} \right)} \cdot e^{-2 \int_M \omega_{\nabla,b} \wedge \gamma_\varepsilon}.$$

We also have (cf. [8, Subsection 5.4])

$$(4.6) \qquad e^{-2 \int_M \omega_{\nabla,b} \wedge \gamma_\varepsilon} = e^{< [\omega_{\nabla,b}, c(\varepsilon)] >} = \pm e^{-2\pi i \langle \mathrm{Arg}_\nabla, c(\varepsilon) \rangle}.$$

From (3.8) and (3.15), we get

$$(4.7) \qquad \left(\frac{\rho_{\mathrm{an}}(\nabla)}{\rho_{\varepsilon,o}(\nabla)} \right)^2 = \pm e^{-2\pi i \langle \mathrm{Arg}_\nabla, c(\varepsilon) \rangle} \cdot e^{-2i\pi \left(\eta(\nabla) - \mathrm{rank}\, E \cdot \eta_{\mathrm{trivial}} \right)}.$$

By combining (4.5) and (4.6) with (4.7), we obtain the result. $\square$

5. Refined analytic torsion of lens spaces

In this section we compute the refined analytic torsion of lens spaces. We begin by recalling the relationship of the acyclic case of the refined analytic torsion with the Ray-Singer torsion and the eta invariants. We then recall the definition of a lens space and the formula for the Ray-Singer torsion of a lens space from [19]. Then we recall the formula for the eta invariant of a lens space from [2]. By combining these results, we obtain the refined analytic torsion of a lens space.

5.1. The acyclic case of the refined analytic torsion. Denote by $\widetilde{M}$ the universal covering of M and by $\pi_1(M)$ the fundamental group of M, viewed as the group of deck transformations of $\widetilde{M} \to M$. For each complex representation $\alpha : \pi_1(M) \to GL(r, \mathbb{C})$, we denote by

$$(5.1) \qquad\qquad E_\alpha := \widetilde{M} \times_\alpha \mathbb{C}^r \to M$$

the flat vector bundle induced by α. Let ∇_α be the flat connection on E_α induced from the trivial connection on $\widetilde{M} \times \mathbb{C}^r$. We also denote by ∇_α the induced differential

$$\nabla_\alpha \ : \ \Omega^\bullet(M, E_\alpha) \to \Omega^{\bullet+1}(M, E_\alpha),$$

where $\Omega^\bullet(M, E_\alpha)$ denotes the space of smooth differential forms of M with values in E_α.

If the representation α is acyclic, i.e., $H^\bullet(M, E_\alpha) = 0$, then the determinant line $\mathrm{Det}(H^\bullet(M, E_\alpha))$ is canonically isomorphic to $\mathbb{C}$. In particular, if α is an acyclic unitary representation of $\pi_1(M)$, then (cf. [5, Section 12])

$$(5.2) \qquad \rho_{\mathrm{an}}(\nabla_\alpha) = \rho_\alpha^{\mathrm{RS}} \cdot e^{-i\pi\eta_\alpha} \cdot e^{i\pi \cdot \mathrm{rank}\,\alpha \cdot \eta_{\mathrm{trivial}}},$$

where $\rho_\alpha^{\mathrm{RS}} := \rho^{\mathrm{RS}}(\nabla_\alpha)$ is the well-known Ray-Singer torsion, [20], and $\eta_\alpha := \eta(\nabla_\alpha)$.

5.2. The lens space. Fix an integer $m \geq 3$ and let G_m denote the cyclic group of order m. We fix a generator $g \in G_m$ so that $G_m = \{1, g, \ldots, g^{m-1}\}$.

Let $p_1, \ldots, p_n$ be integers relatively prime to m. Then the action of G_m on the sphere

$$S^{2n-1} = \left\{ z \in \mathbb{C}^n \mid \|z\| = 1 \right\}$$

defined by

$$(5.3) \qquad g \cdot (z_1, \ldots, z_n) = \left(e^{2\pi i p_1/m} z_1, \ldots, e^{2\pi i p_n/m} z_n \right)$$

is free. The *lens space* $L = L(m; p_1, \ldots, p_n)$ is the orbit space of this action

$$L = L(m; p_1, \ldots, p_n) := S^{2n-1}/G_m.$$

Clearly, $\pi_1(L) = G_m$.

Fix $q \in \mathbb{Z}$ and consider the unitary representation $\alpha = \alpha_q : \pi_1(L) = G_m \to U(1)$, defined by

$$\alpha_q(g) = e^{2\pi i q/m}.$$

We will be interested in the refined analytic torsion $\rho_{\mathrm{an}}(q) = \rho_{\mathrm{an}}(\nabla_{\alpha_q})$ associated to the representation α_q.

5.3. The Ray-Singer torsion of the lens spaces. In this subsection we recall the formula for the Ray-Singer torsion of lens spaces from [19, Section 4]. Note that our definition of the logarithm of the Ray-Singer torsion is negative one half of the logarithm of the Ray-Singer torsion defined in [19].

PROPOSITION 5.1. *Let l_k $(k = 1, \ldots, n)$ be any integers such that $l_k p_k \equiv 1 \,(\mathrm{mod}\, m)$ and let $\rho_{\alpha_q}^{\mathrm{RS}}(L)$ denote the Ray-Singer torsion of the lens space L associated to the nontrivial acyclic representation α_q. Then*

$$\rho_{\alpha_q}^{\mathrm{RS}}(L) = \prod_{k=1}^n \left| e^{\frac{2\pi i q l_k}{m}} - 1 \right|.$$

5.4. The η invariant of the odd signature operator for lens spaces. By a slight modification of Proposition 2.12 in [2], where the eta invariant η_{trivial} of a lens space for trivial representation was computed, we have the following proposition; see also [12, Proposition 4.1].

PROPOSITION 5.2.   Let $\eta_{\alpha_q} = \eta_q$ denote the eta invariant of the odd signature operator $\mathrm{B}_{\mathrm{even}}(\nabla_{\alpha_q}, g^L)$ of the lens space L. Then

$$\eta_q = \frac{i^{-n}}{2m} \sum_{l=1}^{m-1} \left(e^{\frac{2\pi i l q}{m}} \cdot \prod_{j=1}^{n} \cot \frac{\pi l p_j}{m} \right).$$

In particular, when n is even we have

$$\eta_q = \frac{i^{-n}}{2m} \sum_{l=1}^{m-1} \left(\cos \frac{2\pi l q}{m} \cdot \prod_{j=1}^{n} \cot \frac{\pi l p_j}{m} \right),$$

and when n is odd we have

$$\eta_q = \frac{i^{1-n}}{2m} \sum_{l=1}^{m-1} \left(\sin \frac{2\pi l q}{m} \cdot \prod_{j=1}^{n} \cot \frac{\pi l p_j}{m} \right).$$

Note that our η invariant is equal to one half of the η invariant in [2].

Combining these two propositions with (5.2) and Proposition 2.12 in [2], we have:

THEOREM 5.3.   For the lens space $L = L(m; p_1, \ldots, p_n), m \geq 3$, and the nontrivial acyclic representation $\alpha_q : \pi_1(L) = G_m \rightarrow U(1)$ such that $\alpha_q(g) = e^{2\pi i q/m}$, the refined analytic torsion is given by

$$\rho_{\mathrm{an}}(q) = \prod_{k=1}^{n} \left| e^{\frac{2\pi i q l_k}{m}} - 1 \right| \cdot e^{\frac{i^{-(n+1)}\pi}{2m} \sum_{l=1}^{m-1} (e^{\frac{2\pi i l q}{m}} \cdot \prod_{j=1}^{n} \cot \frac{\pi l p_j}{m})}.$$

$$\cdot\, e^{\frac{i^{-(n-1)}\pi}{2m} \sum_{l=1}^{m-1} (\prod_{j=1}^{n} \cot \frac{\pi l p_j}{m})},$$

where l_k $(k = 1, \ldots, n)$ are any integers such that $l_k p_k \equiv 1 \,(\mathrm{mod}\, m)$

6. Comparison between the refined analytic torsion and the Turaev torsion of a lens space

In this section we begin by recalling the definition of the Turaev torsion and computing the Turaev torsion of lens spaces. We then calculate the constant θ^C for the three-dimensional lens space $L(5; 1, 1)$ and the ratio R of the refined analytic torsion and cohomological Turaev torsion of the five-dimensional lens space $L(3; 1, 1, 1)$ and explain how our computations give answers to Questions 1 and 2 of the introduction.

6.1. Torsion of an acyclic chain complex. Let $\mathbb{F}$ be a field of characteristic zero and let

$$C: \quad 0 \longrightarrow C_d \xrightarrow{\partial_{d-1}} C_{d-1} \xrightarrow{\partial_{d-2}} \ldots \xrightarrow{\partial_0} C_0 \longrightarrow 0$$

be a finite dimensional chain complex over $\mathbb{F}$. Assume that the chain complex (C, ∂) is acyclic, i.e., $H_*(C) = 0$. For each i, let c_i be a fixed basis for C_i and

b_i be a sequence of vectors in C_i whose image under ∂_{i-1} is a basis in $\operatorname{Im}\partial_{i-1}$. Then the vectors $\partial_i(b_{i+1}), b_i$ form a basis for C_i. The torsion of C is defined by

$$\tau_{\mathrm{comb}}(C) = \prod_{i=0}^{d} [\,\partial_i(b_{i+1})b_i/c_i\,]^{(-1)^{i+1}},$$

where $[\partial_i(b_{i+1})b_i/c_i]$ is the determinant of the matrix transforming c_i into the basis $\partial_i(b_{i+1}), b_i$ of C_i.

6.2. The Reidemeister torsion. Fix a CW-decomposition $X = \{e_1, \ldots, e_N\}$ of M. For each $j = 1, \ldots, N$, fix a lift $\widetilde{e}_j$, i.e., a cell of the CW-decomposition $\widetilde{X}$ of $\widetilde{M}$, such that $\pi(\widetilde{e}_j) = e_j$. By (5.1), the pull-back of the bundle E_α to $\widetilde{M}$ is the trivial bundle $\widetilde{M} \times \mathbb{C}^n \to \widetilde{M}$. Hence, the set of the cells $\widetilde{e}_1 \ldots \widetilde{e}_N$ identifies the chain complex $C(X, \alpha)$ of the CW-complex X with coefficients in E_α with the complex $C(\widetilde{X}) \otimes_\alpha \mathbb{C}^r$, where $\alpha : \pi_1(X) \to GL(r, \mathbb{C})$ is a representation. Assume that this chain complex $C(X, \alpha)$ is acyclic, i.e.,

$$H_*(\,M, E_\alpha\,) = H_*(\,C(X, \alpha)\,) = 0.$$

Then the Reidemeister torsion is defined as the torsion of this chain complex.

6.3. Combinatorial Euler structures and the homological Turaev torsion. In this subsection we recall the definition of combinatorial Euler structures from [23].

A family $\hat{e} = \{\hat{e}_i\}$ of open cells in the maximal abelian covering

$$\widehat{X} = \widetilde{X}/[\,\pi_1(X), \pi_1(X)\,]$$

of X is called *fundamental* if each open cell e_i in X is covered exactly by one cell $\hat{e}_i$ of $\hat{e}$.

Following Turaev, we denote the operation of any two cells in multiplicative notation. Let

$$\hat{e}'/\hat{e} = \prod_{e_i \in X} (\,\hat{e}_i'/\hat{e}_i\,)^{(-1)^{\dim e_i}} \in H_1(M)$$

for any two fundamental families $\hat{e}$ and $\hat{e}'$, where $\hat{e}_i'/\hat{e}_i \in H_1(M)$. We say that the fundamental families $\hat{e}$ and $\hat{e}'$ are equivalent if $\hat{e}/\hat{e}' = 1$. The equivalence classes are called *combinatorial Euler structures* on M.

Let $\alpha : \pi_1(M) \to GL(r, \mathbb{C})$ be an acyclic representation. Then we can associate to each combinatorial Euler structure ε on M the homological Turaev torsion

$$\tau_\alpha(\,M, \varepsilon\,) = \tau_\alpha(\,M, \hat{e}\,) \in \mathbb{C}/\pm .$$

For each Euler structure ε on M, there is an *Euler class* $c(\varepsilon) \in H_1(M)$ associated to it (cf. [22] or [15, Subsection 5.2]). If $d = \dim M$ is odd, then (in multiplicative notation)

$$(6.1) \qquad\qquad c(h\varepsilon) = h^2 c(\varepsilon)$$

for any $\varepsilon \in \mathrm{Eul}(M)$, $h \in H_1(M)$.

Turaev also introduced the homology orientation to get rid of the sign indeterminacy of Reidemeister torsion. For our purpose it will be enough to consider the Turaev torsion up to sign, so we skip the definition of homology orientation.

6.4. The cohomological Turaev torsion. In this subsection we recall the relationship of the cohomological Turaev torsion with the homological Turaev torsion (cf. [15, Subsection 9.2]).

Let M be a closed oriented manifold of odd dimension $d = 2n-1$, where $n \geq 1$ and $\alpha : \pi_1(M) \to GL(r, \mathbb{C})$ be an acyclic representation of the fundamental group of M. Denote the cohomological Turaev torsion associated to the Euler structure ε by $\rho_\alpha(M, \varepsilon)(= \rho_{\varepsilon,o}(\nabla_\alpha))$. Then (cf. [15, p. 218, (9.2), (9.3)])

$$(6.2) \qquad\qquad \rho_\alpha(M, \varepsilon) = \frac{1}{\tau_{\alpha^*}(M, \varepsilon)},$$

where α^* is the dual representation of α. Recall that, for all $g \in \pi_1(M)$, $\alpha^*(g) = (\alpha(g)^{-1})^t$ (cf. [15, Subsection 4.1]), where t denotes the transpose of matrices. It is clear that for all $g \in \pi_1(M)$ we have $\det(\alpha(g)) \cdot \det(\alpha^*(g)) = 1$.

6.5. The Turaev torsion of lens spaces. In this subsection we compute the Turaev torsion of lens spaces. Let $L = L(m; p_1, \ldots, p_n), m \geq 3$, be the lens space. First we fix a preferred Euler structure ϵ on L. Consider the CW-decomposition $e = \{e_j\}_{j=1,\ldots,2n-1}$ of L such that the CW-decomposition e lifts to a G_m-equivariant CW-decomposition of S^{2n-1}. More precisely, for each $j = 1, \ldots, 2n-1$, let us fix the lift $\widetilde{e}_j$ of e_j to S^{2n-1} such that, for each $i = 1, \ldots, n$,

(6.3)
$$\widetilde{e}_{2i-1} = \left\{ (z_1, \ldots, z_n) \in S^{2n-1} \mid z_{i+1} = \cdots = z_n = 0, 0 < \arg z_i < 2\pi/m \right\}$$

and

$$(6.4) \qquad \widetilde{e}_{2i-2} = \left\{ (z_1, \ldots, z_n) \in S^{2n-1} \mid z_{i+1} = \cdots = z_n = 0, \arg z_i = 0 \right\}.$$

Then

$$\widetilde{e} = \left\{ g^j \cdot \widetilde{e}_{2i-1}, g^j \cdot \widetilde{e}_{2i-2} \right\}_{i=1,\ldots,n, j \in \mathbb{Z}/m\mathbb{Z}}$$

defines a G_m-equivariant CW-decomposition of S^{2n-1} with m cells in each dimension. Note that e has exactly one cell in each dimension. Then by (5.3), we have

$$(6.5) \quad g \cdot \widetilde{e}_{2i-1} = \left\{ (z_1, \ldots, z_n) \in S^{2n-1} \mid \right.$$
$$\left. z_{i+1} = \cdots = z_n = 0, 2\pi p_i/m < \arg z_i < 2\pi(p_i + 1)/m \right\}$$

and

(6.6) $\quad g \cdot \widetilde{e}_{2i-2} = \big\{ (z_1, \ldots, z_n) \in S^{2n-1} \mid$
$$z_{i+1} = \cdots = z_n = 0, \arg z_i = 2\pi p_i/m \big\}.$$

Recall that S^{2n-1} is the universal covering and also the maximal abelian covering of the lens space L, so we can consider the collection of cells $\hat{e} = \{\widetilde{e}_j\}_{1 \le j \le 2n-1}$ in S^{2n-1} as a fundamental family in S^{2n-1}. The equivalence class of this family defines an Euler structure denoted by ϵ.

In the following proposition we will compute the homological Turaev torsion $\tau_\alpha(L, \epsilon)$ of the lens space L and the preferred Euler structure ϵ by using the same argument as for the Reidemeister torsion of lens spaces; see [23, Theorem 10.6, p. 45] for the detailed computation of the Reidemeister torsion of lens spaces.

PROPOSITION 6.1. *Let $L = L(m; p_1, \ldots, p_n), m \ge 3$, be the lens space. Let $g \in \pi_1(L)$ be the generator and let $l_1, \ldots, l_n \in \mathbb{Z}/m\mathbb{Z}$ such that $l_k p_k \equiv 1 \,(\mathrm{mod}\, m)$. Let $\alpha_q : \pi_1(L) = G_m \to U(1)$ be the nontrivial unitary representation such that $\alpha_q(g) = e^{2\pi i q/m}$, $1 \le q \le m - 1$. Also let ϵ be the Euler structure defined as above. Then $H_*(C(L, \alpha_q)) = 0$ and*

(6.7) $\qquad \tau_{\alpha_q}(L, \epsilon) = \prod_{k=1}^{n} \left| e^{2\pi i q l_k/m} - 1 \right|^{-1} \cdot i^n \cdot e^{-\frac{\pi i q \sum_{k=1}^{n} l_k}{m}} \in \mathbb{C}^*/\pm .$

Proof. Assume that $\widetilde{e}_i$ is oriented for each i such that the boundary homomorphism of the chain complex $C(S^{2n-1})$ is given by (cf. (6.3), (6.6)) and recall that $l_i p_i \equiv 1 \,(\mathrm{mod}\, m)$,
$$\partial \widetilde{e}_{2i-1} = g^{l_i} \widetilde{e}_{2i-2} - \widetilde{e}_{2i-2} = (g^{l_i} - 1)\widetilde{e}_{2i-2}$$

and (cf. (6.4) and (6.5)),
$$\partial \widetilde{e}_{2i-2} = \widetilde{e}_{2i-3} + g\widetilde{e}_{2i-3} + \cdots + g^{m-1}\widetilde{e}_{2i-3} = \sum_{j=0}^{m-1} g^j \widetilde{e}_{2i-3}.$$

Since, by assumption, $\alpha_q(g) \ne 1$ and since $\alpha_q(g)^m = 1$, we have that
$$\sum_{j=0}^{m-1} \alpha_q(g)^j = \frac{\alpha_q(g)^m - 1}{\alpha_q(g) - 1} = 0.$$

Hence we have the chain complex
$$C(L, \alpha_q) = C(S^{2n-1}) \otimes_{\alpha_q} \mathbb{C} = \big(\cdots \xrightarrow{0} \mathbb{C}\widetilde{e}_{2i-1} \xrightarrow{e^{2\pi i q l_i/m} - 1} \mathbb{C}\widetilde{e}_{2i-2} \xrightarrow{0} \ldots \big).$$

It is not difficult to see that
$$H_*\big(C(L, \alpha_q)\big) = 0.$$

It follows from the definitions that

(6.8) $$\tau_{\alpha_q}(L, \epsilon) = \pm \prod_{k=1}^{n} \left(e^{2\pi i q l_k/m} - 1 \right)^{-1}.$$

We now compute the phase θ of $\tau_{\alpha_q}(L, \epsilon)$. Set $r = |\tau_{\alpha_q}(L, \epsilon)|$. Then $\tau_{\alpha_q}(L, \epsilon) = re^{i\theta}$ and

(6.9) $$\tau_{\alpha_q}(L, \epsilon) \Big/ \overline{\tau_{\alpha_q}(L, \epsilon)} = e^{2i\theta}.$$

A simple calculation using (6.8) shows that

(6.10) $$\tau_{\alpha_q}(L, \epsilon) \Big/ \overline{\tau_{\alpha_q}(L, \epsilon)} = \prod_{k=1}^{n} \frac{\left(e^{2\pi i q l_k/m} - 1 \right)^{-1}}{\left(e^{-2\pi i q l_k/m} - 1 \right)^{-1}}$$
$$= (-1)^n \cdot e^{-2\pi i q \sum_{k=1}^{n} l_k/m}.$$

From (6.9) and (6.10), we obtain

$$\theta = n\pi/2 - \pi q \sum_{k=1}^{n} l_k/m.$$

Hence the proposition follows.   □

Note that the first component $\prod_{k=1}^{n} |e^{2\pi i q l_k/m} - 1|^{-1}$ of (6.7) is the Reidemeister torsion of L.

Now we compute the cohomological Turaev torsion of lens spaces. We will follow the same notations as in Proposition 6.1.

PROPOSITION 6.2.   *Let $\rho_{\alpha_q}(L, \epsilon)$ denote the cohomological Turaev torsion of the lens space L associated to the preferred Euler structure ϵ. Then*

$$\rho_{\alpha_q}(L, \epsilon) = \prod_{k=1}^{n} \left| e^{2\pi i q l_k/m} - 1 \right| \cdot i^n \cdot e^{-\frac{\pi i q \sum_{k=1}^{n} l_k}{m}} \in \mathbb{C}^*/\pm.$$

Proof. The proposition follows from (6.2) and Proposition 6.1.   □

The following theorem gives the cohomological Turaev torsion of a lens space for an arbitrary Euler structure ε. Recall that the cardinality of the set of the Euler structures $\mathrm{Eul}(L)$ and the cardinality of $H_1(L)$ are the same and equal to m.

THEOREM 6.3.   *Let $\hat{e}$ be a fundamental family of the preferred Euler structure ϵ (cf. Proposition 6.1), and let ε be the Euler structure represented by a fundamental family $\hat{e}'$. Then there exists $s \in \{0, \ldots, m-1\}$, such that $\hat{e}'/\hat{e} = g^s$ (cf. Subsection 6.3), and the cohomological Turaev torsion of lens space L associated to the Euler structure ε is given by*

(6.11) $$\rho_{\alpha_q}(L, \varepsilon) = \prod_{k=1}^{n} \left| e^{2\pi i q l_k/m} - 1 \right| \cdot i^n \cdot e^{\frac{\pi i q (2s - \sum_{k=1}^{n} l_k)}{m}} \in \mathbb{C}^*/\pm.$$

Proof. The theorem follows easily from Proposition 6.2 and the following property of the Turaev torsion (cf. [15, (9.4)]):

$$\rho_{\alpha_q}(L, \hat{e}') = \pm \alpha_q(\hat{e}'/\hat{e}) \rho_{\alpha_q}(L, \hat{e}). \qquad \square$$

6.6. Dependence of the constant θ^C on the representation. An example. Theorem 3.3 does not give any information about the dependence of the constant θ^C on the connected component C. In this subsection we use the results of previous subsection to study this dependence in the case of lens spaces. Our goal is to show that, in general, $\theta^C = \theta^\alpha$ does depend on α, thus providing a positive answer to Question 1 of the introduction.

Let α_q be the representation as before and $L = L(5; 1, 1)$ be the lens space. A direct computation using Theorem 3.4 and Proposition 5.2 shows that

$$\theta^{\alpha_1} = \theta^{\alpha_4} = -3\pi/10 \qquad \text{mod} \quad \pi \mathbb{Z}$$

and

$$\theta^{\alpha_2} = \theta^{\alpha_3} = -7\pi/10 \qquad \text{mod} \quad \pi \mathbb{Z}.$$

Therefore we conclude that *the constant θ^{α_q} depends on the representation α_q*.

6.7. The ratio of the refined analytic torsion and the Turaev torsion. An example. It is natural to ask for which representations α one can find an Euler structure ε and a cohomological orientation o such that $\rho_{\text{an}}(\nabla) = \rho_{\varepsilon,o}(\nabla)$. In this subsection we use Theorem 5.3 and Theorem 6.3 to show that the refined analytic torsion and the cohomological Turaev torsion of the five-dimensional lens space $L(3; 1, 1, 1)$ are never equal.

We compute the ratio R of the two torsions of the lens space $L(3; 1, 1, 1)$ for all nontrivial representations (i.e., $q = 1, 2$) and all Euler structures (i.e., $s = 0, 1, 2$); see Theorem 6.3 for the definition of s. A direct computation gives the following table for the ratio R.

	$s = 0$	$s = 1$	$s = 2$
$q = 1$	$R = \pm e^{\frac{5\pi i}{9}}$	$R = \pm e^{\frac{8\pi i}{9}}$	$R = \pm e^{\frac{2\pi i}{9}}$
$q = 2$	$R = \pm e^{\frac{-5\pi i}{9}}$	$R = \pm e^{\frac{-8\pi i}{9}}$	$R = \pm e^{\frac{-2\pi i}{9}}$

We conclude that *for all Euler structures ε on $L(3; 1, 1, 1)$ and all representations α of the fundamental group of $L(3; 1, 1, 1)$, the refined analytic torsion and the Turaev torsion are not equal.* This provides a partial answer to Question 2 of the introduction.

Added in proof. Since this paper was accepted for publication, additional progress has been made. Burghelea and Haller (D. Burghelea and S. Haller, *Complex valued Ray-Singer torsion, II*, `arXiv:math.DG/0610875`) proved Conjecture 4.1 up to sign. Independently and at the same time Su and Zhang (G. Su and W. Zhang, *A Cheeger-Mueller theorem for symmetric bilinear torsion*, `arXiv:math.DG/0610577`) proved Conjecture 4.1 in full generality. A generalization of Conjecture 4.1 is proposed in the final revision of [8].

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