Illinois Journal of Mathematics
Volume 51, Number 4, Winter 2007, Pages 1265–1274
S 0019-2082

UNIONS OF HYPERPLANES, UNIONS OF SPHERES, AND SOME RELATED ESTIMATES

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Abstract. We give estimates of the sizes of certain unions of hyperplanes or of spheres.

By a hyperplane in $\mathbb{R}^d$ we mean any translate of a $(d-1)$-plane. The collection $\mathcal{H}$ of all hyperplanes P in $\mathbb{R}^d$ can be parametrized by $\Sigma^{(d-1)} \times [0,\infty)$ if one identifies P with (σ, t) whenever $P = \sigma^{\perp} + t\sigma$. Following the capacitarian definition of Hausdorff dimension, we say that a compact set $\mathcal{K}$ of hyperplanes has dimension $\alpha > 0$ if, for each small ϵ, $\mathcal{K}$ carries a Borel probability measure μ such that

$$(1_H) \qquad \int_{\mathcal{K}} \int_{\mathcal{K}} \frac{d\mu(P_1)\ d\mu(P_2)}{(|\sigma_1 - \sigma_2| + |t_1 - t_2|)^{\alpha-\epsilon}} < \infty.$$

Similarly, let $S(x,r)$ stand for the sphere in $\mathbb{R}^d$ with center x and radius r. Identifying the collection of all such spheres with $\mathcal{S} \doteq \mathbb{R}^d \times (0,\infty) \subseteq \mathbb{R}^{d+1}$, we will say that a compact set $\mathcal{K}$ of spheres has dimension $\alpha > 0$ if, for each small ϵ, $\mathcal{K}$ carries a Borel probability measure μ such that

$$(1_S) \qquad \int_{\mathcal{K}} \int_{\mathcal{K}} \frac{d\mu(S_1) d\mu(S_2)}{(|x_1 - x_2| + |r_1 - r_2|)^{\alpha-\epsilon}} < \infty.$$

In both cases we are interested in what can be said about the size of

$$(2) \qquad \bigcup_{T \in \mathcal{K}} T$$

in terms of the Hausdorff dimension of $\mathcal{K}$. Since the dimension of a hyperplane or sphere is $d-1$, intuition suggests the conjectures that

(a) the union (2) should have positive d-dimensional Lebesgue measure whenever $\dim(\mathcal{K}) > 1$, and
(b) if $0 < \alpha < 1$ and $\dim(\mathcal{K}) = \alpha$, then (2) should have dimension at least $d - 1 + \alpha$.

Received April 11, 2006; received in final form February 7, 2007.
2000 *Mathematics Subject Classification.* 28A75, 35B45.
Key words and phrases. Spherical averages, Hausdorff dimension, Strichartz estimate.

In these situations (though not always in similar ones), such intuition appears to be correct. For example, considering hyperplanes and the case $\dim(\mathcal{K}) > 1$, one may define a truncated Radon transform R_0 by

$$R_0 f(\sigma, t) = \int_{\sigma^\perp \cap B(0,1)} f(p + t\sigma)\ d\mathcal{L}^{d-1}(p).$$

The following theorem is from [1].

Theorem 1. *Suppose μ is a nonnegative Borel measure on a compact set $\mathcal{K} \subseteq \mathcal{H}$ and suppose that μ satisfies (1) for $\alpha - \epsilon > 1$. Then*

$$\|R_0 \chi_E\|_{L_\mu^{\alpha-\epsilon,\infty}} \lesssim \mathcal{L}^d(E)^{1/2}$$

for Borel $E \subseteq \mathbb{R}^d$.

Now suppose that $\mathcal{K} \subseteq \mathcal{H}$ and $\dim(\mathcal{K}) = \alpha > 1$. Let μ be a Borel probability measure satisfying (1_H). If E is the set (2) then $R_0 \chi_E(\sigma, t) \geq c > 0$ for each $\sigma^\perp + t\sigma \in \mathcal{K}$, and so it follows from Theorem 1 that $\mathcal{L}^d(E) \geq c^2 > 0$. Thus (a) is true for hyperplanes. For $d \geq 3$ the paper [2] contains an analogue of Theorem 1 for the spherical average operator $Tf(x, r) = \int_{\Sigma^{(d-1)}} f(x - r\sigma) d\sigma$. It therefore follows that, when $d \geq 3$, (a) is also true for spheres. (When $d = 2$ the circle version of (a) is a significantly more difficult question, answered in the affirmative in Wolff's paper [3].) The papers [1] and [2] also contain results which imply the following theorem.

Theorem 2. *Suppose that $\mathcal{K}$ is either a compact set of hyperplanes or, if $d \geq 3$, a compact set of spheres. Suppose that $\dim(\mathcal{K}) = \alpha \in (0, 1)$ and that $\mathcal{K}$ either lies on a smooth curve or has a certain Cantor set structure. Then if $E = \bigcup_{T \in \mathcal{K}} T$ we have $\dim(E) \geq d - 1 + \alpha$.*

Theorem 2 verifies (b) for hyperplanes in case $d = 2$ but applies only in special cases if $d > 2$. Another approach to results like (b) begins by recalling that $E \subseteq \mathbb{R}^d$ has Hausdorff dimension $\beta \in (0, d)$ if and only if, for each $\epsilon > 0$, E carries a Borel probability measure $\widetilde{\mu}$ satisfying

$$\int_{\mathbb{R}^d} \frac{|\widehat{\widetilde{\mu}}(\xi)|^2}{|\xi|^{d-\beta+\epsilon}} d\xi < \infty.$$

That is, $\dim(E) = \beta$ if, for $\epsilon > 0$, E supports a nontrivial nonnegative distribution in the Sobolev space $W^{2,-(d-\beta+\epsilon)/2}$. Thus, for example, (b) is equivalent to the conjecture that, if $0 < \alpha < 1$, $\dim(K) = \alpha$, and $\epsilon > 0$, then $\bigcup_{T \in \mathcal{K}} T$ should support a nonnegative distribution in $W^{2,(\alpha-1)/2-\epsilon}$. On the other hand, the dimension of $\mathcal{H} = \Sigma^{(d-1)} \times [0, \infty)$ is $d \geq 2$ and the dimension of $\mathcal{S} = \mathbb{R}^d \times (0, \infty)$ is $d + 1$ but if $\mathcal{K}$ has dimension as small as $1 + \epsilon$ then we know already that $\bigcup_{T \in \mathcal{K}} T$ has positive measure. It is therefore natural to wonder if more than this (i.e., more than that $\bigcup_{T \in \mathcal{K}} T$ has positive measure) can be said when $\dim(\mathcal{K}) > 1$. In particular, in view of the just-mentioned reformulation of (b), one might conjecture that, no matter the $\alpha \in (0, d)$, if $\dim(\mathcal{K}) = \alpha$, then, for any $\epsilon > 0$, $\bigcup_{T\in\mathcal{K}} T$ should support a nonnegative and nontrivial measure in $W^{2,(\alpha-1)/2-\epsilon}$. Our main result is that this is true in certain cases.

THEOREM 3_H. *If $\mathcal{K} \subseteq \mathcal{H}$ and $\dim(\mathcal{K}) = \alpha \in (0, d]$ then, for $\epsilon > 0$, $\bigcup_{P\in\mathcal{K}} P$ supports a nonnegative measure (function if $\alpha > 1$) in $W^{2,(\alpha-1)/2-\epsilon}$.*

We note that, for hyperplanes, Theorem 3_H implies (a) as well as (b). For spheres our result is less satisfactory.

THEOREM 3_S. *If $\mathcal{K} \subseteq \mathcal{S}$ and $\dim(\mathcal{K}) = \alpha \in (0, (d-1)/2)$ then, for $\epsilon > 0$, $\bigcup_{S\in\mathcal{K}} S$ supports a nonnegative measure in $W^{2,(\alpha-1)/2-\epsilon}$.*

Theorem 3_S implies (a) only when $d \geq 4$ and (b) only when $d \geq 3$ (though, in its range of validity, the partial result for (b) in dimension 2 is a little more general than Wolff's observation in [3] that, for $0 < \alpha < 1$, the union of a set of circles in the plane has dimension at least $1+\alpha$ if the set of centers of those circles has dimension α).

Results like Theorems 3_H and 3_S are often connected with estimates for operators like R and T. That is the case here, and we begin with the Radon transform estimate which goes with Theorem 3_H. Suppose $\psi \in \mathcal{S}(\mathbb{R}^{d-1})$ is a nonnegative radial function with Fourier transform $\widehat{\psi}$ equal to 1 on $B(0, 1)$ and supported in $B(0, 2)$. For $\sigma \in S^{(d-1)}$ fix an orthogonal linear map O_σ from $\sigma^\perp \subseteq \mathbb{R}^d$ to $\mathbb{R}^{d-1}$. Define a Radon transform $\widetilde{R}$ by

$$\widetilde{R}f(\sigma, t) = \int_{\sigma^\perp} f(p + t\sigma)\psi(O_\sigma(p))\, d\mathcal{L}^{d-1}(p).$$

The estimate we have in mind is the following.

THEOREM 4_H. *Suppose μ is a nonnegative Borel measure on a compact set $\mathcal{K} \subseteq \mathcal{H}$ and suppose that μ satisfies the condition (slightly stronger than (1_H))*

$$\mu(\{(\sigma, t) : |\sigma - \sigma_0| + |t - t_0| < \tau\}) \lesssim \tau^\alpha$$

for some $\alpha \in (0, d]$ and for all $(\sigma_0, t_0) \in \mathcal{H}$ and $\tau > 0$. Then, for $\epsilon > 0$,

$$\|\widetilde{R}f\|_{L^{2,\infty}_\mu} \lesssim \|f\|_{W^{2,(1-\alpha)/2+\epsilon}}.$$

If also $\alpha > 1$, then, for small $\epsilon > 0$ and

$$\frac{1}{p} = \frac{1}{2} + \frac{\alpha - 1}{2d} - \epsilon$$

there is the estimate

$$\|\widetilde{R}f\|_{L^{2,\infty}_\mu} \lesssim \|f\|_{L^p(\mathbb{R}^d)}.$$

Here is the corresponding result for spheres.

THEOREM 4_S. *Suppose μ is a nonnegative Borel measure on a compact set $\mathcal{K} \subseteq \mathcal{S}$ and suppose that, for $\alpha \in (0, (d-1)/2)$, μ satisfies the condition*

$$\mu(\{(x,r) : |x - x_0| + |r - r_0| < \tau\}) \lesssim \tau^{\alpha}$$

for all $(x_0, r_0) \in \mathcal{S}$ and $\tau > 0$. Then, for $\epsilon > 0$,

$$\|Tf\|_{L^{2,\infty}_{\mu}} \lesssim \|f\|_{W^{2,(1-\alpha)/2+\epsilon}}.$$

If also $\alpha > 1$, then, for small $\epsilon > 0$ and

$$\frac{1}{p} = \frac{1}{2} + \frac{\alpha - 1}{2d} - \epsilon$$

there is the estimate

$$\|Tf\|_{L^{2,\infty}_{\mu}} \lesssim \|f\|_{L^p(\mathbb{R}^d)}.$$

Proof of Theorem 3_H. Suppose that μ is a measure on $\mathcal{K}$ satisfying

$$\int_{\mathcal{K}} \int_{\mathcal{K}} \frac{d\mu(P_1)\ d\mu(P_2)}{(|\sigma_1 - \sigma_2| + |t_1 - t_2|)^{\alpha}} < \infty.$$

With ψ as above, define a measure $\widetilde{\mu}$ on $\mathbb{R}^d$ by

$$\langle f, \widetilde{\mu} \rangle = \int_{\mathcal{K}} \int_{\sigma^{\perp}} f(p + t\sigma)\psi(O_{\sigma}(p))\ d\mathcal{L}_{d-1}(p)\ d\mu(\sigma, t) = \langle \widetilde{R}f, \mu \rangle.$$

We will show that, for $\epsilon > 0$,

$$\int_{\mathbb{R}^d} |\widehat{\widetilde{\mu}}(\xi)|^2 |\xi|^{\alpha - 1 - 2\epsilon} d\mathcal{L}_d(\xi) < \infty. \tag{3}$$

Replacing α by $\alpha - \epsilon$ then shows that Theorem 3_H is true. Suppose ρ is a nonnegative C^{∞} function supported in $[1/2, 4]$ and equal to one on $[1, 2]$. We will establish (3) by showing that

$$\int_{\mathbb{R}^d} |\widehat{\widetilde{\mu}}(\xi)|^2 \rho^2(2^{-j}|\xi|) d\mathcal{L}_d(\xi) \tag{4}$$

is $\lesssim 2^{-j(\alpha-1)}$. Thus we begin by fixing j. If, for $\sigma \in S^{(d-1)}$, π_{σ} denotes the projection of $\mathbb{R}^d$ into $\sigma^{\perp}$ and $\Pi_{\sigma} = O_{\sigma} \circ \pi_{\sigma}$, then (4) is equal to

$$\begin{aligned} (5) \qquad & \int_{\mathbb{R}^d} \int_{\mathcal{K}} \int_{\mathcal{K}} e^{-i\xi\cdot(t_1\sigma_1 - t_2\sigma_2)} \widehat{\psi}\big(\Pi_{\sigma_1}(\xi)\big) \widehat{\psi}\big(\Pi_{\sigma_2}(\xi)\big) \times \\ & \qquad\qquad \times d\mu(\sigma_1, t_1)\ d\mu(\sigma_2, t_2) \rho^2(2^{-j}|\xi|) d\mathcal{L}_d(\xi) \\ & = \int_{\mathcal{K}} \int_{\mathcal{K}} b(\sigma_1, \sigma_2, t_1\sigma_1 - t_2\sigma_2) d\mu(\sigma_1, t_1)\ d\mu(\sigma_2, t_2) \end{aligned}$$

where

$$b(\sigma_1, \sigma_2, x) = \int_{\mathbb{R}^d} e^{-i\xi\cdot x} \widehat{\psi}(\Pi_{\sigma_1}(\xi)) \widehat{\psi}(\Pi_{\sigma_2}(\xi)) \rho^2(2^{-j}|\xi|) d\mathcal{L}_d(\xi).$$

If $b(\sigma_1, \sigma_2, \cdot)$ is not identically 0, then the tubes of radius 2 through the origin in the directions of σ_1 and σ_2 must intersect at some ξ satisfying $|\xi| \sim 2^j$. This implies that $|\sigma_1 \pm \sigma_2| \lesssim 2^{-j}$. There is no loss of generality in assuming that if (σ_1, t_1) and (σ_2, t_2) are both in the support of μ, then $|\sigma_1 + \sigma_2| \geq 1$ (for this can be achieved by decomposing μ into a finite sum of measures with small supports). Thus we may assume that, unless $b(\sigma_1, \sigma_2, \cdot) \equiv 0$, $|\sigma_1 - \sigma_2| \lesssim 2^{-j}$. Now, with

$$a(\sigma, x) = \int_{\mathbb{R}^d} e^{-i\xi\cdot x} \widehat{\psi}(\Pi_\sigma(\xi)) \rho(2^{-j}|\xi|) d\mathcal{L}_d(\xi),$$

we have $b(\sigma_1, \sigma_2, \cdot) = a(\sigma_1, \cdot) * a(\sigma_2, \cdot)$. Let P_σ be the plate

$$B(0,1) \cap \{x \in \mathbb{R}^d : |x \cdot \sigma| \leq 2^{-j}\}.$$

Assume for the moment the following standard result (which will be proved later):

Lemma 1. *For $N \in \mathbb{N}$ we have*

$$|a(\sigma, \cdot)| \leq C_N 2^j \sum_{n=1}^{\infty} 2^{-nN} \chi_{2^n P_\sigma}. \tag{6}$$

Then it follows that

$$|b(\sigma_1, \sigma_2, \cdot)| \lesssim 2^{2j} \sum_{m,n=1}^{\infty} 2^{-(m+n)N} \chi_{2^n P_{\sigma_1}} * \chi_{2^m P_{\sigma_2}}. \tag{7}$$

If $|\sigma_1 - \sigma_2| \lesssim 2^{-j}$ and $m \leq n$, we have

$$\chi_{2^n P_{\sigma_1}} * \chi_{2^m P_{\sigma_2}} \lesssim 2^{dm-j} \chi_{2^{n+2} P_{\sigma_1}}$$

and so, if $N > d$,

$$\begin{aligned} 2^{2j} \sum_{n=1}^{\infty} \sum_{m=1}^{n} & 2^{-(m+n)N} \chi_{2^n P_{\sigma_1}} * \chi_{2^m P_{\sigma_2}} \\ & \lesssim 2^{2j} \sum_{n=1}^{\infty} \sum_{m=1}^{n} 2^{-(n+m)N} 2^{dm-j} \chi_{2^{n+2} P_{\sigma_1}} \\ & \lesssim 2^j \sum_{n=1}^{\infty} 2^{-nN} \chi_{2^{n+2} P_{\sigma_1}}. \end{aligned}$$

It therefore follows from (7) that (5), and so (4), is controlled by

$$2^j \sum_{n=1}^{\infty} 2^{-nN} \iint_{\{|\sigma_1 - \sigma_2| \lesssim 2^{-j}\}} \chi_{2^{n+2} P_{\sigma_1}} (t_1\sigma_1 - t_2\sigma_2) d\mu(\sigma_1, t_1) d\mu(\sigma_2, t_2). \tag{8}$$

Now if $t_1\sigma_1 - t_2\sigma_2 \in 2^{n+2}P_{\sigma_1}$, then

$$(8) \qquad |t_1 - t_2 + t_2(\sigma_1-\sigma_2)\cdot\sigma_1| = |(t_1\sigma_1 - t_2\sigma_1)\cdot\sigma_1 + t_2(\sigma_1-\sigma_2)\cdot\sigma_1| \\ = |(t_1\sigma_1 - t_2\sigma_2)\cdot\sigma_1| \lesssim 2^{n-j}.$$

If also $|\sigma_1-\sigma_2| \lesssim 2^{-j}$, then $|t_2| \lesssim 1$ gives $|t_1 - t_2| \lesssim 2^{n-j}$ and so

$$|\sigma_1-\sigma_2| + |t_1-t_2| \lesssim 2^{n-j}.$$

Thus (8) is bounded by

$$(9) \qquad \sum_{n=1}^{\infty} 2^{-nN} 2^j \iint_{\{|\sigma_1-\sigma_2|+|t_1-t_2|\lesssim 2^{n-j}\}} d\mu(\sigma_1,t_1)d\mu(\sigma_2,t_2).$$

Since

$$\iint_{\{|\sigma_1-\sigma_2|+|t_1-t_2|\le\tau\}} d\mu(\sigma_1,t_1)d\mu(\sigma_2,t_2) \\ \le \tau^{\alpha}\int_{\mathcal{K}}\int_{\mathcal{K}} \frac{d\mu(\sigma_1,t_1)d\mu(\sigma_2,t_2)}{(|\sigma_1-\sigma_2|+|t_1-t_2|)^{\alpha}} \lesssim \tau^{\alpha},$$

we may bound (9), and so (4), by

$$\sum_{n=1}^{\infty} 2^{-nN}2^j 2^{(n-j)\alpha} \lesssim 2^{-j(\alpha-1)}.$$

This completes the proof of Theorem 3_H. □

Proof of Lemma 1. Without loss of generality let $\sigma = (1,0,\dots,0)$. Writing $\xi = (\xi_1,\xi')$ and identifying $\sigma^{\perp}$ with $\mathbb{R}^{d-1}$, we have

$$(10) \qquad a(\sigma,x) = \iint e^{-i\xi\cdot x}\widehat{\psi}(\xi')\rho(2^{-j}|\xi|)d\mathcal{L}_{d-1}(\xi')d\mathcal{L}_1(\xi_1).$$

Suppose $x \in 2^{n+1}P_{\sigma} \sim 2^n P_{\sigma}$. Writing $x = (x_1,x') \in \mathbb{R}\times\mathbb{R}^{d-1}$, assume first that $|x| \ge 2^n$ so that, if $j > 1$, $|x'| \ge 2^{n-1}$. Then, considering the support of $\widehat{\psi}$,

$$\Big|\int e^{-i\xi'\cdot x'}\widehat{\psi}(\xi')\rho(2^{-j}|\xi|)d\mathcal{L}_{d-1}(\xi')\Big| \\ = \Big|\int_{B(0,2)} e^{-i\xi'\cdot x'}\widehat{\psi}(\xi')\rho(2^{-j}|\xi|)d\mathcal{L}_{d-1}(\xi')\Big|.$$

Integrating by parts N times, this is bounded by $C_N 2^{-nN}$. Thus (10) is bounded by $C_N 2^j 2^{-nN}$ since $|\xi_1| \lesssim 2^j$. Suppose now that $x \in 2^{n+1}P_{\sigma}\backslash 2^n P_{\sigma}$ and $|x| < 2^n$. Then $|x_1| > 2^{n-j}$. Now

$$(11) \qquad \int e^{-i\xi_1 x_1}\rho(2^{-j}|\xi|)d\xi_1 = 2^j\int e^{-i\widetilde{\xi}_1 2^j x_1}\rho\Big(\sqrt{\widetilde{\xi}_1^2 + |2^{-j}\xi'|^2}\Big)d\widetilde{\xi}_1.$$

Since $|2^j x_1| \sim 2^n$, integrating by parts N times bounds (11) by $C_N 2^{j-nN}$. Since $\widehat{\psi}$ is supported in $B(0,2)$, the same bound applies to (10). □

Proof of Theorem 4_H. Theorem 4_H will follow from the estimate

$$\|\widetilde{R}^* \chi_{\mathcal{E}}\|_{W^{2,(\alpha-1)/2-\epsilon}} \lesssim (\mu(\mathcal{E}))^{1/2}, \quad \mathcal{E} \subseteq \mathcal{H},$$

dual to

$$\|\widetilde{R} f\|_{L^{2,\infty}_\mu} \lesssim \|f\|_{W^{2,(1-\alpha)/2+\epsilon}}$$

and, if $\alpha > 1$, the Sobolev embedding theorem. Thus, for Borel $\mathcal{E} \subseteq \mathcal{H}$ and for suitable f, we note that

$$\langle f, \widetilde{R}^* \chi_{\mathcal{E}} \rangle = \langle \widetilde{R} f, \chi_{\mathcal{E}} \mu \rangle = \int_{\mathcal{E}} \int_{\sigma^\perp} f(p + t\sigma) \psi(O_\sigma(p)) \, d\mathcal{L}_{d-1}(p) \, d\mu(\sigma, t).$$

Following the proof of Theorem 3 with μ replaced by $\chi_{\mathcal{E}}\mu$ (see (9)) shows that

$$\|\widetilde{R}^* \chi_{\mathcal{E}}\|^2_{W^{2,(\alpha-1)/2-\epsilon}}$$

is controlled by the sum on j of the terms

$$2^{j(\alpha-1-2\epsilon)} \sum_{n=1}^{\infty} 2^{-nN} 2^j \int_{\mathcal{E}} \int_{\{|\sigma_1-\sigma_2|+|t_1-t_2| \lesssim 2^{n-j}\}} d\mu(\sigma_1, t_1) d\mu(\sigma_2, t_2)$$
$$\lesssim 2^{j(\alpha-1-2\epsilon)} \sum_{n=1}^{\infty} 2^{-nN} 2^j \mu(\mathcal{E}) 2^{\alpha(n-j)} \lesssim 2^{-2j\epsilon} \mu(\mathcal{E}).$$

This yields the desired result. □

Proof of Theorem 3_S. Here we write σ for Lebesgue measure on $S^{(d-1)}$. The proof is generally parallel to that of Theorem 3_H. Thus suppose that μ is a measure on $\mathcal{K}$ satisfying

$$\int_{\mathcal{K}} \int_{\mathcal{K}} \frac{d\mu(S_1) \, d\mu(S_2)}{(|x_1 - x_2| + |r_1 - r_2|)^\alpha} < \infty$$

and define $\widetilde{\mu}$ on $\mathbb{R}^d$ by

$$\langle f, \widetilde{\mu} \rangle = \int_{\mathcal{K}} \int_{S^{(d-1)}} f(x + r\zeta) \, d\sigma(\zeta) \, d\mu(x, r) = \langle \widetilde{T} f, \mu \rangle.$$

With ρ as in the proof of Theorem 3, we would like to show that

$$\int_{\mathbb{R}^d} |\widehat{\widetilde{\mu}}(\xi)|^2 \rho(2^{-j}|\xi|) d\mathcal{L}_d(\xi) \lesssim 2^{-j(\alpha-1)}. \tag{12}$$

We begin by rewriting (12) as

$$\int_{\mathbb{R}^d} \int_{\mathcal{K}} \int_{\mathcal{K}} \widehat{\sigma}(r_1\xi) \, \widehat{\sigma}(r_2\xi) \, e^{-i(x_1-x_2)\cdot\xi} d\mu(x_1, r_1) \, d\mu(x_2, r_2) \rho(2^{-j}|\xi|) d\mathcal{L}_d(\xi).$$

Changing to polar coordinates on $\mathbb{R}^d$ and abusing notation by writing $\widehat{\sigma}(|\xi|)$ to stand for $\widehat{\sigma}(\xi)$, this is

$$\begin{aligned}(13)\qquad \int_{\mathcal{K}}\int_{\mathcal{K}}\int_0^{\infty}\widehat{\sigma}(r_1 r)\ \widehat{\sigma}(r_2 r)\ \widehat{\sigma}(|x_1-x_2|r)\rho(2^{-j}r)r^{d-1}\times\\ \times\, dr\ d\mu(x_1,r_1)\ d\mu(x_2,r_2)\\ =\int_{\mathcal{K}}\int_{\mathcal{K}} b(r_1,r_2,|x_1-x_2|)\ d\mu(x_1,r_1)\ d\mu(x_2,r_2)\end{aligned}$$

if

$$b(r_1,r_2,s)=\int_0^{\infty}\widehat{\sigma}(r_1 r)\ \widehat{\sigma}(r_2 r)\ \widehat{\sigma}(sr)\rho(2^{-j}r)r^{d-1}dr.$$

We will use the following notation: if $S_1 = S(x_1, r_1)$ and $S_2 = S(x_2, r_2)$ are spheres, then $\delta = \delta(S_1, S_2)$ will stand for the distance $|x_1 - x_2| + |r_1 - r_2|$ between S_1 and S_2 while $\Delta = \Delta(S_1, S_2)$ will stand for $||x_1 - x_2| - |r_1 - r_2||$. We also observe that on the compact subset $\mathcal{K}$ of $\mathcal{S}$, r is bounded away from 0. We will estimate (13), and therefore establish (12), by considering the different cases which result from splitting the integral in a certain way.

Case I. $\iint_{\{\Delta<\delta/2\}} b(r_1,r_2,|x_1-x_2|)\ d\mu(x_1,r_1)\ d\mu(x_2,r_2)$.

If $\Delta < \delta/2$ then $\delta \sim |x_1 - x_2|$. Now $|b(r_1, r_2, |x_1 - x_2|)| \lesssim 2^j$ follows from

$$(14)\qquad |\widehat{\sigma}(s)| \lesssim s^{(1-d)/2}$$

(recall that the r_j are bounded away from 0 and that $|\widehat{\sigma}|$ is bounded). Thus the portion of the Case I integral where $|x_1 - x_2| \le 2^{-j}$ is controlled by

$$2^j\iint_{\{\delta\lesssim 2^{-j}\}} d\mu(x_1,r_1)\ d\mu(x_2,r_2)\lesssim 2^{-j(\alpha-1)},$$

where the last inequality follows (as in the proof of Theorem 3_H) from the capacitarian assumption on μ. If $|x_1 - x_2| \gtrsim 2^{-j}$ then (14) and $\delta \sim |x_1 - x_2|$ imply that the relevant integral is controlled by

$$\begin{aligned}\frac{2^j}{(2^j|x_1-x_2|)^{(d-1)/2}}&\lesssim\frac{1}{\delta^{(d-1)/2}2^{j(d-3)/2}}\\ &\lesssim\frac{1}{\delta^{\alpha}2^{-j[(d-1)/2-\alpha]}}\frac{1}{2^{j(d-3)/2}}\\ &=\frac{1}{\delta^{\alpha}2^{j(-1+\alpha)}}.\end{aligned}$$

Here the second inequality follows from $\delta \gtrsim 2^{-j}$ and $\alpha \le (d-1)/2$. Thus the Case I integral is controlled by $2^{-j(\alpha-1)}$.

Case II. $\iint_{\{\delta<4\cdot 2^{-j}\}} b(r_1,r_2,|x_1-x_2|)\ d\mu(x_1,r_1)\ d\mu(x_2,r_2)$.

Since

$$\iint_{\{\delta<4\cdot 2^{-j}\}} d\mu(x_1,r_1)\ d\mu(x_2,r_2)\lesssim 2^{-j\alpha}$$

and $|b(r_1,r_2,|x_1-x_2|)|\lesssim 2^j$, the desired bound of $2^{-j(1-\alpha)}$ is immediate.

Case III. $\iint_{\{4\cdot 2^{-j}\le\delta\le 2\Delta\}} b(r_1,r_2,|x_1-x_2|)\ d\mu(x_1,r_1)\ d\mu(x_2,r_2)$.

Recall that

$$b(r_1,r_2,|x_1-x_2|)=\int_a^b \widehat{\sigma}(r_1r)\ \widehat{\sigma}(r_2r)\ \widehat{\sigma}(|x_1-x_2|r)\rho(2^{-j}r)r^{d-1}dr$$

where $a\gtrsim 2^j$. Utilizing the asymptotic expansion of $\widehat{\sigma}$ and recalling that r_1 and r_2 are bounded away from 0, the principal term in this integral is controlled by the largest of

$$\Big|\int_a^b \frac{e^{i(\pm r_1\pm r_2\pm|x_1-x_2|)r}}{(r|x_1-x_2|)^{(d-1)/2}}dr\Big|. \tag{15}$$

After rescaling and then multiplying μ by a cutoff function of x, we may assume that $r_1,r_2\ge 1/2$ and $|x_1-x_2|\le 1/2$. One can check that then $\Delta=||r_1-r_2|-|x_1-x_2||$ minimizes $|\pm r_1\pm r_2\pm|x_1-x_2||$. An integration by parts bounds (15) by some multiple of

$$|x_1-x_2|^{-(d-1)/2}\Big(\Big|\int_a^b\int_a^r e^{i\Delta s}ds\ r^{-(d+1)/2}dr\Big|+2^{-j(d-1)/2}\Big|\int_a^b e^{i\Delta s}ds\Big|\Big).$$

Since $a\ge 2^j$, it follows that

$$|(15)|\lesssim\frac{2^{-j(d-1)/2}}{\Delta\cdot|x_1-x_2|^{(d-1)/2}}\lesssim\frac{2^{-j(d-1)/2}}{\Delta^{(d+1)/2}}\lesssim\frac{2^{-j(d-1)/2}}{\Delta^{\alpha}\ 2^{-j[(d+1)/2-\alpha]}},$$

where the last inequality follows from $\Delta\gtrsim 2^{-j}$ and $\alpha\le(d-1)/2<(d+1)/2$. Thus

$$\iint_{\{4\cdot 2^{-j}\le\delta\le 2\Delta\}}|(15)|\ d\mu(x_1,r_1)\ d\mu(x_2,r_2)\lesssim 2^{-j(1-\alpha)}$$

by the capacitarian assumption on μ. The nonprincipal terms are controlled similarly. For example, the term coming from the principal terms of $\widehat{\sigma}(r_ir)$ and the second order term from $\widehat{\sigma}(|x_1-x_2|r)$ is controlled by

$$\int_1^b\frac{dr}{(r|x_1-x_2|)^{(d+1)/2}}\lesssim\frac{1}{\Delta^{(d+1)/2}2^{j(d-1)/2}}$$

and so may be treated as was $|(15)|$. This completes the proof of Theorem 3_S. □

The changes to the proof of Theorem 3_S which are required in order to prove Theorem 4_S are analogous to the changes in the proof of Theorem 3_H which yield the proof of Theorem 4_H.

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