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GENERALIZATIONS OF THE THEOREMS OF CARTAN AND GREENE–KRANTZ TO COMPLEX MANIFOLDS

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ABSTRACT. In this article, some generalizations of the theorems of Cartan and Greene–Krantz for the family of biholomorphic mappings on (not necessary bounded) domains of a complex manifold are given. Moreover, a necessary and sufficient condition for strongly complete C-hyperbolicity of domains in a complex manifold with compact quotients is obtained.

1. Introduction

H. Cartan [4] proved the following theorem about compactness of families of biholomorphic mappings (see also [13, Thm. 4, p. 78]).

THEOREM. *Let Ω be a bounded domain in $\mathbf{C}^n$. Suppose $\{f_i\}$ is a sequence of biholomorphic mappings $f_i : \Omega \to \Omega$ which converges uniformly on compact subsets of Ω to a mapping f. Then the following three conditions are equivalent:*

(i) *f is a biholomorphic mapping of Ω onto Ω.*
(ii) *$f(\Omega)$ is not a subset of $\partial\Omega$, the boundary of Ω in $\mathbf{C}^n$.*
(iii) *The Jacobian determinant $\det[f'(z)]$ is not identically zero on Ω.*

Much attention has been given to generalizations of Cartan's theorem. For instance, under some additional hypotheses on the domains and the mappings, S. Bell [3] and W. Klingenberg and S. Pinchuk [10] proved the above theorem with "biholomorphic" replaced by "proper". However, as far as we know, the problem of generalizing Cartan's theorem to unbounded domains in a complex manifold remains open.

The first purpose of this paper is to give several versions of Cartan's theorem for families of biholomorphic mappings on a (not necessary bounded) domain of a complex manifold from the viewpoint of hyperbolic complex analysis. Namely, we will prove the following results:

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THEOREM 2.7. *Let D be a weakly taut domain in a complex manifold M such that D has the (IM) property. Suppose $\{f_i\}$ is a sequence of biholomorphic mappings $f_i : D \to D$ which converges uniformly on compact subsets of D to a mapping f. Then the following three conditions are equivalent:*

(i) *f is a biholomorphic mapping of D onto D.*
(ii) *$f(D)$ is not a subset of ∂D, the boundary of D in M.*
(iii) *There exists a point $a \in D$ such that the Jacobian determinant $\det\big((df)_a\big)$ is non-zero.*

THEOREM 2.10. *Let D be a homogeneous domain in a complex manifold M such that D has the (IM) property. Suppose $\{f_i\}$ is a sequence of biholomorphic mappings $f_i : D \to D$ which converges uniformly on compact subsets of D to a mapping f. Then the following three conditions are equivalent:*

(i) *f is a biholomorphic mapping of D onto D.*
(ii) *$f(D)$ is not a subset of ∂D, the boundary of D in M.*
(iii) *There exists a point $a \in D$ such that the Jacobian determinant $\det\big((df)_a\big)$ is non-zero.*

In [9], D. Kim claimed the following result.

THEOREM. *Let D be a bounded domain in $\mathbf{C}^n$ such that $D/\operatorname{Aut}(D)$ is compact. Then D is strongly complete C-hyperbolic.*

This theorem is cited in [15, Theorem 1.8, p. 71] and [11]. Unfortunately, it seems incorrect. There are some mistakes in his proof, e.g., in the bottom line (which is a key step in his proof) of p. 141 in [9]. The second purpose of this paper is to give a corrected version of this theorem using the results in the first section. Namely, we will prove the following result.

THEOREM 3.2. *Let D be a C-hyperbolic domain in a complex manifold M such that $\operatorname{Aut}(D) \Subset \operatorname{Hol}(D, M)$. Suppose there is a compact subset K of D such that for every $x \in D$ there are a biholomorphic mapping $f \in \operatorname{Aut}(D)$ and a point $a \in K$ such that $f(x) = a$. Then the following assertions are equivalent:*

(i) *D is strongly complete C-hyperbolic.*
(ii) (1) *For every $z \in K$ there exists $r > 0$ such that $\overline{B}_{c_D}(z, r)$ is a compact subset of D.*
(2) *For every finite boundary point $q \in \partial D$ and for every $\varepsilon > 0$, there exists a neighbourhood U of q in M such that $c_D(z, z') < \varepsilon$ for all $z, z' \in U \cap D$.*

As is well known, the scaling process introduced by Pinchuk [14] is a very useful tool in the characterization of domains with noncompact automorphisms group. The essential idea of this method is that the limit of a sequence of biholomorphic maps must be either again a biholomorphic map or a map that has image contained in the boundary. More precisely, we have the following theorem of Greene–Krantz [5, p. 161].

THEOREM. *Suppose* $\{\Omega_i\}_{i=1}^\infty$ *and* $\{A_i\}_{i=1}^\infty$ *are sequences of bounded domains in* $\mathbf{C}^n$ *with* $\lim \Omega_i = \Omega_0$ *and* $\lim A_i = A_0$ *for some (uniquely determined) bounded domains* Ω_0, A_0 *in* $\mathbf{C}^n$. *Suppose also that* $\{f_i : A_i \to \Omega_i\}_{i=1}^\infty$ *is a sequence of biholomorphic maps and that* $\{f_i : A_i \to \mathbf{C}^n\}_{i=1}^\infty$ *converges uniformly on compact subsets of* A_0 *to a limit* $f_0 : A_0 \to \mathbf{C}^n$. *Then one of two mutually exclusive conditions holds: Either* f_0 *maps* A_0 *biholomorphically onto* Ω_0 *or* f_0 *maps* A_0 *into* $\partial\Omega_0$*, the boundary of* Ω_0.

The last purpose of this paper is to generalize the Greene–Krantz theorem to families of biholomorphic mappings on (not necessary bounded) domains of a complex manifold. Namely, we will prove the following result.

THEOREM 4.7. *Let* $\{\Omega_i\}_{i=1}^\infty$ *and* $\{A_i\}_{i=1}^\infty$ *be sequences of pseudo-taut domains in a complex manifold* M *with* $\lim \Omega_i = \Omega_0$ *and* $\lim A_i = A_0$ *for some (uniquely determined) pseudo-taut domains* Ω_0, A_0 *in* M. *Suppose that* $\Omega_i \subset \Omega_0$ *and* $A_i \subset A_0$ $(i \geq 1)$. *Suppose also that* $\{f_i : A_i \to \Omega_i\}_{i=1}^\infty$ *is a sequence of biholomorphic maps. Then one of the following two assertions holds:*

(i) *The sequence* $\{f_i\}$ *is compactly divergent, i.e., for each compact set* $K \subset A_0$ *and each compact set* $L \subset \Omega_0$*, there is an integer* i_0 *such that* $f_i(K) \cap L = \emptyset$ *for* $i \geq i_0$.
(ii) *There exists a subsequence* $\{f_{i_j}\} \subset \{f_i\}$ *such that the sequence* $\{f_{i_j}\}$ *converges uniformly on compact subsets of* A_0 *to a biholomorphic map* $f : A_0 \to \Omega_0$.

In this article complex manifolds are assumed to be connected and a connected open subset of a complex manifold is said to be a domain of this manifold. For complex manifolds X and Y, we denote by $\mathrm{Hol}(X, Y)$ the space of all holomorphic mappings from X to Y, equipped with the open-compact topology. For basic notions and properties of hyperbolic complex analysis we refer to [11] and [12].

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2. Generalizations of Cartan's theorem to complex manifolds

For the sake of convenience, we give the following definition.

DEFINITION 2.1. Let D be a complex manifold and $\mathrm{Aut}(D)$ be the group of all biholomorphic mappings from D onto D. We say that D has the *(IM)* *property* if there exists a metric ρ on D such that ρ induces the standard topology of D and ρ is invariant under $\mathrm{Aut}(D)$ (i.e., $\rho(x,y) = \rho(f(x), f(y))$ for all $x, y \in D$ and all $f \in \mathrm{Aut}(D)$).

The class of complex manifolds having the (IM) property is large. It contains the hyperbolic complex manifolds, and the complex manifolds equipped with Bergman metric. Moreover, we have the following example.

Example 2.2. Consider $\mathbf{C}^* := \mathbf{C} \setminus \{0\}$. It is easy to see that $\mathbf{C}^*$ is not hyperbolic and does not have a Bergman metric. We show that $\mathbf{C}^*$ has the (IM) property.

Indeed, first we determine the group $\mathrm{Aut}(\mathbf{C}^*)$.

Assume that $f \in \mathrm{Aut}(\mathbf{C}^*)$. Set $\Delta^* = \{z \in \mathbf{C} : 0 < |z| < 1\}$. Then $f|_{\Delta^*} : \Delta^* \to \mathbf{C}$ is holomorphic and $\mathrm{card}\big(\mathbf{C} \setminus f(\Delta^*)\big) > 2$. By the Big Picard Theorem, f extends meromorphically to $\mathbf{C}$, and hence $f(z) = z^n \cdot g(z)$, where $n \in \mathbf{Z}$ and g is an entire function vanishing nowhere on $\mathbf{C}$.

We consider two cases.

(a) $n \geq 0$: Then f extends holomorphically to 0. By the Second Main Theorem of Nevanlinna, it follows that every finite holomorphic map from $\mathbf{C}$ into $\mathbf{C}$ is a polynomial. Hence $f(z) = az, a \neq 0$.

(b) $n < 0$: Then the function $1/f(z) = z^{-n}/g(z)$ extends holomorphically to 0. Using the above argument, we have $f(z) = a/z$, $a \neq 0$. Then $\mathrm{Aut}(\mathbf{C}^*)$ consists exactly of the functions of the forms az and a/z with $a \neq 0$.

We now construct an invariant metric ρ on $\mathbf{C}^*$. Consider the function $h : \Delta = \{z \in \mathbf{C} : |z| < 1\} \to \mathbf{C}^*$ given by $h(z) = e^z$ for $z \in \Delta$. Then h is biholomorphic from Δ onto $h(\Delta)$. Take an arbitrary $\omega \in \mathbf{C}^*$. We put

$$\rho(1,\omega) = \begin{cases} |z| & \text{if } \omega = e^z \in h(\Delta), \\ 1 & \text{if } \omega \notin h(\Delta), \end{cases}$$

and define $\rho(\omega_1, \omega_2) = \rho(1, \omega_2/\omega_1)$ for all $\omega_1, \omega_2 \in \mathbf{C}^*$. It is easy to see that ρ is a metric on $\mathbf{C}^*$ induced the topology of $\mathbf{C}^*$ and ρ is invariant under $\mathrm{Aut}(\mathbf{C}^*)$.

Definition 2.3. Let D be a domain in a complex manifold M.

We say that D is *weakly taut* if for every sequence $\{f_n\}$ in $\mathrm{Aut}(D)$, either

(a) there is a subsequence $\{f_{n_k}\}$ which converges in $\mathrm{Hol}(D, M)$, or
(b) the sequence $\{f_n\}$ is compactly divergent, i.e., for each compact set $K \subset D$ and each compact set $L \subset D$ there is an integer n_0 such that $f_n(K) \cap L = \emptyset$ for $n > n_0$.

Examples 2.4. We now give some examples of classes of weakly taut domains.

2.4.1. If D is a taut domain in a complex manifold M then D is obviously a weakly taut domain in M. However, the converse assertion is not true in general. Indeed, it is easy to see that $\mathbf{C}^*$ is a weakly taut domain in $\mathbf{C}$, but not hyperbolic.

2.4.2. If M is a hyperbolic complex manifold then each relatively compact domain D of M is weakly taut. This follows immediately from Arzela-Ascoli's theorem.

Since every bounded domain in $\mathbf{C}^n$ is hyperbolic, we have the following consequence of 2.4.2.

2.4.3. Every bounded domain in $\mathbf{C}^n$ is weakly taut.

2.4.4. Let D be a non-bounded hyperbolic domain in $\mathbf{C}^n$ such that the following condition is satisfied: If $\{z_k\}_{k=1}^{\infty} \subset D$ is an arbitrary sequence such that $\lim_{k\to\infty}\|z_k\| = \infty$, then $\lim_{k\to\infty} d_D(z_0, z_k) = \infty$, where z_0 is some point of D, and $\|.\|$ is the Euclidean norm of $\mathbf{C}^n$ and d_D is the Kobayashi distance of D.

Then D is a weakly taut domain in $\mathbf{C}^n$. Indeed, assume that $\{f_\nu\}$ is an arbitrary sequence in $\mathrm{Aut}(D)$ which does not satisfy condition (b) in Definition 2.3. Without loss of generality we may assume that, for each $\nu \geq 1$, there are $a_\nu \in K$ and $b_\nu \in L$ such that $f_\nu(a_\nu) = b_\nu$, where K and L are compact subsets of D. By taking a subsequence we also may assume that $\lim_{\nu\to\infty} a_\nu = a_0 \in K$ and $\lim_{\nu\to\infty} b_\nu = b_0 \in L$.

Now we must prove that there is a subsequence $\{f_{\nu_k}\} \subset \{f_\nu\}$ which converges in $\mathrm{Hol}(D, \mathbf{C}^n)$. By Montel's theorem, it suffices to show that, for any compact set $\widetilde{K} \subset D$, there exists $M > 0$ so that $\|f_\nu(z)\| < M$ for $z \in \widetilde{K}, \nu \geq 1$.

Suppose this does not hold, i.e., there exists a compact set $\widetilde{K} \subset D$ such that $\limsup_{\nu\to\infty} \sup_{\widetilde{K}} \|f_\nu(z)\| = \infty$. Then there exists $\widetilde{a_\nu} \in \widetilde{K}$ such that $\limsup_{\nu\to\infty} \|f_\nu(\widetilde{a_\nu})\| = \infty$. By taking a subsequence of the sequence $\{\widetilde{a_\nu}\}$, we may assume that $\lim_{\nu\to\infty} \widetilde{a_\nu} = \widetilde{a_0} \in \widetilde{K}$ and $\lim_{\nu\to\infty} \|\widetilde{f}_\nu(\widetilde{a_\nu})\| = \infty$. We have

$$\begin{aligned} d_D(f_\nu(\widetilde{a_\nu}), b_0) &\leq d_D(f_\nu(\widetilde{a_\nu}), f_\nu(a_\nu)) + d_D(b_\nu, b_0) \\ &= d_D(\widetilde{a_\nu}, a_\nu) + d_D(b_\nu, b_0) \to d_D(\widetilde{a_0}, a_0) \text{ as } \nu \to \infty. \end{aligned}$$

This is impossible.

Remark 2.5. We believe that hyperbolicity of a domain D in a complex manifold M does not imply weak tautness of this domain. Unfortunately, we do not know of any counterexample.

Definition 2.6. Let M, M_1 be two complex manifolds of dimension n and $f : M \to M_1$ a holomorphic mapping and a an arbitrary point of M. Then, locally, f can be considered as a holomorphic mapping from B^n to B^n, where B^n is the open unit ball in $\mathbf{C}^n$. This allows us to compute the Jacobian determinant $\det\big((df)_a\big)$. Note that the non-vanishing property of $\det\big((df)_a\big)$ is defined independently of the chosen local coordinate systems of M and M_1. Thus we can say that the Jacobian determinant $\det\big((df)_a\big)$ is non-vanishing.

In order to prove Theorem 2.7 we need the following lemmas.

LEMMA 2.7.1 ([13, Prop. 5, p. 79]). *Let D be a domain in a complex manifold M. Let $\{\varphi_i\}$ be a sequence of continuous open mappings from D into M. Suppose that $\{\varphi_i\}$ converges uniformly on compact subsets of D to a mapping $\varphi : D \to M$. Suppose that, for some $a \in D$, a is an isolated point of $\varphi^{-1}\big(\varphi(a)\big)$. Then, for any neighbourhood U of a, there is an i_0 such that $\varphi(a) \in \varphi_i(U)$ for each $i \geq i_0$.*

LEMMA 2.7.2 ([13, Cor., p. 80]). *Suppose M is a complex manifold and $\{f_i\}$ is a sequence of holomorphic functions on M, converging uniformly on compact subsets of M to a holomorphic function f. Then, if $f_i(z) \neq 0$ for all z, i and f is nonconstant, we have $f(z) \neq 0$ for all $z \in M$.*

The following lemma plays an essential role in the proof of Theorems 2.7 and 2.10 and can be viewed as a generalization of Cartan's theorem to complex manifolds.

LEMMA 2.7.3. *Let D be a domain in a complex manifold M such that D has the (IM) property. Suppose $\{f_i\} \subset \mathrm{Aut}(D)$ converges uniformly on compact subsets of D to a mapping f. Then the following three conditions are equivalent:*

(i) *$f(D)$ is an open subset of D and f is a biholomorphic mapping of D onto $f(D)$.*
(ii) *$f(D) \not\subset \partial D$.*
(iii) *There exists a point $a \in D$ such that the Jacobian determinant $\det\big((df)_a\big)$ is non-zero.*

Proof. (i)⇒(ii): Obvious.

(ii)⇒(iii): Clearly, $f(D) \subset \overline{D}$. Thus assertion (ii) implies that there is $a \in D$ such that $f(a) = b \in D$. Take a neighbourhood V of a in D such that $f(V) \subset D$. Without loss of generality we may assume that V is an open subset of $\mathbf{C}^n$ and $f(V) \subset \mathbf{C}^n$.

We now show that $f|_V$ is injective. Indeed, assume that $f(z_1) = f(z_2)$, where $z_1, z_2 \in V$. By the hypothesis, there exists a metric ρ of D such that ρ induces the standard topology of D and ρ is invariant under $\mathrm{Aut}(D)$. We have

$$\begin{aligned}\rho\big(f(z_1), f(z_2)\big) = 0 &= \rho\big(\lim f_i(z_1), \lim f_i(z_2)\big)\\ &= \lim \rho\big(f_i(z_1), f_i(z_2)\big)\\ &= \lim \rho(z_1, z_2) = \rho(z_1, z_2).\end{aligned}$$

This implies that $z_1 = z_2$.

By Osgood's theorem [13], the image $f(V)$ is open in $\mathbf{C}^n$ and the restricted mapping $f|_V : V \to f(V)$ is biholomorphic. Thus the Jacobian determinant $\det\big((df)_a\big)$ is non-zero.

(iii)⇒(i): Put $U = \{z \in D : \det\big((df)_z\big) \neq 0\}$. Then $a \in U$ and U is open in D. We now show that U is closed in D.

Suppose that this is false. Then there exists a point $a_0 \in \partial U \cap D$ such that $\det\big((df)_{a_0}\big) = 0$. Take a small connected neighbourhood V of a_0 in D such that the holomorphic mappings $f_i|_V$ and $f|_V$ can be considered as holomorphic mappings from $V \subset \mathbf{C}^n$ into $\mathbf{C}^n$.

Consider the holomorphic functions $J_i : V \to \mathbf{C}$ and $J : V \to \mathbf{C}$ given by $J_i(z) = \det\big((df_i)_z\big)$ and $J(z) = \det\big((df)_z\big)$. Then the sequence $\{J_i\}$ converges uniformly on compact subsets of V to J. Also $J_i(z) \neq 0$ for all i and all z since $f_i \in \operatorname{Aut}(D)$. On the other hand, since $J(a_0) = 0$ and $V \cap U \neq \emptyset$, it follows that J is nonconstant. According to Lemma 2.7.2, we have $J(z) \neq 0$ for all $z \in V$. This is a contradiction. Thus U is closed in D, and hence $U = D$. By the inverse function theorem, $f : D \to M$ is an open mapping and any $z \in D$ is isolated in $f^{-1}\big(f(z)\big)$. According to Lemma 2.7.1, we have $f(D) \subset \bigcup_{i\geq 1} f_i(D) = D$. By using an invariant metric ρ of D and repeating the above argument, we see that f is injective. By the Osgood theorem it follows that the mapping $f : D \to f(D)$ is biholomorphic. □

Proof of Theorem 2.7. By Lemma 2.7.3, it suffices to prove the implication (iii)⇒ (i).

According to Lemma 2.7.3, $f \in \operatorname{Hol}(D, D)$ and the mapping $f : D \to f(D)$ is biholomorphic.

Take $a \in D$ and put $b = f(a)$. Take compact neighbourhoods U, V of a, b, respectively, such that $f(U) \subset \overset{o}{V}$. By Lemma 2.7.1, since the sequence $\{f_i\}$ converges uniformly to f, there is an i_0 such that $f_i(U) \subset \overset{o}{V}$ and $b \in f_\nu(U)$ for each $i \geq i_0$. We put $g_i = f_i^{-1}$ for $i \geq 1$. Then $a \in U = g_i\big(f_i(U)\big)$.

Choose $K = \overline{\bigcup_{i\geq i_0} f_i(U)}$ and $L = \{a\}$. It is easy to see that K, L are compact subsets of D and $g_i(K) \cap L \neq \emptyset$ for each $i \geq i_0$. Then the weak tautness of D implies that there is a subsequence of the sequence $\{g_i\}$ which converges uniformly on compact subsets of D to a holomorphic mapping $g : D \to M$. Without loss of generality we may assume that the sequence $\{g_i\}$ converges uniformly to g.

Take $b_i \in K$ such that $g_i(b_i) = a$. Without loss of generality we may assume that $\lim b_i = b_0 \in K$. Then $g(b_0) = a$, i.e., $g(D) \not\subset \partial D$. According to Lemma 2.7.3, $g \in \operatorname{Hol}(D, D)$ and the mapping $g : D \to g(D)$ is also biholomorphic. It is easy to see that $f_o g = g_o f = \operatorname{Id}_D$, and hence $f \in \operatorname{Aut}(D)$. □

Corollary 2.8. *Let D be a domain in a complex manifold M such that D has the (IM) property. Let $D^* = D \cup \{\infty\}$ be the one-point compactification* *of D. Let $C(D, D^*)$ denote the space of continuous maps from D into D^* equipped with the compact-open topology and let $\{\infty\}$ denote the constant map which sends D to ∞. Then D is weakly taut if and only if* $\mathrm{Aut}(D) \cup \{\infty\}$ *is a compact subset of* $C(D, D^*)$.

Proof. "$\Rightarrow$": Assume that $\{f_i\} \subset \mathrm{Aut}(D) \cup \{\infty\}$. Without loss of generality we may assume that $f_i \neq \infty$ for each $i \geq 1$. Consider two cases:

(i) The sequence $\{f_i\}$ satisfies condition (b) of Definition 2.3: Then $\{f_i\}$ converges uniformly on compact subsets of D to the mapping ∞ in $C(D, D^*)$.

(ii) The sequence $\{f_i\}$ does not satisfy condition (b) of Definition 2.3: Then there exist compact subsets K, L of D and a subsequence $\{f_{i_k}\}$ of the sequence $\{f_i\}$ such that $f_{i_k}(K) \cap L \neq \emptyset$ for each $k \geq 1$ and $\{f_{i_k}\}$ converges uniformly on compact subsets of D to a mapping $f \in \mathrm{Hol}(D, M)$.

Take $a_{i_k} \in K, b_{i_k} \in L$ such that $f_{i_k}(a_{i_k}) = b_{i_k}$ for each $k \geq 1$. Without loss of generality we may assume that $\lim_{k\to\infty} a_{i_k} = a_0 \in K$ and $\lim_{k\to\infty} b_{i_k} = b_0 \in L$. Then $f(a_0) = b_0 \in D$, and hence $f(D) \not\subset \partial D$. By Theorem 2.7, it follows that $f \in \mathrm{Aut}(D)$.

"$\Leftarrow$": This is immediate from the definition of weak tautness. □

COROLLARY 2.9. *Cartan's theorem holds in the following cases:*

(i) *D is a taut domain of a complex manifold M.*
(ii) *D is a relatively compact domain of a hyperbolic complex manifold M; in particular, D is a bounded domain in $\mathbf{C}^n$.*
(iii) *D is a non-bounded hyperbolic domain in $\mathbf{C}^n$ satisfying the following condition: If $\{z_k\}_{k=1}^{\infty} \subset D$ is an arbitrary sequence such that $\lim_{k\to\infty} \|z_k\| = \infty$, then $\lim_{k\to\infty} d_D(z_0, z_k) = \infty$, where z_0 is some point of D, $\|.\|$ is the Euclidean norm of $\mathbf{C}^n$ and d_D is the Kobayashi distance of D.*

Recall that a complex manifold M is said to be homogeneous if for any pair of points $x, y \in M$ there is $\sigma \in \mathrm{Aut}(M)$ such that $\sigma(x) = y$.

In order to prove Theorem 2.10 we need the following lemma.

LEMMA 2.10.1 ([11, Thm. 5.5.1, p. 268]). *Let ρ be a metric on a complex manifold D such that ρ induces the topology of D and ρ is invariant under* $\mathrm{Aut}(D)$. *Let a be a point of D. Assume that $f : D \to D$ is a holomorphic mapping satisfying the following conditions:*

(i) $f(a) = a$.
(ii) $|\det\big((df)_a\big)| = 1$.
(iii) *f is an isometry for ρ, i.e., $\rho(x, y) = \rho\big(f(x), f(y)\big)$ for $x, y \in D$.*

Then f is a biholomorphic mapping.

Proof of Theorem 2.10. In view of Lemma 2.7.3 it suffices to prove the implication (iii)⇒(i). Also, by Lemma 2.7.3, since ρ is invariant under $\mathrm{Aut}(D)$, f is an isometry for ρ.

On the other hand, since $\mathrm{Aut}(D)$ acts transitively on D, we may assume that $f(a) = a$. It now remains to prove $|\det\big((df)_a\big)| = 1$.

Since our problem is local, we may assume that D is a domain in $\mathbf{C}^n$. By Lemma 2.7.3, there exists $r_0 > 0$ such that $B(a, r_0) \subset D \cap f(D)$, where $B(a, r) = \{z \in \mathbf{C}^n : \|z - a\| < r\}$ for each $r > 0$. It is easy to see that $f(B(a, r)) = B(a, r)$ for each $0 < r \leq r_0$.

Assume that $|\det\big((df)_a\big)| < 1$. Then there exists $r_1 \in (0, r_0)$ such that $|\det\big((df)_z\big)| < \alpha < 1$ for each $z \in B(a, r_1)$. This implies that $\mathrm{vol}\, B(a, r_1) < \alpha^2.volB(a, r_1)$. This is a contradiction.

Similarly, if $|\det\big((df)_a\big)| > 1$, we also get a contradiction. □

3. On strongly complete C-hyperbolicity of domains in a complex manifold with compact quotients

Let X be a complex space. We denote its Caratheodory pseudodistance by c_X. Even if c_X is a distance, it does not in general induce the complex space topology of X (see [7] and also [6]).

A complex space X is said to be Caratheodory-hyperbolic or C-hyperbolic if c_X is a distance and induces the complex space topology of X. A C-hyperbolic space X is said to be complete (resp. strongly complete) if X is Cauchy complete with respect c_X (resp. if all closed balls with respect to c_X are compact). Janicki, Pflug, and Vigué [8] exhibited a domain X in $\mathbf{C}^3$ such that X is complete C-hyperbolic, but not strongly complete C-hyperbolic.

We now give the following definition.

Definition 3.1. Let D be a domain in a complex manifold M and let $q \in \partial D$ be a boundary point of D. We say that q is *a finite boundary point* of D if there exist a point $a \in D$ and a sequence $\{z_n\} \subset D$ such that $\{z_n\} \to q$ and $\liminf_{z_n \to q} c_D(a, z_n) < \infty$.

It is easy to see that $q \in \partial D$ is a finite boundary iff there exists a sequence $\{z_n\} \subset D$ which converges to q such that $\liminf_{z_n \to q} c_D(a, z_n) < \infty$ for any point $a \in D$.

In order to prove Theorem 3.2 we need the following lemma.

Lemma 3.3. *Let D be a domain in a complex manifold M such that D has the (IM) property and* $\mathrm{Aut}(D) \Subset \mathrm{Hol}(D, M)$. *Suppose that there exists a compact subset K of D such that for every $x \in D$ there is a biholomorphic mapping $f \in$* $\mathrm{Aut}(D)$ *and a point $a \in K$ such that $f(x) = a$. Then the following assertions are equivalent:*

(i) *For every $z \in D$ and for every $r > 0$, $\overline{B}_{c_D}(z, r)$ is a compact subset of D.*

(ii) (1) *For every $z \in K$ there exists $r > 0$ such that $\overline{B}_{c_D}(z, r)$ is a compact subset of D.*

(2) *For every finite boundary point $q \in \partial D$ and for every $\varepsilon > 0$ there exists a neighbourhood U of q in M such that $c_D(z, z') < \varepsilon$ for all $z, z' \in U \cap D$.*

Proof. (i)⇒(ii): It is easy to see that if $\overline{B}_{c_D}(z, r)$ is a compact subset of D for every $z \in D$ and $r > 0$, then there is no finite boundary point of D. Thus the implication is proved.

(ii)⇒(i): Assume that there exist $p \in D$ and $r^* > 0$ such that the closed ball $\overline{B}_{c_D}(p, r^*)$ is not a compact subset of D. Then there exists a sequence $\{\widetilde{x}_n\} \subset \overline{B}_{c_D}(p, r^*)$ such that

$$\{\widetilde{x}_n\} \to q \in \partial D. \tag{1}$$

Let $x \in D$ and $y \in \overline{D}$. We define

$$\widetilde{c}_D(x, y) = \liminf_{\substack{y' \in D \\ y' \to y}} c_D(x, y').$$

Then, from (1), we get

$$\widetilde{c}_D(x, q) < \infty \text{ for every } x \in D. \tag{2}$$

Define the function $\varphi : D \to [0, \infty)$ by

$$\varphi(x) = \sup\{r \geq 0 : \overline{B}_{c_D}(x, r) \Subset D\} \text{ for each } x \in D.$$

Then φ is continuous on D, and hence $\min_K \varphi(x) = \widetilde{r} > 0$. This implies that

$$\infty > \inf_{\substack{x \in K \\ y \in \partial D}} \widetilde{c}_D(x, y) = r_0 \geq \widetilde{r} > 0. \tag{3}$$

Therefore there exist $a \in K$ and $q_0 \in \partial D$ such that $r_0 \leq \widetilde{c}_D(a, q_0) < 2r_0 < \infty$, i.e., q_0 is a finite boundary point of D.

By hypothesis, there exists a neighbourhood U of q_0 in M such that $c_D(z, z') < r_0/2$ for all $z, z' \in U \cap D$. Take a sequence $\{x_n\} \subset U \cap D$ such that $\{x_n\} \to q_0$. Then $c_D(x_1, x_n) < r_0/2$ for all $n \geq 1$.

By hypothesis, for every $n \geq 1$, there exists $f_n \in \mathrm{Aut}(D)$ such that $f_n(x_n) = a_n \in K$. Without loss of generality we may assume that $\{a_n\} \to a_0 \in K$. Since $\mathrm{Aut}(D) \Subset \mathrm{Hol}(D, M)$, without loss of generality we may assume that the sequence $\{f_n\}$ converges locally uniformly to a mapping $f \in \mathrm{Hol}(D, M)$. Then we have $\{f_n(x_1)\} \to f(x_1)$ and $r_0/2 \geq c_D(x_1, x_n) = c_D\big(f_n(x_1), f_n(x_n)\big) = c_D(f_n(x_1), a_n)$ for all n. Since $\{a_n\} \to a_0 \in D$, it follows that

$$c_D(f_n(x_1), a_0) < \frac{2r_0}{3} \text{ for all } n \geq n_0.$$

Then $\widetilde{c}_D(a_0, f(x_1)) \leq 2r_0/3 < r_0$, and hence, by (3), we get $f(x_1) \in D$, i.e., $f(D) \not\subset \partial D$. By Theorem 2.7, it follows that f is a biholomorphism from D onto D.

Since $\operatorname{Aut}(D) \Subset \operatorname{Hol}(D, M)$, it is easy to see that the sequence $\{f_n^{-1}\}$ converges locally uniformly to the mapping f^{-1} in $\operatorname{Aut}(D)$. This implies that $\lim_{n\to\infty} f_n^{-1}(a_n) = f^{-1}(a_0) \in D$. But this is absurd since $\lim_{n\to\infty} f_n^{-1}(a_n) = \lim_{n\to\infty} x_n = q_0 \in \partial D$. □

Proof of Theorem 3.2. Since D is a C-hyperbolic domain in a complex manifold M, D has the (IM) property for c_M. Hence Theorem 3.2 follows immediately from Lemma 3.3. □

From Theorem 3.2 we get the following corrected version of the above-mentioned result of Kim [9].

Corollary 3.4. *Let D be a bounded domain in $\mathbf{C}^n$ such that $D/\operatorname{Aut}(D)$ is compact. Then D is strongly complete C-hyperbolic if and only if for every finite boundary point $q \in \partial D$ and for every $\varepsilon > 0$, there exists a neighbourhood U of q in $\mathbf{C}^n$ such that $c_D(z, z') < \varepsilon$ for all $z, z' \in U \cap D$.*

Corollary 3.4 is deduced from Theorem 3.2 and the following lemma.

Lemma 3.5. *Let D be a bounded domain in $\mathbf{C}^n$. Then for each $a \in D$, there is $r > 0$ such that $\overline{B}_{c_D}(a, r)$ is compact in D.*

Proof. Let $a \in D$ be given.

Take a ball $B(0, R) = \{z \in \mathbf{C}^n : \|z\| < R\} \subset \mathbf{C}^n$ such that $\overline{D} \subset B(0, R) = B$. Since B is C-hyperbolic, there exists $r_0 > 0$ such that $\overline{B}_{c_B}(a, r_0) \subset D$. Since B is strongly complete C-hyperbolic, it follows that $\overline{B}_{c_B}(a, r_0)$ is a compact subset of D.

Suppose that $Cl_B \overline{B}_{c_D}(a, r) \cap \partial D \neq \emptyset$ for any $0 < r < r_0$, where $Cl_X Y$ is the closure of Y in X. This means that there exists a sequence $\{x_n\}_{n=n_0}^{\infty} \subset \partial D \cap Cl_B \overline{B}_{c_D}(a, 1/n)$.

Take $y_n \in B_{c_B}(x_n, 1/n) \cap \overline{B}_{c_D}(a, 1/n)$ for all $n \geq n_0$. Then $c_B(x_n, a) \leq c_B(x_n, y_n) + c_B(y_n, a) < 1/n + c_D(y_n, a) < 2/n$ for each $n \geq n_0$. Since ∂D is a compact subset of B, without loss of generality we may assume that $\{x_n\} \to x_0 \in \partial D$. Then $c_B(x_0, a) = 0$, i.e., $a = x_0 \in \partial D$. This is a contradiction. Hence there is $r \in (0, r_0)$ such that $\overline{B}_{c_D}(a, r)$ is a closed subset of B. But $\overline{B}_{c_D}(a, r) \subset \overline{B}_{c_B}(a, r_0)$, i.e., $\overline{B}_{c_D}(a, r)$ is a compact subset of D. □

Remark 3.6. By Montel's theorem, if D is a bounded domain in $\mathbf{C}^n$ then $\operatorname{Aut}(D) \Subset \operatorname{Hol}(D, \mathbf{C}^n)$. Unfortunately, this assertion does not hold for unbounded domains, even for complete hyperbolic (unbounded) domains in $\mathbf{C}^n$. Indeed, we have the following counterexample. Put $H = \{z \in \mathbf{C} : \operatorname{Im} z > 0\}$. Then H is complete hyperbolic. Define the mapping $f_n : H \to H$ given by $f_n(z) = z + n$ for each $z \in H$. It is easy to see that $\{f_n\} \subset \mathrm{Aut}(H)$ and the sequence $\{f_n\}$ is compactly divergent. This implies that $\mathrm{Aut}(H)$ is not relatively compact in $\mathrm{Hol}(H, \mathbf{C})$.

4. Generalization of the Greene–Krantz theorem to complex manifolds

First we give the following definitions.

DEFINITION 4.1. Let $\{\Omega_i\}_{i=1}^{\infty}$ be a sequence of open sets in a complex manifold M and let Ω_0 be an open set of M. The sequence $\{\Omega_i\}_{i=1}^{\infty}$ is said to converge to Ω_0, and we write $\lim \Omega_i = \Omega_0$, iff

(a) for every compact set $K \subset \Omega_0$ there is a $j = j(K)$ such that $i \geq j$ implies $K \subset \Omega_i$, and
(b) if K is a compact set which is contained in Ω_i for all sufficiently large i, then $K \subset \Omega_0$.

DEFINITION 4.2. Let $\mathbf{B}^n$ be the open unit ball in $\mathbf{C}^n$ and ρ_n the Bergman distance in $\mathbf{B}^n$. Let M be a complex manifold of dimension n. Denote by $\mathcal{F}(\mathbf{B}^n, M)$ the set of all mappings $f : \mathbf{B}^n \to M$ such that f is biholomorphic onto its image, i.e., the mapping $f : \mathbf{B}^n \to f(\mathbf{B}^n)$ is biholomorphic.

Given two points p, q of M, we consider a chain of holomorphic balls from p to q, that is, a chain of points $p = p_0, p_1, \ldots, p_k = q$ of M, points $a_1, \ldots, a_k$ of $\mathbf{B}^n$, and holomorphic mappings $f_1, \ldots, f_k \in \mathcal{F}(\mathbf{B}^n, M)$ such that $f_1(0) = p, f_i(a_i) = f_{i+1}(0), \ldots, f_k(a_k) = q$. Denoting this chain by α, we define its length $l(\alpha)$ by

$$l(\alpha) = \sum_{i=1}^{k} \rho_n(0, a_i)$$

and we define the pseudodistance $k_M^n(p, q) = \inf l(\alpha)$, where the infimum is taken over all chains α of holomorphic balls from p to q. The pseudodistance k_M^n is called the *n-Kobayashi pseudodistance* of M. If k_M^n is a distance, i.e., if $k_M^n(p, q) > 0$ for every pair $p, q \in M$ with $p \neq q$, then M is said to be *n-hyperbolic.*

Using the same argument as in [11], we have the following result.

PROPOSITION 4.3.

(i) *Let M be a complex manifold of dimension n. Then k_M^n is inner and continuous on $M \times M$.*
(ii) *If a complex manifold M is* n-hyperbolic *then k_M^n induces the standard topology of M.*
(iii) *Let M, N be two complex manifolds of dimension n. Assume that $f : M \to N$ is a holomorphic mapping such that the mapping f :*

$M \to f(M)$ is biholomorphic. Then $k^n_M(p,q) \geq k^n_N(f(p), f(q))$ for all $p, q \in M$. Therefore, k^n is invariant under biholomorphisms.

(iv) *Every hyperbolic manifold of dimension n is n-hyperbolic.*

Definition 4.4. Let D be a domain in a complex manifold M of dimension n. Denote by $\mathcal{F}(\mathbf{B}^n, D)$ the set of all mappings $f : \mathbf{B}^n \to D$ such that f is biholomorphic onto its image, i.e., the mapping $f : \mathbf{B}^n \to f(\mathbf{B}^n)$ is biholomorphic. The domain D is said to be *pseudo-taut* if for every sequence $\{f_k\}$ in $\mathcal{F}(\mathbf{B}^n, D)$, either

(i) the sequence $\{f_k\}$ is compactly divergent, i.e., for each compact set $K \subset \mathbf{B}^n$ and each compact set $L \subset D$, there is an integer k_0 such that $f_k(K) \cap L = \emptyset$ for $k \geq k_0$, or

(ii) there are a subsequence $\{f_{k_j}\}$ and a closed subset $S \subset \mathbf{B}^n$ with $\overline{\mathbf{B}^n \setminus S} = \mathbf{B}^n$ (which may depend on $\{f_{k_j}\}$) such that the sequence $\{f_{k_j} : \mathbf{B}^n \to M\}$ converges uniformly on compact subsets of $\mathbf{B}^n$ to a limit $f : \mathbf{B}^n \to M$ with $S = f^{-1}(\partial D)$.

Examples.

(i) Every bounded domain in $\mathbf{C}^n$ is pseudo-taut.

(ii) Every taut complex manifold is pseudo-taut.

(iii) Put $D = \Delta \times \Delta \setminus \{(0,0)\} \subset \mathbf{C}^2$. It is easy to see that D is pseudo-taut, but not taut.

Proposition 4.5. *Let D be a pseudo-taut domain in a complex manifold M of dimension n. Then D is $n-$hyperbolic.*

Proof. We start with the following lemma.

Lemma. *Let U, V, W and U' be open subsets of D such that $\overline{U} \cap \overline{U}' = \emptyset$ and $W \Subset V \Subset U$ and U is n-hyperbolic. Assume that there exists a positive number $\delta < 1$ such that, for every $f \in \mathcal{F}(\mathbf{B}^n, D)$ with $f(0) \in V$, we have $f(\mathbf{B}^n_\delta) \subset U$, where $\mathbf{B}^n_\delta = \{z \in \mathbf{C}^n : ||z|| < \delta\}$. Then $k^n_D(W, U') > 0$.*

The proof of this lemma is the same as that of [11, Lemma 5.1.4, p. 241].

We can now prove Proposition 4.5.

Assume that there are $p, q \in D$ with $p \neq q$ such that $k^n_D(p,q) = 0$. Take open neighbourhoods U, V, W of p and U' of q such that $W \Subset V \Subset U$ and U is n-hyperbolic and $\overline{U} \cap \overline{U}' = \emptyset$. Then, by the above lemma, it follows that, for each $k \geq 1$, there is $f_k \in \mathcal{F}(\mathbf{B}^n, D)$ with $f_k(0) \in V$ such that $f_k(\mathbf{B}^n_{1/k}) \nsubseteq U$. Since the sequence $\{f_k\}$ is not compactly divergent, it follows that there are a subsequence $\{f_{k_j}\}$ and a closed subset $S \subset \mathbf{B}^n$ with $\overline{\mathbf{B}^n \setminus S} = \mathbf{B}^n$ such that $\{f_{k_j} : \mathbf{B}^n \to M\}$ converges uniformly on compact subsets of $\mathbf{B}^n$ to a holomorphic mapping $f : \mathbf{B}^n \to M$ with $S = f^{-1}(\partial D)$.

Since $\{f_k(0)\} \subset V$, it follows that $0 \notin S$, i.e., $0 \in \mathbf{B}^n \setminus S$ and $f(0) \in \overline{V} \subset U$. Take k_0 large enough such that $\mathbf{B}^n_{1/k_0} \subset \mathbf{B}^n \setminus S$ and $f(\mathbf{B}^n_{1/k_0}) \subset U$. Then, for j large enough, we get $f_{k_j}(\mathbf{B}^n_{1/k_j}) \subset U$. This is impossible. □

PROPOSITION 4.6. *Let D be a domain in a complex manifold M of dimension n. Then D is pseudo-taut if and only if, for each complex manifold X of dimension n and for each sequence $\{f_k\} \subset \mathcal{F}(X, D)$, either*

(i) *the sequence $\{f_k\}$ is compactly divergent, i.e., for each compact set $K \subset X$ and each compact set $L \subset D$, there is an integer k_0 such that $f_k(K) \cap L = \emptyset$ for $k \geq k_0$, or*

(ii) *there are a subsequence $\{f_{k_j}\}$ and a closed subset $S \subset X$ with $\overline{X \setminus S} = X$ (which may depend on $\{f_{k_j}\}$) such that the sequence $\{f_{k_j} : X \to M\}$ converges uniformly on compact subsets of X to a limit $f : X \to M$ with $S = f^{-1}(\partial D)$, where $\mathcal{F}(X, D)$ is the set of all mappings $f : X \to D$ such that f is biholomorphic onto its image, i.e., the mapping $f : X \to f(X)$ is biholomorphic.*

Proof. The sufficiency of the condition is obvious, so it remains to prove the necessity. We divide the argument into three steps.

Step 1. By Lemma 4.5, D is n-hyperbolic.

Assume that X is a complex manifold of dimension n and $\{f_k\} \subset \mathcal{F}(X, D)$. Assume that the sequence $\{f_k\}$ is not compactly divergent. Then there exist compact subsets $K \subset X, L \subset D$ such that $f_{k_j}(K) \cap L \neq \emptyset (j \geq 1)$. Without loss of generality, we may assume that $f_k(K) \cap L \neq \emptyset (k \geq 1)$. Then there exists $\{x_k\} \subset K$ such that $\{f_k(x_k)\} \subset L$. Without loss of generality, we may also assume that $\{x_k\} \to p \in K$ and $\{f_k(x_k)\} \to z_0 \in L$. Take a neighbourhood V of L in D. Since $k^n_D(f_k(x_k), f_k(p)) \leq k^n_X(x_k, p) \to 0$ as $k \to \infty$, it follows that $f_k(p) \in V$ for k large enough.

Let q be any point of X. We consider a chain of holomorphic balls from p to q, that is, a chain of points $p = p_0, p_1, \ldots, p_t = q$ of X, points $a_1, \ldots, a_t$ of $\mathbf{B}^n$, and holomorphic mappings $h_1, \ldots, h_t \in \mathcal{F}(\mathbf{B}^n, X)$ such that $h_1(0) = p, h_i(a_i) = h_{i+1}(0) = p_i, h_t(a_t) = q$. Clearly, $\{f_k \circ h_1\} \subset \mathcal{F}(\mathbf{B}^n, D)$ and $f_k \circ h_1(0) = f_k(p) \in V$ for k large enough. By the pseudo-tautness of D there exist a sequence $\{a_1^{(\mu)}\} \subset \mathbf{B}^n$ which converges to a_1 and an infinite subset $\mathbf{N}_1$ of $\mathbf{N}$ such that $\{(f_k \circ h_1)(a_1^{(\mu)})\}_{k \in \mathbf{N}_1} \Subset D$ for all $\mu \geq 1$. We see that $\{h_1(a_1^{(\mu)})\} \to h_1(a_1) = h_2(0)$. Thus there is μ_0 large enough such that $h_1(a_1^{(\mu_0)}) \in h_2(\mathbf{B}^n)$. Take $z_1 \in \mathbf{B}^n$ such that $h_1(a_1^{(\mu_0)}) = h_2(z_1)$, and hence, $\{(f_k \circ h_2)(z_1)\}_{k \in \mathbf{N}_1} \Subset D$. By the pseudo-tautness of D there exist a sequence $\{a_2^{(\mu)}\} \subset \mathbf{B}^n$ which converges to a_2 and an infinite subset $\mathbf{N}_2$ of $\mathbf{N}_1$ such that $\{(f_k \circ h_2)(a_2^{(\mu)})\}_{k \in \mathbf{N}_2} \Subset D$ $(\mu \geq 1)$.

Continuing this process, we obtain a sequence $\{a_t^{(\mu)}\} \subset \mathbf{B}^n$ which converges to a_t and an infinite subset $\mathbf{N}_q$ of $\mathbf{N}$ such that $\{(f_k \circ h_t)(a_t^{(\mu)})\}_{k\in\mathbf{N}_q} \Subset D$ ($\mu \geq 1$). Roughly speaking, for each $q \in X$, there exist a sequence $\{q^{(\mu)}\}_{\mu=1}^{\infty} \subset X$ and an infinite subset $\mathbf{N}_q$ of $\mathbf{N}$ such that $\{f_k(q^{(\mu)})\}_{k\in\mathbf{N}_q} \Subset D$ ($\mu \geq 1$).

By the Cantor diagonalization process, it follows that there exist an infinite subset $\mathbf{N}_0$ of $\mathbf{N}$ and a countable subset X_1 such that $\overline{X}_1 = X$ and $\{f_k(z)\}_{k\in\mathbf{N}_0} \Subset D$ for each $z \in X_1$.

Denote by $\tilde{S}$ the set of all points $z \in X$ such that $\{f_k(z)\}_{k\in\mathbf{N}_0}$ is not relatively compact in D. Then $X \setminus \tilde{S}$ is open. Indeed, take $z_0 \in X \setminus \tilde{S}$. Then $\{f_k(z_0)\}_{k\in\mathbf{N}_0} \Subset D$. Since $k_D^n(f_k(z_0), f_k(z)) \leq k_X^n(z_0, z)$ ($z \in X$), it follows that $\{f_k(z)\}_{k\in\mathbf{N}_0} \Subset D$ for each $z \in X$ closed enough to z_0.

On the other hand, since $X_1 \subset X \setminus \tilde{S}$, we have $\overline{X \setminus \tilde{S}} = X$.

We cover $X \setminus \tilde{S}$ by a family $\{X_i\}_{i=1}^{\infty}$ of open subsets of X, where each X_i is biholomorphic to $\mathbf{B}^n$.

Since D is n-hyperbolic, the family $\{f_{k|X_1}\}_{k\in\mathbf{N}_0}$ is equicontinuous. Since $\{f_k(z)\}_{k\in\mathbf{N}_0} \Subset D$ for each $z \in X_1$, by the Arzela-Ascoli theorem, there exists an infinite subset $\mathbf{N}_0^{(1)}$ of $\mathbf{N}_0$ such that the sequence $\{f_{k|X_1}\}_{k\in\mathbf{N}_0^{(1)}}$ converges uniformly in $\mathrm{Hol}(X_1, D)$.

Consider the family $\{f_{k|X_2}\}_{k\in\mathbf{N}_0^{(1)}}$. Repeating the above argument, we obtain an infinite subset $\mathbf{N}_0^{(2)}$ of $\mathbf{N}_0^{(1)}$ such that the sequence $\{f_{k|X_2}\}_{k\in\mathbf{N}_0^{(2)}}$ converges uniformly in $\mathrm{Hol}(X_2, D)$. Continuing this process and using the Cantor diagonalization process, we obtain an infinite subset $\widetilde{\mathbf{N}}_0$ of $\mathbf{N}_0$ such that the sequence $\{f_{k|X\setminus\tilde{S}}\}_{k\in\widetilde{\mathbf{N}}_0}$ converges uniformly on compact subsets of $X \setminus \tilde{S}$ to a mapping $f \in \mathrm{Hol}(X \setminus \tilde{S}, D)$.

Step 2. Cover X by a family $\{U_i\}_{i=1}^{\infty}$ of open subsets of X, where each $\alpha_i : \mathbf{B}^n \to U_i$ is biholomorphic.

Consider the family $\{f_k \circ \alpha_1\}_{k\in\widetilde{\mathbf{N}}_0} \subset \mathrm{Hol}(\mathbf{B}^n, D)$. Since this family is not compactly divergent, by the pseudo-tautness of D it follows that there exists an infinite subset $\widetilde{\mathbf{N}}_0^{(1)}$ of $\widetilde{\mathbf{N}}_0$ such that the sequence $\{f_k \circ \alpha_1\}_{k\in\widetilde{\mathbf{N}}_0^{(1)}}$ converges uniformly in $\mathrm{Hol}(\mathbf{B}^n, M)$. This implies that the sequence $\{f_k\}_{k\in\widetilde{\mathbf{N}}_0^{(1)}}$ converges uniformly in $\mathrm{Hol}(U_1, M)$.

Consider the family $\{f_k \circ \alpha_2\}_{k\in\widetilde{\mathbf{N}}_0^{(1)}} \subset \mathrm{Hol}(\mathbf{B}^n, D)$. Repeating the above argument, there exists an infinite subset $\widetilde{\mathbf{N}}_0^{(2)}$ of $\widetilde{\mathbf{N}}_0^{(1)}$ such that the sequence $\{f_k\}_{k\in\widetilde{\mathbf{N}}_0^{(2)}}$ converges uniformly in $\mathrm{Hol}(U_2, M)$. Continuing this process and using the Cantor diagonalization process, we obtain an infinite subset $\widetilde{\widetilde{\mathbf{N}}}_0$ of $\widetilde{\mathbf{N}}_0$ such that the sequence $\{f_k\}_{k\in\widetilde{\widetilde{\mathbf{N}}}_0}$ converges uniformly on compact subsets of X to a mapping $F \in \mathrm{Hol}(X, M)$. It is easy to see that $F_{|X\setminus\tilde{S}} = f$.

Step 3. Put $S = F^{-1}(\partial D)$. Then S is closed in X and $S \subset \tilde{S}$. Hence there exists a subsequence $\{f_k\}_{k\in\widetilde{\widetilde{\mathbf{N}}}_0}$ of the sequence $\{f_k\}$ and a closed subset $S \subset X$ with $\overline{X \setminus S} = X$ such that the sequence $\{f_k : X \to M\}_{k\in\widetilde{\widetilde{\mathbf{N}}}_0}$ converges uniformly on compact subsets of X to a limit $F : X \to M$ with $S = F^{-1}(\partial D)$. □

Proof of Theorem 4.7. Take a sequence of open subsets $A_0^1 \Subset A_0^2 \Subset \cdots$ of A_0 such that $A_0 = \bigcup_{j=1}^\infty A_0^j$. Without loss of generality we may assume that $A_0^1 \subset A_j$ for each $j \geq 1$. Assume that the sequence $\{f_i\}$ is not compactly divergent. Then there exist compact subsets $K \subset A_0, L \subset \Omega_0$ such that $f_{i_j}(K) \cap L \neq \emptyset$ for all $j \geq 1$. Without loss of generality, we may assume that $K \subset A_0^1$ and $f_i(K) \cap L \neq \emptyset$ $(i \geq 1)$. Then there exists $\{x_i\} \subset K$ such that $\{f_i(x_i)\} \subset L$. Without loss of generality, we may also assume that $\{x_i\} \to p \in K$ and $\{f_i(x_i)\} \to q \in L$.

Consider the sequence $\{f_{i|A_0^1}\}_{i=1}^\infty \subset \mathrm{Hol}(A_0^1, \Omega_0)$. Then this sequence is not compactly divergent. By the pseudo-tautness of Ω_0, it follows that there are an infinite subset $\mathbf{N}_1$ of $\mathbf{N}$ and a closed subset $S_1 \subset A_0^1$ with $\overline{A_0^1 \setminus S_1} = A_0^1$ such that the sequence $\{f_i : A_0^1 \to M\}$ converges uniformly on compact subsets of A_0^1 to a limit $F_1 : A_0^1 \to M$ with $S_1 = F_1^{-1}(\partial\Omega_0)$.

Consider the sequence $\{f_{i|A_0^2}\}_{i\in\mathbf{N}_1} \subset \mathrm{Hol}(A_0^2, \Omega_0)$. Repeating the above argument, we see that there are an infinite subset $\mathbf{N}_2$ of $\mathbf{N}_1$ and a closed subset $S_2 \subset A_0^2$ with $\overline{A_0^2 \setminus S_2} = A_0^2$ such that the sequence $\{f_i : A_0^2 \to M\}$ converges uniformly on compact subsets of A_0^2 to a limit $F_2 : A_0^2 \to M$ with $S_2 = F_2^{-1}(\partial\Omega_0)$. It is easy to see that $F_{2|A_0^1} = F_1$ and $S_2 \cap A_0^1 = S_1$.

Continuing this process and using the Cantor diagonalization process, we obtain an infinite subset $\mathbf{N}_0$ of $\mathbf{N}$ and a closed subset $S = \bigcup_{i=1}^\infty S_i$ of A_0 with $\overline{A_0 \setminus S} = A_0$ such that the sequence $\{f_i\}_{i\in\mathbf{N}_0}$ converges uniformly on compact subsets of A_0 to a mapping $F \in \mathrm{Hol}(A_0, M)$ with $S = F^{-1}(\partial\Omega_0)$. It is easy to see that $F(p) = q \in \Omega_0$.

Put $g_i = f_i^{-1}$ for all $i \in \mathbf{N}_0$. We now show that $\{g_i(q) = p_i\}_{i\in\mathbf{N}_0} \to p$. Indeed, take a relatively compact open neighbourhood U of q in Ω_0. Then, for all i large enough, we have

$$\begin{aligned} k^n_{A_0}(p, p_i) &\leq k^n_{A_i}(p, p_i) = k^n_{\Omega_i}(f_i(p), f_i(p_i)) \\ &= k^n_{\Omega_i}(f_i(p), q) \leq k^n_U(f_i(p), q) \to 0 \text{ as } i \to \infty. \end{aligned}$$

Since A_0 is n-hyperbolic, it follows that $\{p_i\} \to p$, i.e., $\{g_i(q)\}_{i\in\mathbf{N}_0} \to p$. This implies that the sequence $\{g_i\}_{i\in\mathbf{N}_0}$ is not compactly divergent.

Repeating the above argument, we obtain an infinite subset $\widetilde{\mathbf{N}}_0$ of $\mathbf{N}_0$ and a closed subset $T \subset \Omega_0$ with $\overline{\Omega_0 \setminus T} = \Omega_0$ such that the sequence $\{g_i : \Omega_0 \to M\}_{i\in\widetilde{\mathbf{N}}_0}$ converges uniformly on compact subsets of Ω_0 to a limit $G : \Omega_0 \to M$ with $T = G^{-1}(\partial A_0)$.

Take a neighbourhood V of p in A_0 such that $F(V) \subset \Omega_0$. Repeating the proof of (*), we see that $\{g_i(F(z))\}_{i\in\mathbf{N}_0} \to z$ for each $z \in V$. This implies that $G(F(z)) = z$ $(z \in V)$. Hence $F_{|V}$ is injective. By the Osgood theorem, the mapping $F_{|V} : V \to F(V)$ is biholomorphic.

Consider the holomorphic functions $J_i : A_i \to \mathbf{C}$ and $J : A_0 \to \mathbf{C}$ given by $J_i(z) = \det\big((df_i)_z\big)$ and $J(z) = \det\big((dF)_z\big)$. Then $J(z) \neq 0$ $(z \in V)$ and, for each $i \in \mathbf{N}_0$, the function J_i is non-vanishing on A_i. Moreover, the sequence $\{J_i\}_{i\in\mathbf{N}_0}$ converges uniformly on compact subsets of A_0 to J. By Hurwitz's theorem, it follows that J never vanishes. This implies that the mapping $F : A_0 \to M$ is open and any $z \in A_0$ is isolated in $F^{-1}\big(F(z)\big)$. According to Lemma 2.7.1, we have $F(A_0) \subset \Omega_0$.

Repeating this argument, we see that $G(\Omega_0) \subset A_0$. But then uniform convergence allows us to conclude that for all $z \in A_0$ we have $G \circ F(z) = \lim_{i\in\widetilde{\mathbf{N}}_0, i\to\infty} G_i \circ F_i(z) = z$ and likewise that for all $w \in \Omega_0$ we have $F \circ G(w) = \lim_{i\in\widetilde{\mathbf{N}}_0, i\to\infty} F_i \circ G_i(w) = w$.

This proves that F and G are each one-to-one and onto, hence in particular that F is a biholomorphic mapping. □

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