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SELF-SIMILAR SETS IN DOUBLING SPACES

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Abstract. We extend the well-known Moran-Hutchinson Theorem, which says that under the open set condition the Hausdorff dimension of self-similar sets coincides with the similarity dimension, to the more general setting of doubling metric spaces and prove its reverse implication. We also provide examples of iterated function systems in doubling metric spaces where our results apply.

1. Introduction

A large class of fractals containing many well-known examples may be defined by a family of similarity contractions $\{f_1, \ldots, f_N\}$ on a metric space (X, d) as the unique nonempty compact invariant set K with respect to $\{f_i\}_{1 \leq i \leq N}$, i.e., $K = \bigcup_{i=1}^{N} f_i(K)$. If K has fractal properties (which is mostly the case), it is called a *self-similar* fractal defined by the iterated function system (IFS) $\{f_i\}_{1 \leq i \leq N}$. It has been known for a long time that the similarity dimension and the Hausdorff dimension of K coincide if the 'pieces' $f_i(K)$ are pairwise disjoint. Moran [20] showed that this result remains true for *Euclidean spaces* if the overlap of the 'pieces' $f_i(K)$ is not 'too strong'. The result of Moran was put into a larger context by Hutchinson [17], who introduced the notion of the *open set condition*. Schief [21] showed conversely that if $0 < \mathcal{H}^\alpha(K) < \infty$, where α is the similarity dimension of $K \subseteq \mathbb{R}^n$, then the open set condition holds.

Although the open set condition is a theoretically useful criterion to decide whether or not the Hausdorff dimension and the similarity dimension coincide, it may be hard to verify it. There were therefore attempts to find conditions equivalent to the open set condition which are easier to verify. Kigami [19] gave another, analytical condition under which the two dimensions coincide, which holds in the general metric setting.

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In this paper, we extend the results by Schief and Kigami to the more general setting of *doubling* metric spaces. Note that Schief [22] showed that in the general context of complete metric spaces the Euclidean results are no longer true. However, we prove in the presence of the doubling property the equivalence of the open set condition, Schief's measure-theoretic condition and Kigami's analytical condition. Motivation for extending these results to doubling metric spaces is provided by recent works on IFS in the setting of the Heisenberg group [6], [7], [8] where our results apply. We also give further examples of non-Euclidean doubling metric spaces.

The structure of the paper is as follows: In Section 2, we introduce the notation and recall some well-known facts about metric spaces and about self-similar fractals. In Section 3, we state and prove our main theorem. The next two sections provide examples of iterated function systems in non-Euclidean doubling metric spaces: In Section 4, we discuss the case of the (first) Heisenberg group and in Section 5, we present a general construction for induced doubling metric spaces. In Section 6 we give final remarks and formulate some questions arising from this work.

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2. Background and notation

Let (X, d) denote the metric space X equipped with the metric d. Given a subset $A \subseteq X$ we write $A^c := X \setminus A$ for the complement and $\overline{A}$ for the closure of A. $\operatorname{diam}(A) := \sup\{d(x, y) : x, y \in A\}$ denotes the diameter and $\#A$ the cardinality of A. The (open) ball with centre $x \in X$ and radius $r \in \mathbb{R}^+$ is denoted by $B(x, r) := \{y \in X : d(x, y) < r\}$ and the r-neighbourhood of A by $U(A, r) := \{y \in X : \exists x \in A \text{ with } d(x, y) < r\}$. For the distance between $x \in X$ and $A \subseteq X$ or between $A, B \subseteq X$, we write $D(x, A) := \inf\{d(x, y) : y \in A\}$ and $D(A, B) := \inf\{d(x, y) : x \in A, y \in B\}$, respectively. Observe that $D(A, B)$ is different from the usual Hausdorff distance $d_H(A, B)$ between non-empty compact subsets $A, B \subseteq X$ defined as

$$d_H(A, B) := \inf\{\delta : A \subseteq U(B, \delta), B \subseteq U(A, \delta)\}.$$

A collection $\{U_i\}_{i \in \mathbb{N}}$ with $0 < \operatorname{diam}(U_i) \leq \delta$ for each i and with $A \subseteq \bigcup_{i=1}^{\infty} U_i$ is called a *δ-covering* of $A \subseteq X$. We denote the s-dimensional Hausdorff measure of $A \subseteq X$ by

$$\mathcal{H}^s(A) := \lim_{\delta \to 0} \inf \left\{ \sum_{i=1}^{\infty} \operatorname{diam}(U_i)^s : \{U_i\} \ \delta\text{-covering of } A \right\}$$

and its Hausdorff dimension by

$$\dim_H(A) := \inf\{s : \mathcal{H}^s(A) = 0\} = \sup\{s : \mathcal{H}^s(A) = \infty\}.$$

For a more detailed discussion of the Hausdorff measure and dimension, see, for example, Falconer [11]. Note that the Hausdorff measure and the Hausdorff dimension depend on a metric d. In this sense, if we need to specify which metric we are looking at, we will use the notation of $\dim_H(A, d)$ instead of $\dim_H(A)$.

In the increasing literature of analysis in metric spaces, doubling spaces play an important role as shown by the celebrated theorem of Assouad [3] (see also [16, Theorem 12.1]), stating that if (X, d) is doubling and $0 < s < 1$, then the 'snowflaked' space (X, d^s) is bilipschitz equivalent to a subset of some $\mathbb{R}^p$, where p and the bilipschitz constant depend only on s and the doubling constant.

A metric space (X, d) is called *doubling*, if and only if there exists a number $C \in \mathbb{N}$ such that for all $x \in X$ and all $r > 0$ there exist $\{x_1, \ldots, x_C\} \subseteq X$ such that

$$B(x, r) \subseteq \bigcup_{i=1}^{C} B\left(x_i, \frac{r}{2}\right).$$

Note that C, called the *doubling constant* of (X, d), is universal for X and not dependent on the choice of x or r.

A mapping $f_i : X \to X$ is called a *similarity contraction* with *contraction ratio* r_i if and only if $d(f_i(x), f_i(y)) = r_i \cdot d(x, y)$ holds for all $x, y \in X$. Given a family $\{f_1, \ldots, f_N\}$ of similarity contractions, define $r_{\min} := \min_{1 \le i \le N}\{r_i\}$, $r_{\max} := \max_{1 \le i \le N}\{r_i\}$ and $f(A) := \bigcup_{i=1}^{N} f_i(A)$ for any $A \subseteq X$ and denote the unique compact invariant set with respect to $\{f_i\}_{1 \le i \le N}$ by K. Formally,

$$K = f(K) = \bigcup_{i=1}^{N} f_i(K).$$

The similarity dimension α of the invariant set K is the unique solution of the equation

$$\sum_{i=1}^{N} r_i^{\alpha} = 1. \tag{2.1}$$

Our main objective is to give conditions under which the similarity dimension and the Hausdorff dimension coincide. Such a condition is the so-called *open set condition*. It holds for $\{f_i\}_{1 \le i \le N}$ if and only if there exists a bounded non-empty open set $O \subseteq X$ such that $\bigcup_{i=1}^{N} f_i(O) \subseteq O$ and $f_i(O) \cap f_j(O) = \emptyset$ for $i \neq j$.

We conclude this part by recalling some notations of symbolic dynamics, following the exposition of Kigami [19]. For $N \in \mathbb{N}$ and for $k \geq 1$, define

$$W_k^N := \{1, \dots, N\}^k = \{w_1 \dots w_k : w_j \in \{1, \dots, N\}, 1 \leq j \leq k\},$$

called the set of *words of length* k with symbols $\{1, \dots, N\}$. Also, for $k = 0$, set $W_0^N := \{\emptyset\}$ and call $\emptyset$ the *empty word*. Moreover, define

$$W_*^N := \bigcup_{k=0}^{\infty} W_k^N,$$

the *set of finite sequences* with symbols $\{1, \dots, N\}$. To ease notation, we write W_k and W_* instead of W_k^N and W_*^N, respectively.

For $A \subseteq X$, $\{f_i\}_{1 \leq i \leq N}$ and $w \in W_*$, define

$$A_w := f_w(A) := f_{w_1} \circ \dots \circ f_{w_k}(A)$$

and $r_w := r_{w_1} \cdots r_{w_k}$. If $A = K$, the K_w are the scaled 'copies' of K. The diameters of the K_w may vary strongly, especially if the difference between $r_{\min}$ and $r_{\max}$ or k are large. In order to obtain a grouping of the K_w with approximately the same size, we define for $0 < a < 1$

$$\Lambda(a) := \{w : w = w_1 \dots w_k \in W_*, r_{w_1 \dots w_{k-1}} > a \geq r_w\}.$$

Two words $u, v \in W_*$ are called incomparable if and only if u is not an initial word of v or vice versa, i.e., if there exists no $w \in W_*$ such that $u = vw$ or $v = uw$, where $uv := u_1 \dots u_k v_1 \dots v_l$ denotes the concatenation of u and v. Note that any two words $u, v \in \Lambda(a)$ are incomparable by the definition of $\Lambda(a)$.

3. The theorem

THEOREM 3.1. *Let (X, d) be a complete doubling metric space and let further $K \subseteq X$ be the invariant set with respect to the IFS of similarity contractions $\{f_i\}_{1 \leq i \leq N}$. Then the following statements are equivalent.*

(1) *The open set condition (OSC) holds for $\{f_i\}_{1 \leq i \leq N}$.*
(2) *Kigami's condition (KC) holds, i.e., there are positive constants c_1, c_2, c_* and M such that*

$$\operatorname{diam}(K_w) \leq c_1 \cdot r_w$$

for all $w \in W_*$ and

$$\#\{w : w \in \Lambda(a), D(x, K_w) \leq c_2 \cdot a\} \leq M$$

for any $x \in K$ and any $a \in (0, c_*)$.
(3) $0 < \mathcal{H}^\alpha(K) < \infty$, *where α is the similarity dimension from (2.1).*

REMARK 3.2. Note that, by Theorem 3.1, if (i) or (ii) hold, then $\dim_H(K) = \alpha$.

The implication (ii) $\Rightarrow$ (iii) was proved by Kigami [18, Theorem 2.4], [19, Theorem 1.5.7]. We start with (iii) $\Rightarrow$ (i), following the idea of Schief [21], who proved the implication in the setting of Euclidean spaces. We mention that this implication was already proved by Schief in [22]. We include nevertheless the proof which becomes simpler in the context of doubling spaces. Finally, we prove (i) $\Rightarrow$ (ii). We note here that Kigami [18, Proposition 2.8], [19, Proposition 1.5.8] proved the implication (i) $\Rightarrow$ (ii) in the context of Euclidean spaces. Our result is an extension of Kigami's statement to the more general setting of doubling spaces.

LEMMA 3.3. *Let (X,d) be a doubling metric space with doubling constant C, and let $\varrho < r$. Then there exists a constant $C' = C'(C, \frac{r}{\varrho}) \in \mathbb{N}$ (independent of x and r) such that the number of disjoint balls $B(y_i, \varrho) \subseteq B(x,r) \subseteq X$ is bounded by C'.*

Proof. Assume first that $\varrho \geq r/2$. Consider any collection of disjoint balls $B(x_i, \varrho)$ with centres in $B(x,r)$ and denote it with $\mathcal{B} := \{B_i\}_{i \in \mathcal{I}}$. Applying the doubling property of X twice, we obtain balls $B(y_i, \frac{r}{4})$, $1 \leq i \leq C^2$, such that

$$B(x,r) \subseteq \bigcup_{j=1}^{C^2} B\left(y_j, \frac{r}{4}\right).$$

Therefore for all $i \in \mathcal{I}$ there exists an index j such that $x_i \in B(y_j, \frac{r}{4})$ and thus $B(y_j, \frac{r}{4}) \subseteq B_i$. As the elements of $\mathcal{B}$ are pairwise disjoint, we have for $C' := C^2$ that $\#\mathcal{I} \leq C' < \infty$.

Now let $\frac{r}{2^n} \leq \varrho < \frac{r}{2^{n-1}}$. Then the same argument as above works with C^{n+1} balls $B(y_i, \frac{r}{2^{n+1}})$, $1 \leq i \leq C^{n+1}$. This completes the proof. □

LEMMA 3.4. *Let α denote the similarity dimension of K. Then:*

(i) *For measurable subsets $A \subseteq K$ the Hausdorff measure $\mathcal{H}^\alpha(A)$ coincides with the outer measure*

$$\mathcal{H}^\alpha_*(A) := \inf \left\{ \sum_{i=1}^{\infty} \operatorname{diam}(U_i)^\alpha : \{U_i\} \text{ open covering of } A \right\}.$$

(ii) *$\mathcal{H}^\alpha(f_v(K) \cap f_w(K)) = 0$ for incomparable $v, w \in W_*$.*

A proof of this can be found in [5, Proposition 3]. The rest of the preparation follows Schief [21] adapted to the general setting of a doubling metric space. We give the whole proof for the sake of completeness.

LEMMA 3.5. *Suppose $\mathcal{H}^\alpha(K) > 0$ and let $c > 0$. Then there exists $n \geq 1$ and open sets $\{U_1, \ldots, U_n\}$ such that*

$$U := \bigcup_{i=1}^{n} U_i \supseteq K$$

and

$$\sum_{i=1}^{n} \operatorname{diam}(U_i)^\alpha \le (1+c^\alpha)\cdot\mathcal{H}^\alpha(K).$$

Proof. This follows directly from Lemma 3.4 (i) and the compactness of K. $\square$

LEMMA 3.6. *Let $c>0$ and U be the open set from Lemma 3.5, and let $\delta := D(K,U^c)$. Then for all incomparable $u,v\in W_*$ such that $c\cdot r_u < r_v$ holds*

$$d_H(K_u,K_v) \ge \delta\cdot r_u.$$

Proof. Assume that $d_H(K_u,K_v) < \delta\cdot r_u$. As a consequence of the relation $D(K_u, f_u(U)^c) = r_u\cdot D(K,U^c) = \delta\cdot r_u$, we have $K_v \subseteq U(K_u,\delta\cdot r_u) \subseteq f_u(U)$. This implies

$$\begin{aligned}\mathcal{H}^\alpha(K)\cdot r_u^\alpha\cdot(1+c^\alpha) &< \mathcal{H}^\alpha(K)\cdot(r_u^\alpha + r_v^\alpha) = \mathcal{H}^\alpha(K_u)+\mathcal{H}^\alpha(K_v)\\ &= \mathcal{H}^\alpha(K_u\cup K_v) \le \sum\nolimits_{i=1}^{n} \operatorname{diam}(f_u(U_i))^\alpha\\ &= \sum\nolimits_{i=1}^{n} r_u^\alpha\cdot\operatorname{diam}(U_i)^\alpha \le \mathcal{H}^\alpha(K)\cdot r_u^\alpha\cdot(1+c^\alpha),\end{aligned}$$

where the second to last inequality in the proof above is due to Lemma 3.4 (i) and the second equality follows from Lemma 3.4 (ii). $\square$

The key step in the proof is the following proposition.

PROPOSITION 3.7. *For $v,w\in W_*$ and $0<\varepsilon<1/3$, define $G_w := U(K_w,\varepsilon r_w)$, $I(w) := \{v\in\Lambda(\operatorname{diam}(G_w)) : K_v\cap G_w \neq \emptyset\}$ and $\gamma := \sup_w \#I(w)$. Then $\gamma<\infty$.*

Proof. Without loss of generality we may assume $\operatorname{diam}(K)$ to be small enough to ensure $\operatorname{diam}(G_w)\le 1$ for all $w\in W_*$. Since $r_v > a\cdot r_{\min} \ge r_u\cdot r_{\min}$ holds for $u,v\in\Lambda(a)$, we may and do apply Lemma 3.6 for $c := r_{\min}$ to get $\delta>0$ such that $d_H(K_u,K_v)\ge\delta\cdot r_u$ for arbitrary a and different $u,v\in\Lambda(a)$.

This in turn implies the existence of $x\in K$ such that

$$d(f_u(x), f_v(x)) \ge \delta\cdot r_u. \tag{3.1}$$

Given another point $y\in K$ such that $d(x,y) < \delta\cdot r_{\min}/3$, it follows immediately that $d(f_u(x),f_u(y)) < r_u\cdot\delta\cdot r_{\min}/3 < \delta\cdot r_u/3$ and $d(f_v(x),f_v(y)) < r_v\cdot\delta\cdot r_{\min}/3 < \delta\cdot r_u/3$. Applying the triangle inequality shows that (3.1) holds also for y with $\delta/3$ instead of δ.

As K is a compact subset of the doubling metric space (X,d) (and thus contained in $B(x,\operatorname{diam}(K))$ for some $x\in K$), by Lemma 3.3 there exists a (minimal) number $p = p(C,\operatorname{diam}(K)/\delta)\in\mathbb{N}$ such that

$$Y := \{y_i\}_{1\le i\le p} \subseteq K \text{ and } \bigcup_{i=1}^{p} B\left(y_i,\frac{\delta}{3}\right) \supseteq K.$$

Now let $d_w := \operatorname{diam}(G_w)$. Then for all $w \in W_*$ and any two different $u, v \in I(w)$ there exists $y_i \in Y$ such that

$$d(f_u(y_i), f_v(y_i)) \geq \frac{\delta \cdot r_u}{3} \geq \frac{\delta \cdot d_w \cdot r_{\min}}{3}. \tag{3.2}$$

For each $y_i \in Y$, $1 \leq i \leq p$, we define $I_i(w) \subseteq I(w) \times I(w)$ as the subset of all pairs $(u, v) \in I(w) \times I(w)$ such that (3.2) is fulfilled. As seen above, each pair $(u, v) \in I(w) \times I(w) \setminus \triangle(I(w))$ is contained in at least one of the $I_i(w)$, where $\triangle(I(w)) := \{(v, v) : v \in I(w)\}$ is the diagonal of $I(w) \times I(w)$. This implies that

$$\bigcup_{i=1}^{p} I_i(w) = I(w) \times I(w) \setminus \triangle(I(w)).$$

To finish the proof, we show that the $\#I_i(w)$ are all bounded by a constant independent of w: First, note that by (3.2)

$$B\left(f_u(y_i), \frac{\delta \cdot d_w \cdot r_{\min}}{6}\right) \cap B\left(f_v(y_i), \frac{\delta \cdot d_w \cdot r_{\min}}{6}\right) = \emptyset \tag{3.3}$$

for all $u, v \in I_i(w)$ and

$$B\left(f_v(y_i), \frac{\delta \cdot d_w \cdot r_{\min}}{6}\right) \subseteq B(x, 3 \cdot d_w) \tag{3.4}$$

for some $x \in G_w$ and for any $v \in I_i(w)$. As the ratio between the radii of the balls in (3.4) is always the same for arbitrary $w \in W_*$ (namely $\frac{18}{\delta \cdot r_{\min}}$), Lemma 3.3 together with (3.3) and (3.4) implies the existence of a number $C' = C'(C, \frac{18}{\delta \cdot r_{\min}}) \in \mathbb{N}$ such that $\#I_i(w) \leq C'$ for all $1 \leq i \leq p$. Hence it follows that

$$\begin{aligned}\gamma = \sup\nolimits_w \#I(w) &\leq \sup\nolimits_w \#(I(w) \times I(w) \setminus \triangle(I(w))) \\ &\leq \sum\nolimits_{i=1}^{p} \sup\nolimits_w \#I_i(w) \leq p \cdot C' < \infty,\end{aligned}$$

proving the claim. $\square$

Lemma 3.8. *Let $w \in W_*$ such that $\gamma = \#I(w)$. Then for arbitrary $v \in W_*$*

$$I(vw) = \{vu : u \in I(w)\}.$$

Proof. As γ is maximal by definition, it is sufficient to show the inclusion $\{vu : u \in I(w)\} \subseteq I(vw)$. Let $u \in I(w)$ arbitrary. From $\emptyset \neq K_u \cap G_w$ it follows that

$$\begin{aligned}\emptyset \neq f_v(K_u \cap G_w) &= f_v(K_u) \cap f_v(G_w) = K_{vu} \cap f_v(U(K_w, \varepsilon \cdot r_w)) \\ &= K_{vu} \cap U(K_{vw}, \varepsilon \cdot r_{vw}) = K_{vu} \cap G_{vw},\end{aligned}$$

which implies that $vu \in I(vw)$. $\square$

DEFINITION 3.9. For $w \in W_*$ such that $\gamma = \#I(w)$ we define

$$G_w^* := U\left(K_w, \frac{\varepsilon \cdot r_w}{2}\right)$$

and

$$U := \bigcup_{v \in W_*} G_{vw}^*.$$

As we shall see, the set U as defined above will serve as the separating open set for the open set condition.

Proof of Theorem 3.1. (iii) $\Rightarrow$ (i): Assume $0 < \mathcal{H}^\alpha(K) < \infty$ and let U be the open set from Definition 3.9. To prove that U satisfies the required OSC, we check first that $f_i(U) \subseteq U$ for all $1 \leq i \leq N$. From $K_w \subseteq G_w^* \subseteq U$ it follows that for each $1 \leq i \leq N$

$$f_i(U) = f_i\left(\bigcup_{v \in W_*} G_{vw}^*\right) = \bigcup_{v \in W_*} f_i(G_{vw}^*) = \bigcup_{v \in W_*} G_{ivw}^* \subseteq U.$$

To check that $f_i(U) \cap f_j(U) = \emptyset$ for all $i \neq j$ let $v \in W_*$ for which $i \neq v_1$. Since K_i is covered by $\{K_u : u \in \Lambda(\mathrm{diam}(G_{vw})), u_1 = i\}$, it follows from Lemma 3.6 that

$$D(K_{vw}, K_i) \geq \varepsilon \cdot r_{vw}. \tag{3.5}$$

Now assume that there are $i \neq j$ such that $f_i(U) \cap f_j(U) \neq \emptyset$. Hence there are $u, v \in W_*$ such that $G_{iuw}^* \cap G_{jvw}^* \neq \emptyset$ and $r_{iuw} \geq r_{jvw}$. If x is an element of this intersection, there exist $x_1 \in K_{iuw}$ and $x_2 \in K_{jvw}$ such that $d(x, x_1) < \frac{\varepsilon \cdot r_{iuw}}{2}$ and $d(x, x_2) < \frac{\varepsilon \cdot r_{jvw}}{2}$ which implies $d(x_1, x_2) < \varepsilon \cdot r_{iuw}$. It follows that

$$D(K_{iuw}, K_j) < \varepsilon \cdot r_{iuw},$$

which contradicts (3.5).

Note that U satisfies the so-called strong OSC (SOSC) in the sense that $K \cap U \neq \emptyset$. This follows directly from $K_w \subseteq K$, $K_w \subseteq G_w^* \subseteq U$ and $K_w \neq \emptyset$.

(i) $\Rightarrow$ (ii): Let $O \subseteq X$ be an open set for which the open set condition holds. For any compact set $A \subseteq X$ the sequence $\{f^n(A)\}_{n \in \mathbb{N}}$ converges to K in the Hausdorff metric [11], [17]. Since O is bounded and X is doubling, it follows that $\overline{O}$ is compact. Letting $A = O$, from the above consideration we obtain $K \subseteq \overline{O}$, and hence $K_w \subseteq \overline{O}_w$ for any $w \in W_*$. Without loss of generality, we may assume that $\mathrm{diam}(O) \leq 1$ and we prove Kigami's condition with $c_1 = c_2 = 1$. Then, for all $w \in W_*$,

$$\mathrm{diam}(K_w) \leq \mathrm{diam}(\overline{O}_w) \leq r_w,$$

which is the first part of Kigami's condition. To check the second one, set $\Lambda_{a,x} := \{w : w \in \Lambda(a), d(x, K_w) \leq a\}$. By the definition of $\Lambda(a)$, $r_w \leq a$ holds and thus

$$\bigcup_{w\in\Lambda_{a,x}} O_w \subseteq B(x, 2\cdot a).$$

The openness of O implies the existence of $y \in O$ and $\varepsilon > 0$ such that $B(y, \varepsilon) \subseteq O$. Therefore for all $w \in \Lambda_{a,x}$

$$f_w(B(y,\varepsilon)) = B(f_w(y), \varepsilon\cdot r_w) \subseteq O_w$$

and

$$B(f_w(y), \varepsilon\cdot r_{\min}\cdot a) \subseteq B(f_w(y), \varepsilon\cdot r_w) \subseteq O_w \subseteq B(x, 2\cdot a). \tag{3.6}$$

By the open set condition it follows that $O_v \cap O_w = \emptyset$ for different $v, w \in \Lambda_{a,x}$, which implies

$$B(f_v(y), \varepsilon\cdot r_{\min}\cdot a) \cap B(f_w(y), \varepsilon\cdot r_{\min}\cdot a) = \emptyset. \tag{3.7}$$

Now by Lemma 3.3 together with (3.6) and (3.7), it follows that there exists $M \in \mathbb{N}$ such that

$$\#\Lambda_{a,x} \leq \#B(f_w(y), \varepsilon\cdot r_{\min}\cdot a) \leq M,$$

which completes the proof of Theorem 3.1. □

We conclude our paper with two examples of iterated function systems defined on doubling metric spaces in which our results apply. In Section 4, we recall that the (first) Heisenberg group, which is the $\mathbb{R}^3$ with a different group operation, equipped with the Heisenberg metric [4], is doubling. This is an example of a more general class of Carnot groups and Carnot-Carathéodory spaces [15] which are also doubling (or locally doubling, respectively). In Section 5, we present a method to induce a new metric from an existing one and state a condition under which the induced metric is doubling, provided the original one has this property.

4. The Heisenberg group

The (first) Heisenberg group $(\mathbb{H}, *) = (\mathbb{H}^1, *)$ has the space $\mathbb{R}^3$ as its underlying space and the group operation is given by

$$(x_1, x_2, t) * (x_1', x_2', t') := (x_1 + x_1', x_2 + x_2', t + t' + 2(x_2\cdot x_1' - x_2'\cdot x_1)).$$

The Heisenberg distance $d_{\mathbb{H}}$ of two points $q_1, q_2 \in \mathbb{H}$ is defined by

$$d_{\mathbb{H}}(q_1, q_2) := \|q_1^{-1} * q_2\|_{\mathbb{H}},$$

where the Heisenberg norm $\|q\|_{\mathbb{H}}$ is given by

$$\|q\|_{\mathbb{H}} := ((x_1^2 + x_2^2)^2 + t^2)^{\frac{1}{4}}$$

for $q = (x_1, x_2, t) \in \mathbb{H}$. For fixed $q_0 \in \mathbb{H}$ the (left) translation by q_0 is the group automorphism

$$\tau_{q_0} : \mathbb{H} \to \mathbb{H}, \quad \tau_{q_0}(q) := q_0 * q,$$

and for fixed $\alpha \in \mathbb{R}^+$ the dilation by α is the group automorphism

$$\delta_\alpha : \mathbb{H} \to \mathbb{H}, \quad \delta_\alpha(q) = (\alpha \cdot x_1, \alpha \cdot x_2, \alpha^2 \cdot t),$$

where $q = (x_1, x_2, t) \in \mathbb{H}$.

It is not hard to see that $d_{\mathbb{H}}$ is a homogeneous, left invariant metric on $\mathbb{H}$, i.e., that $d_{\mathbb{H}}(\tau_{q_0}(q_1), \tau_{q_0}(q_2)) = d_{\mathbb{H}}(q_1, q_2)$ and $d_{\mathbb{H}}(0, \delta_\alpha(q)) = \alpha \cdot d_{\mathbb{H}}(0, q)$ for any $q_0, q_1, q_2, q \in \mathbb{H}$ and $\alpha \in \mathbb{R}^+$.

Let $|\cdot|$ denote the usual Euclidean volume measure in $\mathbb{R}^3$. It follows immediately from the left invariance and the homogeneity that there exists a constant $C \in \mathbb{R}^+$ such that for an arbitrary ball $B(p, r)$, $p \in \mathbb{H}$, $r \in \mathbb{R}^+$, in the Heisenberg metric we have

$$|B(p, r)| = C \cdot r^4.$$

This implies that the Heisenberg group equipped with the Heisenberg metric $(\mathbb{H}, d_{\mathbb{H}})$ is a doubling metric space and our results from the previous section apply.

Let us mention at this point that, while the Euclidean and Heisenberg metrics generate the same topology, they are not bilipschitz equivalent. The Hausdorff dimension of a given set with respect to the Euclidean metric is in general different from the Hausdorff dimension of the same set with respect to the Heisenberg metric. The whole space, for example, has Hausdorff dimension 4 and the vertical axis has Hausdorff dimension 2 with respect to the Heisenberg metric. However, the Hausdorff dimension of an arbitrary set does not always increase like that. It is in fact an interesting problem (see [7]) to study the relation between the two Hausdorff dimensions. IFS and their invariant sets in the Heisenberg group have been studied in recent papers [6], [8]. In what follows we apply Theorem 3.1 to obtain a simpler proof of the dimension formula for self-similar IFS from [8].

Recall [6] that if $F : \mathbb{R}^3 \to \mathbb{R}^3$ is an affine map of the form

$$F(x, t) := (Ax + t \cdot a + b, d^T x + c \cdot t + \tau), \tag{4.1}$$

where $x = (x_1, x_2)$, A is a real 2×2 matrix, $a, b, d \in \mathbb{R}^2$ and $c, \tau \in \mathbb{R}$, then F is Lipschitz with respect to the Heisenberg metric $d_{\mathbb{H}}$ if and only if the relations

$$a = (0, 0)\ , \quad d = -2A^T Jb \quad \text{and} \quad c = \det(A) \tag{4.2}$$

hold, where $J : \mathbb{R}^2 \to \mathbb{R}^2$ denotes the map $J(x_1, x_2) := (-x_2, x_1)$. The mapping F is a similarity with respect to the Heisenberg metric if and only if the above relations hold and A is a similarity matrix of $\mathbb{R}^2$, i.e.,

$$A^T A = c^2 I \quad \text{or} \quad A^T A = -c^2 I,$$

where $c = \det(A)$ and I is the identity matrix of dimension 2.

Take now a self-similar IFS $\{f_i\}_{1 \le i \le N}$ in $\mathbb{R}^2$, where $f_i(x) = A_i x + b_i$ have similarity ratios r_i for $1 \le i \le N$. Consider the lifted IFS on the Heisenberg group $\{F_i\}_{1 \le i \le N}$ given by (4.1) and (4.2). Then the new system is again self-similar with respect to the Heisenberg metric $d_{\mathbb{H}}$ with the same similarity ratios r_i. The following statement is an immediate consequence of Theorem 3.1.

THEOREM 4.1. *Suppose that $\{F_i\}_{1\leq i\leq N}$ is an IFS of Heisenberg similarities with ratios r_i as described above which satisfies the OSC. Then the Hausdorff dimension of the invariant set $K_{\mathbb{H}}$ with respect to the Heisenberg metric is*

$$\dim_H(K_{\mathbb{H}}, d_{\mathbb{H}}) = \alpha,$$

where α is the similarity dimension of $\{F_i\}_{1\leq i\leq N}$ defined by $\sum_{i=1}^N r_i^\alpha = 1$.

As an interesting application of the above result we obtain a dimension formula for certain self-affine IFS in the Euclidean space. To do this, observe that $\{F_i\}_{1\leq i\leq N}$ can also be viewed as an IFS in $\mathbb{R}^3$. Note, however, that the new system is no longer self-similar with respect to the Euclidean metric but merely self-affine. Dimension formulae for self-affine IFS were studied by Falconer [10], [12]. In our setting, we can also obtain a dimension formula for $K_{\mathbb{H}}$ (with respect to the Euclidean metric d_E), which is much simpler than Falconer's result, as follows.

COROLLARY 4.2. *Suppose that $\{f_i\}_{1\leq i\leq N}$ is a planar IFS of similarities with ratios r_i satisfying the OSC. Let $\{F_i\}_{1\leq i\leq N}$ be a lift of $\{f_i\}_{1\leq i\leq N}$ satisfying (4.1) and (4.2) whose invariant set is $K_{\mathbb{H}}$. Then we have*

$$\dim_H(K_{\mathbb{H}}, d_E) = \dim_H(K_{\mathbb{H}}, d_{\mathbb{H}}) = \alpha,$$

where $\sum_{i=1}^N r_i^\alpha = 1$.

Proof. Denote by $K \subseteq \mathbb{R}^2$ the invariant set of $\{f_i\}_{1\leq i\leq N}$. Due to the OSC, the Hausdorff dimension of K is equal to its similarity dimension α.

Furthermore, we use the fact that the OSC is inherited from $\{f_i\}_{1\leq i\leq N}$ to the lifted IFS $\{F_i\}_{1\leq i\leq N}$ [6, Proposition 3.14]. Applying Theorem 4.1 we obtain $\dim_H(K_{\mathbb{H}}, d_{\mathbb{H}}) = \alpha$. Recall now that the Euclidean dimension of an arbitrary set $A \subseteq \mathbb{R}^3$ is always less than or equal to its Heisenberg dimension [7], and so

$$\dim_H(K_{\mathbb{H}}, d_E) \leq \dim_H(K_{\mathbb{H}}, d_{\mathbb{H}}) = \alpha. \tag{4.3}$$

Consider the natural projection $\pi : \mathbb{R}^3 \to \mathbb{R}^2$, where $\pi(x, t) := x$. Then it is easy to see that $\pi(K_{\mathbb{H}}) = K$. Since π is Euclidean 1-Lipschitz, we obtain that

$$\dim_H(K_{\mathbb{H}}, d_E) \geq \dim_H(K, d_E) = \alpha. \tag{4.4}$$

Combining (4.3) and (4.4) gives that $\dim_H(K_{\mathbb{H}}, d_E) = \alpha$. □

The above statement can be used to create subsets of $\mathbb{R}^3$ for which the Hausdorff dimension with respect to the Heisenberg and the Euclidean metric coincide. A particularly illustrative example is the so-called Heisenberg square, which is 2-dimensional with respect to both the Heisenberg metric and the Euclidean metric. In order to appreciate the example, let us recall that by Pansu's isoperimetric inequality [15] any continuous surface in $\mathbb{R}^3$ has dimension 3 with respect to the Heisenberg metric. This shows the difficulty of constructing a set which is 2-dimensional with respect to both metrics.

To define such an object let us start from the planar IFS $\{f_1, f_2, f_3, f_4\}$ with the mappings $f_1(x_1, x_2) := \frac{1}{2} \cdot (x_1, x_2)$, $f_2(x_1, x_2) := \frac{1}{2} \cdot (x_1, x_2 + 1)$, $f_3(x_1, x_2) := \frac{1}{2} \cdot (x_1 + 1, x_2)$ and $f_4(x_1, x_2) := \frac{1}{2} \cdot (x_1 + 1, x_2 + 1)$, whose invariant set is the unit square $K = Q = [0, 1]^2$.

One possible Heisenberg lift is the system $\{F_1, F_2, F_3, F_4\}$, where F_i are given by

$$\begin{aligned}
F_1(x_1, x_2, t) &:= \left(\frac{1}{2} \cdot x_1, \frac{1}{2} \cdot x_2, \frac{1}{4} \cdot t\right), \\
F_2(x_1, x_2, t) &:= \left(\frac{1}{2} \cdot (x_1 + 1), \frac{1}{2} \cdot x_2, \frac{1}{4} \cdot t - \frac{1}{2} \cdot x_2\right), \\
F_3(x_1, x_2, t) &:= \left(\frac{1}{2} \cdot x_1, \frac{1}{2} \cdot (x_2 + 1), \frac{1}{4} \cdot t + \frac{1}{2} \cdot x_1\right), \\
F_4(x_1, x_2, t) &:= \left(\frac{1}{2} \cdot (x_1 + 1), \frac{1}{2} \cdot (x_2 + 1), \frac{1}{4} \cdot t + \frac{1}{2} \cdot x_1 - \frac{1}{2} \cdot x_2\right).
\end{aligned}$$

Figure 4.1 shows the invariant set $Q_{\mathbb{H}}$ associated with the lifted system $\{F_1, F_2, F_3, F_4\}$, having the property $\dim_H(Q_{\mathbb{H}}, d_E) = \dim_H(Q_{\mathbb{H}}, d_{\mathbb{H}}) = 2$.

We refer to [6] and [8] for a detailed analysis of the properties of this set and more related examples.

5. An induced metric

We start with a preliminary statement about real concave functions; see [14, p. 117].

Lemma 5.1. *Let $\varphi : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ be an increasing concave function such that $\varphi(0) = 0$. Then φ is subadditive, i.e.,*

$$\varphi(a + b) \le \varphi(a) + \varphi(b) \qquad \text{for } a, b \ge 0.$$

Using φ as above, we can induce a new metric as our next statement shows. The proof is left to the reader (see also [9, Proposition 2.3]).

Proposition 5.2. *Let (X, d) be a metric space and $\varphi : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ be an increasing concave function satisfying $\varphi(0) = 0$ and $\varphi(t) > 0$ for $t > 0$. Then (X, d_φ) is also a metric space, where $d_\varphi(x, y) := \varphi(d(x, y))$.*

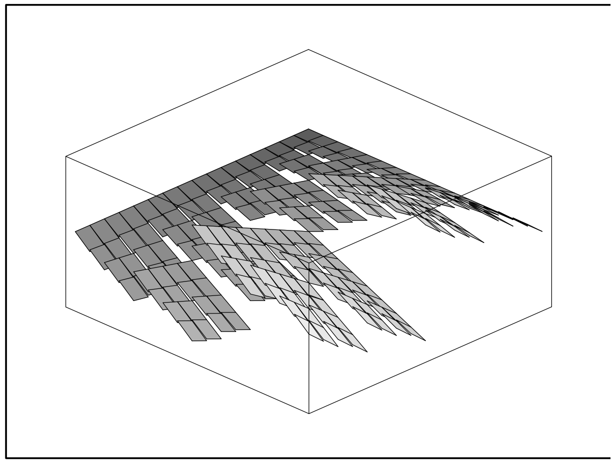

FIGURE 4.1. Invariant set of a lifted IFS.

REMARK 5.3. It may be shown that each φ with the above properties induces a functor Φ from the category of metric spaces to itself (where we take the morphisms to be the continuous maps). As for the objects, Φ maps a metric space (X, d) to (X, d_φ), and on the morphisms, Φ is the identity.

It is of interest to see under which conditions on φ the metric space (X, d_φ) will be doubling. In this direction, we have the following result.

THEOREM 5.4. *Let (X, d) be a* doubling *metric space, and $\varphi : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ a strictly increasing function which fulfils the conditions of Proposition 5.2.*

(i) *If φ satisfies*

$$\sup_{\varrho<\varphi(\mathrm{diam}(X))} \left\{ \frac{\varphi(\frac{1}{2}\cdot\varphi^{-1}(\varrho))}{\varrho} \right\} = \vartheta < 1, \tag{5.1}$$

then (X, d_φ) is a doubling metric space.

(ii) *If (X, d) is connected,* $\mathrm{diam}(X) > 0$ *and φ satisfies*

$$\sup_{\varrho<\varphi(\mathrm{diam}(X))} \left\{ \frac{\varphi(\frac{1}{2}\cdot\varphi^{-1}(\varrho))}{\varrho} \right\} = 1, \tag{5.2}$$

then (X, d_φ) is not doubling.

REMARK 5.5. Observe that "$\leq$" in (5.1) is trivially fulfilled: Indeed, the inequality $\frac{1}{2}\cdot\varphi^{-1}(\varrho) \leq \varphi^{-1}(\varrho)$ holds for all $\varrho \in \mathbb{R}_0^+$ and all $\varphi : \mathbb{R}_0^+ \to \mathbb{R}_0^+$. As φ is strictly increasing, $\varphi(\frac{1}{2}\cdot\varphi^{-1}(\varrho)) \leq \varrho$ for all $\varrho \in \mathbb{R}_0^+$ follows immediately.

Proof. (i) Let C be the doubling constant of (X,d). It follows from (5.1) that $\vartheta\cdot\varrho \geq \varphi(\frac{\varphi^{-1}(\varrho)}{2})$ for all $\varrho < \varphi(\operatorname{diam}(X))$. Therefore

$$\begin{aligned} B_{d_\varphi}(x,\varrho) = B_d(x,\varphi^{-1}(\varrho)) &\subseteq \bigcup_{i=1}^{C} B_d\left(x_i, \frac{\varphi^{-1}(\varrho)}{2}\right) \\ &= \bigcup_{i=1}^{C} B_{d_\varphi}\left(x_i, \varphi\left(\frac{\varphi^{-1}(\varrho)}{2}\right)\right) \subseteq \bigcup_{i=1}^{C} B_{d_\varphi}(x_i, \vartheta\cdot\varrho). \end{aligned}$$

Iterating this argument, we obtain

$$B_{d_\varphi}(x,\varrho) \subseteq \bigcup_{i=1}^{C^k} B_{d_\varphi}\left(x_i', \vartheta^k\cdot\varrho\right).$$

With k large enough such that $\vartheta^k \leq \frac{1}{2}$, it follows that (X, d_φ) is doubling.

(ii) Let (X,d) be a connected doubling metric space and suppose (5.2). To ease notation, we write r instead of $\varphi^{-1}(\varrho)$, so that (5.2) reads

$$\sup_{r<\operatorname{diam}(X)} \left\{\frac{\varphi(\frac{1}{2}\cdot r)}{\varphi(r)}\right\} = 1,$$

which is equivalent to the existence of a sequence $\{r_n\}_{n\in\mathbb{N}}$ with

$$\lim_{n\to\infty}\left(\frac{\varphi(\frac{1}{2}\cdot r_n)}{\varphi(r_n)}\right) = 1. \tag{5.3}$$

By choosing a subsequence, if necessary, we can assume

$$\frac{\varphi(\frac{1}{2}\cdot r_n)}{\varphi(r_n)} \geq 1 - \frac{1}{n^2} \tag{5.4}$$

for all $n \in \mathbb{N}$. Note that the limiting value 1 in (5.3) can only be reached if $\lim_{n\to\infty} r_n = 0$ or $\lim_{n\to\infty} r_n = \infty$. Indeed, suppose $\lim_{n_k\to\infty} r_{n_k} = r_*$ for some $0 < r_* < \infty$ and a subsequence $\{r_{n_k}\}_k$. As φ is strictly increasing, we have $\varphi(\frac{1}{2}\cdot r_*) < \varphi(r_*)$ or equivalently $\frac{\varphi(\frac{1}{2}\cdot r_*)}{\varphi(r_*)} < 1$, which is a contradiction. (If (X,d) is bounded, only $\lim_{n\to\infty} r_n = 0$ remains.) Assume first that (X,d) is unbounded. Let $x_0 \in X$. For r_n and for $1 \leq i \leq n$ there exist $x_i \in X$ such that

$$d(x_0, x_i) = \frac{i}{2}\cdot r_n, \tag{5.5}$$

for otherwise

$$X = \left(X \cap B\left(x_0, \frac{i}{2}\cdot r_n\right)\right) \cup \left(X \setminus \overline{B}\left(x_0, \frac{i}{2}\cdot r_n\right)\right)$$

would be an open dissection of X, which is a contradiction to the connectedness of X. Using the triangle inequality for x_0, x_i and x_j, it is easy to see that

$$d_\varphi(x_i,x_j)=\varphi(d(x_i,x_j))\geq\varphi(d(x_0,x_j)-d(x_0,x_i))\geq\varphi\left(\frac{1}{2}\cdot r_n\right)=:\frac{1}{2}\cdot\varrho_n \tag{5.6}$$

for all $1\leq i<j\leq n$ and

$$d_\varphi(x_0,x_i)=\varphi\left(\frac{i}{2}\cdot r_n\right)\leq\varphi\left(\frac{n}{2}\cdot r_n\right)$$

for all $1\leq i\leq n$.

As

$$r_n=\frac{n-2}{n-1}\cdot\frac{1}{2}\cdot r_n+\frac{1}{n-1}\cdot\frac{n}{2}\cdot r_n,$$

the concavity of φ implies

$$\varphi(r_n)\geq\frac{n-2}{n-1}\cdot\varphi\left(\frac{1}{2}\cdot r_n\right)+\frac{1}{n-1}\cdot\varphi\left(\frac{n}{2}\cdot r_n\right).$$

Using (5.4) in the second and in the last inequality, we have

$$\begin{aligned}\varphi\left(\frac{n}{2}\cdot r_n\right)&\leq(n-1)\cdot\varphi(r_n)-(n-2)\cdot\varphi\left(\frac{1}{2}\cdot r_n\right)\\&\leq(n-1)\cdot\varphi(r_n)-(n-2)\cdot\left(1-\frac{1}{n^2}\right)\cdot\varphi(r_n)\\&=\left(1+\frac{n-2}{n^2}\right)\cdot\varphi(r_n)\leq\frac{n^2+n-2}{n^2-1}\cdot\varphi\left(\frac{1}{2}\cdot r_n\right)\end{aligned}$$

and, for n large enough,

$$\varphi\left(\frac{n}{2}\cdot r_n\right)\leq 2\cdot\varphi\left(\frac{1}{2}\cdot r_n\right)=\varrho_n,$$

which implies that

$$d_\varphi(x_0,x_i)=\varphi\left(\frac{i}{2}\cdot r_n\right)\leq\varphi\left(\frac{n}{2}\cdot r_n\right)\leq\varrho_n \tag{5.7}$$

for all $1\leq i\leq n$. As a consequence, we obtain a collection of n points $\{x_1,\ldots,x_n\}\subseteq B_{d_\varphi}(x_0,\varrho_n)$ with the property that $d_\varphi(x_i,x_j)\geq\frac{1}{2}\cdot\varrho_n$. Since n can be arbitrarily large, this will contradict the doubling property.

Assume that X is bounded. Notice that for $x_0\in X$ we can find $y_0\in X$ such that

$$d(x_0,y_0)>\frac{1}{3}\cdot\operatorname{diam}(X).$$

Let $\lim_{n\to\infty}r_n=0$ such that $\frac{\varphi(\frac{1}{2}\cdot r_n)}{\varphi(r_n)}\geq 1-\frac{1}{n^2}$. By selecting a subsequence, if necessary, we can assume without loss of generality that

$$\frac{n}{2}\cdot r_n<\frac{1}{3}\cdot\operatorname{diam}(X).$$

We can now repeat the argument of the previous case to obtain a contradiction to the doubling property. □

Before considering functions φ which induce doubling spaces, we show by means of two examples how the doubling property in (X, d_φ) can be lost: In the first example it is destroyed by the very large balls and in the second one by the very small balls.

Example 5.6. Let $\varphi : \mathbb{R}_0^+ \to \mathbb{R}_0^+, \varphi(t) := \log(t+1)$, and let (X, d) be a doubling metric space. Let (X, d_φ) be the induced space, where again $d_\varphi(x, y) = \varphi(d(x, y))$. If (X, d) is bounded, then (X, d_φ) is doubling; if (X, d) is unbounded and connected, (X, d_φ) is not doubling. Indeed, it is easy to see that φ is strictly increasing and that $\varphi^{-1} : \mathbb{R}_0^+ \to \mathbb{R}_0^+, \varphi^{-1}(t) = e^t - 1$ is the inverse function of φ.

A simple consideration of the derivative shows that φ is strictly increasing on $\mathbb{R}_0^+$. Together with

$$\lim_{\varrho\to\infty}\left(\frac{\varphi(\frac{1}{2}\cdot\varphi^{-1}(\varrho))}{\varrho}\right) = \lim_{\varrho\to\infty}\left(\frac{\log(\frac{1}{2}\cdot(e^\varrho+1))}{\varrho}\right) = \lim_{\varrho\to\infty}\left(\frac{e^\varrho}{e^\varrho+1}\right) = 1,$$

using L'Hôpital's rule, it follows immediately that

$$\sup_{0<\varrho<s}\left\{\frac{\varphi(\frac{1}{2}\cdot\varphi^{-1}(\varrho))}{\varrho}\right\} < 1$$

for all $s \in \mathbb{R}_0^+$. Now Theorem 5.4 yields the claim.

Example 5.7. Let $\varphi : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ and $\varphi^{-1} : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ be defined as follows.

$$\varphi(t) := \begin{cases} 0 & t = 0 \\ \sqrt{-\frac{1}{\log(t)}} & 0 < t \le e^{-e^3} \\ t + e^{-\frac{3}{2}} - e^{-e^3} & t > e^{-e^3} \end{cases},$$

$$\varphi^{-1}(t) := \begin{cases} 0 & t = 0 \\ e^{-\frac{1}{t^2}} & 0 < t \le e^{-\frac{3}{2}} \\ t - e^{-\frac{3}{2}} + e^{-e^3} & t > e^{-\frac{3}{2}} \end{cases}.$$

It is not hard to see that φ^{-1} is indeed the inverse function of φ and that φ is increasing and concave. Let further (X, d) be a connected doubling metric space. Then the induced metric space (X, d_φ), where $d_\varphi(x, y) = \varphi(d(x, y))$, is not doubling due to Theorem 5.4 and the following calculation.

$$\begin{aligned}\sup_{\varrho\in\mathbb{R}^+}\left\{\frac{\varphi(\frac{1}{2}\cdot\varphi^{-1}(\varrho))}{\varrho}\right\} &\geq \lim_{\varrho\to 0}\left(\frac{\varphi(\frac{1}{2}\cdot\varphi^{-1}(\varrho))}{\varrho}\right) = \lim_{\varrho\to 0}\left(\frac{\sqrt{-\frac{1}{\log\left(\frac{1}{2}\cdot e^{-\frac{1}{\varrho^2}}\right)}}}{\varrho}\right)\\ &= \lim_{\varrho\to 0}\left(\frac{\sqrt{\frac{\varrho^2}{\varrho^2\cdot\log(2)+1}}}{\varrho}\right) = \lim_{\varrho\to 0}\left(\frac{1}{\sqrt{\varrho^2\cdot\log(2)+1}}\right)\\ &= 1.\end{aligned}$$

Remark 5.8. The inequality in the above calculation is in fact an equality due to Remark 5.5.

We consider next a standard example where the doubling property is preserved from (X, d) to (X, d_φ).

Example 5.9 (Snowflaking). Let $\varphi : \mathbb{R}_0^+ \to \mathbb{R}_0^+, \varphi(t) := t^s$ with $s \in (0, 1)$. If (X, d) is doubling, so is (X, d_φ), where $d_\varphi(x, y) = \varphi(d(x, y)) =: d^s(x, y)$: It is easy to see that φ is increasing and concave. Furthermore, the equation

$$\frac{\varphi(\frac{1}{2}\cdot\varphi^{-1}(\varrho))}{\varrho} = \frac{\left(\frac{1}{2}\cdot\varrho^{\frac{1}{s}}\right)^s}{\varrho} = \frac{\frac{1}{2^s}\cdot\varrho}{\varrho} = \frac{1}{2^s}$$

holds for all $\varrho \in \mathbb{R}^+$. Hence it follows directly that

$$\sup_{\varrho\in\mathbb{R}^+}\left\{\frac{\varphi(\frac{1}{2}\cdot\varphi^{-1}(\varrho))}{\varrho}\right\} = \frac{1}{2^s} < 1,$$

so that Theorem 5.4 yields the claim. Note that the similarity contractions in (X, d^s) are the same as those in (X, d), but with contraction ratio r^s instead of r. It is therefore immediate that the similarity dimension α_{d^s} of an invariant set K with respect to d^s is equal to

$$\alpha_{d^s} = \frac{\alpha_d}{s},$$

where α_d is the similarity dimension of K with respect to d. It is an exercise to check that the Hausdorff dimension of any set $A \subseteq X$ with respect to the metric d^s is

$$\dim_H(A, d^s) = \frac{\dim_H(A, d)}{s}.$$

The snowflaking functor is—up to a constant factor—the only transformation from (X, d) to (X, d_φ) which preserves the class of all similarity contractions, as the following proposition shows. Let us start with the following definition.

DEFINITION 5.10. Let (X,d) be a metric space and $f : X \to X$ a similarity with respect to d. We say that the increasing concave function $\varphi : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ *preserves* f if f is also a similarity with respect to d_φ, i.e., that if f has the property that

$$d(f(x), f(y)) = r \cdot d(x,y) \qquad \text{for all } x, y \in X \tag{5.8}$$

for some $r \in \mathbb{R}^+$, then there exists $\tilde{r} \in \mathbb{R}^+$ such that

$$d_\varphi(f(x), f(y)) = \tilde{r} \cdot d_\varphi(x,y) \qquad \text{for all } x, y \in X. \tag{5.9}$$

We say that φ preserves a family $\mathcal{F}$ of similarities if φ preserves each similarity $f \in \mathcal{F}$.

REMARK 5.11. (i) Note that if $f : X \to X$ is a similarity *contraction* with respect to d preserved by φ, then f is also a similarity *contraction* with respect to d_φ.

(ii) Note that if φ preserves a family $\mathcal{F}$ of similarities, then it preserves the whole subgroup $\langle \mathcal{F} \rangle$ of similarities generated by $\mathcal{F}$ as well.

PROPOSITION 5.12. *Let (X,d) be a connected metric space,* $\operatorname{diam}(X) > 0$, *and let $\varphi : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ be an increasing concave function which preserves the class of all similarity contractions. Then φ is a snowflaking of the form $\varphi(t) = c \cdot t^s$, where $c \in \mathbb{R}^+$ and $s \in (0,1)$.*

Proof. First note that $\varphi(0) = 0$ and $\varphi(t) > 0$ for $t > 0$ in order that d_φ is a metric. We may assume $d(x,y) > 0$. By (5.8) and the definition of d_φ, the following equations are all equivalent to (5.9).

$$\begin{aligned}
\varphi(d(f(x), f(y))) &= \tilde{r} \cdot \varphi(d(x,y)),\\
\varphi(r \cdot d(x,y)) &= \tilde{r} \cdot \varphi(d(x,y)),\\
\varphi(r \cdot d(x,y)) &= \tilde{\varphi}(r) \cdot \varphi(d(x,y)),
\end{aligned}$$

where $\tilde{\varphi}(r) = \frac{\varphi(r \cdot d(x,y))}{\varphi(d(x,y))}$. By fixing x and y and letting r vary we see that $\tilde{\varphi}$ is a continuous function. To ease notation, we write d instead of $d(x,y)$. Hence

$$\log(\varphi(r \cdot d)) = \log(\tilde{\varphi}(r)) + \log(\varphi(d)),$$

and with $r' := \log(r)$ and $d' := \log(d)$

$$\log(\varphi(e^{r'+d'})) = \log(\tilde{\varphi}(e^{r'})) + \log(\varphi(e^{d'})).$$

Setting $\psi := \log \circ \varphi \circ \exp$ and $\tilde{\psi} := \log \circ \tilde{\varphi} \circ \exp$, respectively, we have

$$\psi(r' + d') = \tilde{\psi}(r') + \psi(d'). \tag{5.10}$$

Note that ψ and $\tilde{\psi}$ are continuous functions. As $\operatorname{diam}(X) > 0$ and X is connected, we have $\{d(x,y) : x, y \in X\} \supseteq [0, \operatorname{diam}(X)) \neq \emptyset$, and by definition of d' and r' it follows that (5.10) holds for all $d' \in (-\infty, \log(\operatorname{diam}(X)))$ and all $r' \in \mathbb{R}^-$.

Now Pexider's functional equation for restricted domains [1], [2] implies that

$$\tilde{\psi}(t) = s\cdot t \qquad \text{and} \qquad \psi(t) = s\cdot t + c'$$

with arbitrary $s, c' \in \mathbb{R}$. Therefore

$$\varphi(t) = e^{\psi(\log(t))} = e^{s\cdot\log(t)+c'} = e^{c'}\cdot t^s$$

and

$$\tilde{\varphi}(t) = e^{\tilde{\psi}(\log(t))} = e^{s\cdot\log(t)} = t^s,$$

respectively. Considering that $s \in (0,1)$ for φ to be concave and letting $c := e^{c'}$ completes the proof. □

In fact, an even more general assumption suffices for the result of Proposition 5.12 as the following statement shows. This is especially interesting in the context of IFS with various contraction ratios.

THEOREM 5.13. *Let X be connected,* $\operatorname{diam}(X) > 0$*, and let $f_1, f_2 : X \to X$ be similarity contractions with respect to a metric d and with contraction ratios r_1 and r_2, respectively, such that there is no $q \in \mathbb{Q}$ such that $r_1^q = r_2$. Let further $\varphi : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ be an increasing concave function which preserves f_1 and f_2. Then φ is a snowflaking of the form $\varphi(t) = c\cdot t^s$, where $c \in \mathbb{R}^+$ and $s \in (0,1)$.*

Proof. Let $\mathcal{F}$ denote the free group of similarities generated by $\{f_1, f_2\}$: For $f_3 := f_1^{-1}$ and $f_4 := f_2^{-1}$ define

$$\mathcal{F}_k := \{f_w = f_{w_1} \circ \cdots \circ f_{w_k} : w_j \in \{1,2,3,4\}, 1 \le j \le k\}$$

and $\mathcal{F} := \bigcup_{k=0}^{\infty} \mathcal{F}_k$ as well as $r_w := r_{w_1} \cdots r_{w_k}$. Note that if φ preserves f_1 and f_2, then it preserves all $f_w \in \mathcal{F}$. Indeed, if (5.9) holds for f_1 and f_2 it follows immediately that

$$d_\varphi(f_w(x), f_w(y)) = \tilde{r}_w \cdot d_\varphi(x,y)$$

for all $x, y \in X$, where $\tilde{r}_w = \tilde{r}_{w_1} \cdots \tilde{r}_{w_k}$. In order to ease notation, we write r instead of r_w in the following.

The same considerations as in the proof of Proposition 5.12 yield

$$\varphi(r\cdot d(x,y)) = \tilde{\varphi}_{\mathcal{R}}(r)\cdot\varphi(d(x,y)), \tag{5.11}$$

where $\tilde{\varphi}_{\mathcal{R}} : \mathcal{R} \to \mathbb{R}^+$, $\tilde{\varphi}_{\mathcal{R}}(r) = \varphi(r\cdot d(x,y))/\varphi(d(x,y))$ and $\mathcal{R} := \{r = r_1^{l_1}\cdot r_2^{l_2} \in \mathbb{Z}\}$. Observe that $\mathcal{R} \subseteq \mathbb{R}^+$ is dense because of the assumption about r_1 and r_2. A consequence of this density is that (for fixed x and y) $\tilde{\varphi}_{\mathcal{R}}$ has a unique continuous extension $\tilde{\varphi} : \mathbb{R}^+ \to \mathbb{R}^+$, $\tilde{\varphi}(r) = \varphi(r\cdot d(x,y))/\varphi(d(x,y))$, such that (5.11) holds also for $\tilde{\varphi}$. This implies, following the proof of Proposition 5.12, that

$$\psi(r' + d') = \tilde{\psi}(r') + \psi(d')$$

for all $d' \in (-\infty, \log(\mathrm{diam}(X)))$ and all $r' \in \mathbb{R}$, such that the final consideration in the proof of Proposition 5.12 yields the claim. $\square$

Interestingly, there exist non-trivial $\varphi : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ which preserve the class of similarity contractions whose contraction ratios are of the form r^q for a fixed $r \in (0,1)$ and $q \in \mathbb{Q}^+$. Note that this condition applies to many well-known IFS, particularly to those consisting of similarity contractions with one single contraction ratio.

THEOREM 5.14. *Let X be connected, $\mathrm{diam}(X) > 0$, and let $\{f_i\}_{1\le i\le N}$ be a family of similarity contractions with respect to a metric d and with contraction ratios $\{r_i\}_{1\le i\le N}$ such that there exist $r \in (0,1)$ and $m_i \in \mathbb{N}$ such that*

$$r_i = r^{m_i}, \quad 1 \le i \le N. \tag{5.12}$$

Then there exists an increasing concave function $\varphi : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ that preserves all f_i, $1 \le i \le N$, which is not of the snowflaking form $\varphi(t) = c\cdot t^s$.

Proof. Without loss of generality we may assume that the greatest common divisor of $\{m_i\}_{1\le i\le N}$ is 1. It suffices to construct an increasing concave function $\varphi : \mathbb{R}_0^+ \to \mathbb{R}_0^+$, which is not of the form $\varphi(t) = c\cdot t^s$, such that for all f_i, $1 \le i \le N$, there exists $\tilde{r}_i \in (0,1)$ such that

$$d_\varphi(f_i(x), f_i(y)) = \tilde{r}_i \cdot d(x,y) \tag{5.13}$$

for all $x, y \in X$. We will call this φ the 'perturbated snowflaking functor'.

For $N+1 \le j \le 2\cdot N$ let $f_j := f_{j-N}^{-1}$, define

$$\mathcal{F}_k := \{f_w = f_{w_1} \circ \cdots \circ f_{w_k} : w_j \in \{1, \dots, 2\cdot N\}, 1 \le j \le k\},$$

$\mathcal{F} := \bigcup_{k=0}^\infty \mathcal{F}_k$ and denote $r_w := r_{w_1} \cdots r_{w_k}$. Note that

$$\mathcal{R} := \{r_w = r_1^{l_1} \cdots r_N^{l_N} : l_i \in \mathbb{Z}, 1 \le i \le N\}$$

is the discrete multiplicative Abelian group generated by r, $\mathcal{R} = \{r^l : l \in \mathbb{Z}\}$. As considered before, (5.13) implies that φ preserves all elements of the free group of similarities $\mathcal{F}$ such that, using the same definitions as in the proof of Proposition 5.12, we have

$$\varphi(r_w \cdot d(x,y)) = \tilde{\varphi}(r_w)\cdot\varphi(d(x,y)) \tag{5.14}$$

for all $x, y \in X$ and for all $r_w \in \mathcal{R}$.

Let $\bar{\varphi} : \mathbb{R}_0^+ \to \mathbb{R}_0^+$, $\bar{\varphi}(t) = t^s$ with fixed $s \in (0,1)$, denote the snowflaking functor. For $0 \le a < b$ define the 'chord function' $\hat{\varphi}_{a,b} : [a,b] \to \mathbb{R}^+$ by

$$\hat{\varphi}_{a,b}(t) := \frac{b-t}{b-a}\cdot\bar{\varphi}(a) + \frac{t-a}{b-a}\cdot\bar{\varphi}(b).$$

Define also $I_k := [r^{k+1}, r^k]$ for $k \in \mathbb{Z}$ and $\hat{\varphi} : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ by $\hat{\varphi}(t) := \hat{\varphi}_{r^{k+1},r^k}(t)$ for $t \in I_k$ and $\hat{\varphi}(0) := 0$. Now consider any strictly increasing and concave interpolation $\varphi_0 : I_0 \to \mathbb{R}^+$ of $\hat{\varphi}$ and $\bar{\varphi}$, i.e.,

$$\hat{\varphi}(t) \le \varphi_0(t) \le \bar{\varphi}(t)$$

for all $t \in I_0$ and define $\varphi : \mathbb{R}_0^+ \to \mathbb{R}_0^+$,

$$\varphi(t) := \bar{\varphi}(r^k)\cdot\varphi_0\left(\frac{t}{r^k}\right)$$

for $t \in I_k$ and $\varphi(0) := 0$. From this it follows that

$$\hat{\varphi}(t) \le \varphi(t) \le \bar{\varphi}(t) \tag{5.15}$$

for $t \in \mathbb{R}_0^+$ and $\varphi(r^k) = \hat{\varphi}(r^k) = \bar{\varphi}(r^k)$ for $k \in \mathbb{Z}$ and that φ is therefore well-defined and a continuous and strictly increasing extension of φ_0 which obviously fulfils (5.14).

It remains to show the concavity of φ: Let $0 \le a \le t \le b$, $a \ne b$. We prove $\varphi(t) = \varphi(\lambda\cdot a + (1-\lambda)\cdot b) \ge \lambda\cdot\varphi(a) + (1-\lambda)\cdot\varphi(b)$, where $\lambda = \frac{b-t}{b-a}$.

(i) If $a, t, b \in I_k$ for some $k \in \mathbb{Z}$, there is nothing to show as φ is concave on I_k by definition.

(ii) Let $a, t \in I_k$ and $b \in I_l$ for some $k > l$. Assume first that $t = r^k$. Then

$$\varphi(t) = \bar{\varphi}(t) \ge \lambda\cdot\bar{\varphi}(a) + (1-\lambda)\cdot\bar{\varphi}(b) \ge \lambda\cdot\varphi(a) + (1-\lambda)\cdot\varphi(b), \tag{5.16}$$

where the first inequality is due to the concavity of $\bar{\varphi}$ and the second one due to (5.15). Using first the concavity of φ on I_k and then (5.16) we have for arbitrary $a \le t \le b$

$$\begin{aligned}\varphi(t) &\ge \frac{r^k - t}{r^k - a}\cdot\varphi(a) + \frac{t-a}{r^k - a}\cdot\varphi(r^k) \ge \frac{b-t}{b-a}\cdot\varphi(a) + \frac{t-a}{b-a}\cdot\varphi(b)\\ &= \lambda\cdot\varphi(a) + (1-\lambda)\cdot\varphi(b).\end{aligned}$$

A similar consideration holds for $a \in I_k$ and $t, b \in I_l$.

(iii) Now let $a \in I_k$, $t \in I_l$ and $b \in I_m$ for some $k > l > m$. Recalling the definition of $\hat{\varphi}_{a,b}$ and using the definition of φ, several times the concavity of $\bar{\varphi}$ and (5.15), we have

$$\begin{aligned}\varphi(t) &\ge \hat{\varphi}_{r^{l+1},r^l}(t) \ge \hat{\varphi}_{a,r^l}(t) \ge \hat{\varphi}_{a,b}(t) = \lambda\cdot\bar{\varphi}(a) + (1-\lambda)\cdot\bar{\varphi}(b)\\ &\ge \lambda\cdot\varphi(a) + (1-\lambda)\cdot\varphi(b),\end{aligned}$$

which completes the proof. □

6. Final remarks

REMARK 6.1. In the last section, we considered the case of connected metric spaces. The connectivity assumption was used in an essential way in the proofs. It would be interesting to study iterated function systems defined on non-connected spaces.

Remark 6.2. As pointed out by the referee of this paper, it would be interesting to generalise our results to the case of only asymptotical similarity contractions, i.e., mappings for which there exist positive constants c_1 and c_2 such that for any word $w \in W_*$ and any $x, y \in X$ we have

$$c_1 \cdot r_w \leq d(f_w(x), f_w(y)) \leq c_2 \cdot r_w.$$

In the Euclidean space, the theory of the thermodynamical formalism [13] provides a method to calculate the Hausdorff dimension of such general invariant sets.

It would be interesting to extend the results of the thermodynamical formalism to the setting of doubling metric spaces. A major difficulty in this respect is the lack of a notion of a differential for mappings between general metric spaces.

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