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DECOMPOSITION OF THE ζ-DETERMINANT FOR THE LAPLACIAN ON MANIFOLDS WITH CYLINDRICAL END

PAUL LOYA AND JINSUNG PARK

ABSTRACT. In this paper we combine elements of the b-calculus and elliptic boundary problems to solve the decomposition problem for the (regularized) ζ-determinant of the Laplacian on a manifold with cylindrical end into the ζ-determinants of the Laplacians with Dirichlet conditions on the manifold with boundary and on the half infinite cylinder. We also compute all the contributions to this formula explicitly.

1. Introduction

We investigate the 'Mayer-Vietoris' or 'cut and paste' decomposition formula of the ζ-determinant for a Laplacian on a manifold with cylindrical end into the ζ-determinants of the Laplacians with Dirichlet conditions on the manifold with boundary and on the half infinite cylinder. We also introduce a new method to attack such surgery problems by comparing the problem to a corresponding *model problem*. This approach works for compact manifolds as well as manifolds with cylindrical ends and will be used to solve related decomposition problems for the spectral invariants of Dirac type operators in [12], [13]. We remark that the noncompactness of the underlying manifold introduces many new facets and obstacles not found in the compact case, as we will explain later. We begin with a brief account of zeta determinants.

The ζ-determinant of a Laplace type operator was pioneered in the seminal paper [21] by Ray and Singer. They were seeking an analytic version of the so-called Reidemeister torsion, a combinatorial-topological invariant introduced by Reidemeister [22] and Franz [6]. They conjectured that their analytic invariant was the same as the Reidemeister torsion. Later, this conjecture was proved independently by Cheeger [4] and Müller [17]. The ζ-determinants have also been of great use in quantum field theory where they are being used to develop rigorous models for Feynman path integral techniques [10]. Because of their use in differential topology and quantum field theory, much work has been done on understanding the nature of ζ-determinants, especially

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their behavior under 'cutting and pasting' of manifolds. This was initiated on compact manifolds by Burghelea, Friedlander, and Kappeler [2]. Their method can be considered as a modification of Forman's [5] variation argument of the ζ-determinant to the decomposition problem of the ζ-determinant. Recently the second author and Wojciechowski proved the adiabatic decomposition formula of the ζ-determinant [19], where adiabatic means that the length of the neck near the cutting hypersurface is stretched longer and longer and the limit of the ratio of the ζ-determinants of the whole manifold and the decomposed manifolds under this process is investigated. This approach has a close relation with scattering theory over the manifold with cylindrical end obtained in the limit. In a similar way, one can examine ζ-determinants on manifolds with boundary by attaching a half infinite cylinder to the boundary and considering the corresponding invariant for the manifold with cylindrical end; one must then be able to relate this new invariant to the invariant for the original manifold with boundary. This approach was employed in the seminal book of Melrose [15] in the framework of index theory for manifolds with cylindrical end and further developed in joint work with Hassell and Mazzeo [8] where extensive analytic tools were developed to study 'analytic surgery,' cf. [14]. Using these analytic tools, Piazza [20] derived surgery formulas for determinant bundles and Hassell [7] proved a b-surgery formula for the b-analytic torsion. The present paper falls into this 'b-category' approach, as we now explain.

Let X be a manifold with cylindrical end of arbitrary dimension, that is, we have a decomposition

$$X = M \cup Z,$$

where M is a manifold with boundary Y and $Z = [0,\infty) \times Y$ is a half infinite cylinder. We also assume that M has a tubular neighbourhood $N = [-1,0]\times Y$ of Y. Let Δ_X be a Laplace type operator acting on $C^\infty(X,E)$, where E is a Hermitian vector bundle over X. We assume that the Riemannian metric over X and the Hermitian metric of E have product structures over $\hat{Z} := N \cup Z = [-1,\infty)_u \times Y$, where u is the cylindrical variable. Hence Δ_X has the following form over $\hat{Z}$:

$$\Delta_X|_{\hat{Z}} \; = \; -\partial_u^2 + \Delta_Y,$$

where Δ_Y is a Laplace type operator over Y. Then, by restriction, Δ_X induces Laplace operators over M, Z and we denote these operators by Δ_M, Δ_Z, respectively. For Δ_M, Δ_Z, we impose Dirichlet boundary conditions and denote the resulting operators by $\Delta_{M,d}$, $\Delta_{Z,d}$, respectively. Finally, we assume that the Dirichlet operator on M, $\Delta_{M,d}$, is invertible. This last assumption is satisfied by many operators, for example, if X is connected and Δ_X is a Dirac Laplacian such as the Hodge Laplacian acting on $C^\infty(X, \wedge^q T^*X \otimes V_\rho)$, where (ρ, V_ρ) is a unitary representation of $\pi_1(X)$.

The ζ-determinant of $\Delta_{M,d}$ is defined in the standard way,

$$\det{}_{\zeta}\Delta_{M,d} := \exp\Big(- \frac{d}{ds}\Big|_{s=0}\zeta(s,\Delta_{M,d})\Big),$$

where the ζ-function $\zeta(s,\Delta_{M,d})$ is defined by means of the heat operator through the integral

$$\zeta(s,\Delta_{M,d}) := \frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}\,\mathrm{Tr}(e^{-t\Delta_{M,d}})\;dt.$$

Here the ζ-function $\zeta(s,\Delta_{M,d})$ is a priori defined for $\Re s \gg 0$ and has a meromorphic extension over $\mathbb{C}$ with the origin as a regular point. For the manifold with cylindrical end X, the heat operator $e^{-t\Delta_X}$ is not of trace class. To define the corresponding ζ-determinant, it is therefore necessary to introduce an appropriate regularization of the trace. One natural regularization is Melrose's b-trace [15], which leads to the b-determinant. Thus, ${}^b\mathrm{Tr}\, e^{-t\Delta_X}$ is well-defined and, see Section 2, this b-heat trace has asymptotic expansions in half-integer powers of t as $t \to 0$ and $t \to \infty$. The ${}^b\zeta$-function ${}^b\zeta(s,\Delta_X)$ is defined as the sum of the meromorphic extensions of the functions

$$\frac{1}{\Gamma(s)}\int_0^1 t^{s-1}\;\; {}^b\mathrm{Tr}(e^{-t\Delta_X})\;dt,$$

defined a priori for $\Re s \gg 0$, and

$$\frac{1}{\Gamma(s)}\int_1^\infty t^{s-1}\;\; {}^b\mathrm{Tr}(e^{-t\Delta_X})\;dt,$$

defined a priori for $\Re s \ll 0$. Then the (b-) ζ-determinant of Δ_X is defined as

$$\det{}_{{}^b\zeta}\Delta_X := \exp\Big(- \frac{d}{ds}\Big|_{s=0}{}^b\zeta(s,\Delta_X)\Big).$$

Similarly, one can define the b-determinant $\det_{{}^b\zeta}\Delta_{Z,d}$. An elementary introduction to the b-trace is presented in Section 2.

The main concern of this paper is the decomposition of $\det_{{}^b\zeta}\Delta_X$ in terms of $\det_\zeta\Delta_{M,d}$ and $\det_{{}^b\zeta}\Delta_{Z,d}$, which can be considered as a decomposition of the ζ-determinant over X into contributions from the compact part M and the cylindrical part Z in the spirit of Burghelea, Friedlander and Kappeler [2]. A related idea can be found in the paper of Hassell and Zelditch [9], where they considered a similar problem for an exterior domain in $\mathbb{R}^2$.

To derive the decomposition formula of the ζ-determinant over X, we develop a new method by introducing an *auxiliary model problem* over the cylindrical part. That is, we consider the corresponding problem for the decomposition of $\hat{Z} = N \cup Z$ into N and Z and the core of our approach is to compare the original problem with this model problem. One advantage of our approach is that the difference of the operators on X and the operators of the model problem is of trace class; this allows us to avoid certain trace class issues in [2]. Moreover, the explicit computations of the ζ-determinants over the cylindrical part enable us to get the explicit value of the (a priori unknown) *BFK constant* present in the original formula found in [2].

There are new features of our final result not present in the compact case due to the noncompactness of X, but before explaining this we recall some results in [2]. When the Laplacian Δ acting on functions over a compact 2-dimensional manifold without boundary has a nontrivial kernel (which is automatically an L^2-solution over the compact manifold), the formula proved in [2] contains an additional term originating from the kernel of Δ. The corresponding phenomenon appears in the case considered in [9], where the bounded solution of the Laplacian—the constant function over $\mathbb{R}^2$—contributes to their final formula. Hence one may conjecture that both L^2-solutions and bounded solutions of Δ_X would contribute to our formula; in fact, this conjecture is indeed true. To discuss this phenomenon, we introduce some more notations. Let $\{u_j\}$ be an orthonormal basis for the kernel of Δ_X on $L^2(X,E)$ and let $\{U_j\}$ be a basis of the 'extended L^2-solutions' (bounded solutions of $\Delta_X U_j = 0$) such that at ∞ on the cylinder, $\{U_j(\infty)\}$ are orthonormal in $L^2(Y,E_0)$ where $E_0 := E|_Y$. Let $v_j = u_j|_Y$ and $V_j = U_j|_Y$ be the restrictions of u_j and U_j, respectively, to the hypersurface $\{0\}\times Y$. By Lemma A.3 to be established in the Appendix, the sections $\{v_j, V_j\}$ are linearly independent in $L^2(Y,E_0)$; therefore both operators

$$L = \sum_j v_j \otimes v_j^*, \qquad \widetilde{L} = \sum_j V_j \otimes V_j^* \tag{1.1}$$

are nonnegative linear operators on the finite-dimensional vector space $V = \operatorname{span}\{v_j,\ V_j\} \subset L^2(Y,E_0)$. Since the set $\{v_j,\ V_j\}$ is a linearly independent set spanning V, the operator

$$L + \widetilde{L} : V \longrightarrow V$$

is positive. In particular, $\det(L+\widetilde{L})$ is nonzero.

The final ingredient we need is an operator $\mathcal{R}$ over Y, which is defined to be the sum of the Dirichlet to Neumann operators for Δ_M and Δ_Z. In Theorem A.4, we prove that $\mathcal{R}$ is a nonnegative first order elliptic classical pseudodifferential operator, so that its ζ-regularized determinant is well defined.

We can now state our main theorem:

Theorem 1.1. *When $\Delta_{M,d}$ is invertible, the following decomposition formula holds:*

$$\frac{\det_{{}^b\zeta}\Delta_X}{\det_\zeta \Delta_{M,d}\cdot \det_{{}^b\zeta}\Delta_{Z,d}} \;=\; 2^{-\zeta(0,\Delta_Y)-h_Y}\,\frac{\det_\zeta \mathcal{R}}{\det(L+\widetilde{L})}, \tag{1.2}$$

where $\zeta(s,\Delta_Y)$ is the ζ-function of Δ_Y and $h_Y := \dim\ker\Delta_Y$.

There are a couple of ways to rewrite formula (1.2). First, one can explicitly compute that $\det_{{}^b\zeta}\Delta_{Z,d} = e^{-\log\det_\zeta \Delta_Y/4}$ (see Equation (2.6)), so our main formula can be written

$$\frac{\det_{{}^b\zeta}\Delta_X}{\det_\zeta \Delta_{M,d}} \;=\; 2^{-\zeta(0,\Delta_Y)-h_Y}\,\frac{\det_\zeta \mathcal{R}}{\det(L+\widetilde{L})}\cdot e^{-\frac{1}{4}\log\det_\zeta \Delta_Y}.$$

This theorem can be recast in terms of relative determinants studied by Müller [18]. One can show that $\det_{{}^b\zeta}\Delta_X/\det_{{}^b\zeta}\Delta_{Z,d} = \det_\zeta(\Delta_X, \Delta_{Z,d})$, the relative determinant of the pair $(\Delta_X, \Delta_{Z,d})$. Thus, our main formula can be written

$$\frac{\det_\zeta(\Delta_X, \Delta_{Z,d})}{\det_\zeta \Delta_{M,d}} \;=\; 2^{-\zeta(0,\Delta_Y)-h_Y}\,\frac{\det_\zeta \mathcal{R}}{\det(L+\widetilde{L})}. \tag{1.3}$$

In the recent preprint [16], Müller and Müller derived this relative version using Carron's relative determinant formula [3, Theorem 1.4] which implies that there is a polynomial P such that for $\lambda \notin (-\infty, 0]$,

$$\frac{\det_\zeta(\Delta_X+\lambda, \Delta_{Z,d}+\lambda)}{\det_\zeta(\Delta_{M,d}+\lambda)} \;=\; e^{P(\lambda)}\det_\zeta \mathcal{R}(\lambda).$$

The formula (1.3) is derived in [16] from Carron's formula by taking $\lambda \to 0^+$. However, our proof is independent of this result and our proof can be adapted to solve decomposition problems involving pseudodifferential boundary problems for Dirac operators [12], [13].

Remark 1.2. We can modify the proof of Theorem 1.1 to derive a similar formula when a manifold X with cylindrical ends is decomposed into two manifolds with cylindrical ends. More precisely, suppose that the Laplace type operator Δ_X over X is of product type on a collar neighbourhood of a cutting hypersurface H. If Δ_d denotes the corresponding Laplacian with Dirichlet boundary conditions over the decomposed manifolds with cylindrical ends and has no bounded solutions, then

$$\frac{\det_{{}^b\zeta}\Delta_X}{\det_{{}^b\zeta}\Delta_d} \;=\; 2^{-\zeta(0,\Delta_H)-h_H}\,\frac{\det_\zeta \mathcal{R}}{\det(L+\widetilde{L})},$$

where $\mathcal{R}$, L and $\widetilde{L}$ are the corresponding operators on H defined as above. This problem will be studied elsewhere.

We now explain the structure of this paper. In Section 2 we provide an elementary introduction to the b-integral and b-trace. In Section 3, we introduce basic material about elliptic boundary problems for Laplace type operators. In Section 4, we introduce the *model operators* over the cylindrical part and study their relations with the original operators. In Section 5, we combine the variation argument of the ζ-determinant and the comparison with the objects on the cylinder to get the basic equality for our main result. In Section 6, we state and combine all the ingredients necessary to prove our main result. In Section 7, we explicitly compute the ζ-determinants over the cylindrical part. In doing this, we get the explicit value of the *BFK constant*, that is,

$2^{-\zeta(0,\Delta_Y)-h_Y}$ in our case. In Section 8, for a parameter $\lambda \in \mathbb{R}^+$ we consider the asymptotics of $\det_{{}^b\zeta}(\Delta_X + \lambda)$, $\det_{{}^b\zeta}(\Delta_{Z,d} + \lambda)$, and $\det_\zeta \mathcal{R}(\lambda)$ as $\lambda \to 0^+$, where $\mathcal{R}(\lambda)$ is the operator for $\Delta_X + \lambda$. The asymptotics of $\det_\zeta \mathcal{R}(\lambda)$ determine the contribution $\det(L + \widetilde{L})$. Finally, in Appendix A, we discuss the analytic properties of $\mathcal{R}(\lambda)$ for $\lambda \in [0, \infty)$ that are used in the main body of the paper.

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2. Introduction to the b-integral and b-trace

The heat operator $e^{-t\Delta_X}$ has a simple structure on the collar of X described as follows. On the collar $Z = [0,\infty)_u \times Y$ of X, we have

$$e^{-t\Delta_X}(u, u', y, y') = \frac{1}{\sqrt{4\pi t}}\, e^{-(u-u')^2/(4t)} e^{-t\Delta_Y} + h(t, u, u', y, y'),$$

where, for fixed $t > 0$, $h(t, u, u', y, y') = O(e^{-u/2}\, e^{-u'/2})$. This can be proved in various ways; for instance, one can construct the heat kernel 'by hand' as in [1] or one can appeal to the theory of b-pseudodifferential operators [15]. Restricting this Schwartz kernel to the diagonal, we obtain

$$\operatorname{tr} e^{-t\Delta_X}|_{\mathrm{Diag}} = \frac{1}{\sqrt{4\pi t}} \operatorname{tr} e^{-t\Delta_Y}(y, y) + h(t, u, u, y, y). \tag{2.1}$$

Although the second term $h(t, u, u, y, y) = O(e^{-u})$, which is integrable on the infinite cylinder Z, the first term is constant with respect to u, so is *not* integrable on the infinite cylinder. In particular, the integral of (2.1) over X diverges, so the heat trace defined via the Lidskiĭ [11] trace formula is not defined. This shows that in order to develop *heat kernel methods* on manifolds with cylindrical ends, we need another notion of trace. One such notion was provided by Melrose [15] and is called the b-trace described as follows. Let f be a locally integrable function on X and suppose that on the infinite cylinder Z we have $f(u, y) = c + \tilde{f}(u, y)$, where c is a constant and $\tilde{f}$ is integrable. Then we see that the constant c is exactly the obstruction to f being integrable on X. We define the b-integral of f by simply killing this obstruction:

$${}^b\!\!\int_X f := \int_M f + \int_Z \tilde{f}(u, y)\, du\, dy,$$

where dy is the measure on Y. The *b-trace* of the heat operator $e^{-t\Delta_X}$ is defined in terms of the b-integral via

$${}^b\mathrm{Tr}\, e^{-t\Delta_X} := {}^b\!\!\int_X \operatorname{tr} e^{-t\Delta_X}|_{\mathrm{Diag}},$$

that is, using the decomposition (2.1),

$$ {}^b\mathrm{Tr}\, e^{-t\Delta_X} = \int_M \mathrm{tr}\, e^{-t\Delta_X}|_{\mathrm{Diag}} + \int_Z h(t,u,u,y,y)\, du\, dy. $$

In [15] it is proved that the b-trace of the heat operator has the usual short time asymptotic expansion:

$$ {}^b\mathrm{Tr}\, e^{-t\Delta_X} \sim \sum_{k=0}^{\infty} a_k\, t^{(k-n)/2} \qquad \text{as } t \to 0, \tag{2.2} $$

where $n = \dim X$; the pointwise trace of the heat kernel on the diagonal also has such an expansion. There is a related long time asymptotic expansion (see [7, Appendix]):

$$ {}^b\mathrm{Tr}\, e^{-t\Delta_X} \sim \sum_{k=0}^{\infty} b_k\, t^{-k/2} \qquad \text{as } t \to \infty, \tag{2.3} $$

where $b_0 = h_X + \frac{p}{2} - \frac{h_Y}{4}$ with h_X and p the dimensions of the L^2 and extended L^2 kernels of Δ_X, respectively.

We can also apply the b-trace to the heat operator $e^{-t\Delta_{Z,d}}$. In this case, we know that

$$ e^{-t\Delta_{Z,d}} = \frac{1}{\sqrt{4\pi t}} \Big[e^{-(u-u')^2/(4t)} - e^{-(u+u')^2/(4t)} \Big] e^{-t\Delta_Y}, \tag{2.4} $$

where Δ_Y denotes the Laplacian over Y as before. Thus, restricting to the diagonal, we obtain

$$ e^{-t\Delta_{Z,d}}\big|_{\mathrm{Diag}} = \frac{1}{\sqrt{4\pi t}} \Big[1 - e^{-u^2/t} \Big] e^{-t\Delta_Y}(y,y). $$

The b-trace, by definition, kills the constant term in u, so

$$ {}^b\mathrm{Tr}\, e^{-t\Delta_{Z,d}} = -\left(\int_0^{\infty} \frac{1}{\sqrt{4\pi t}} e^{-u^2/t} du \right) \cdot \mathrm{Tr}\, e^{-t\Delta_Y} = -\frac{1}{4}\, \mathrm{Tr}\, e^{-t\Delta_Y}. \tag{2.5} $$

It follows that ${}^b\zeta(s, \Delta_{Z,d}) = -\frac{1}{4}\zeta(s,\Delta_Y)$, and hence

$$ \det{}_{{}^b\zeta} \Delta_{Z,d} = e^{-\log \det_\zeta \Delta_Y / 4}. \tag{2.6} $$

3. Elliptic boundary problems for Laplace type operators

In this section, we recall some basic material concerning elliptic boundary problems for Laplace type operators that will be used in the following sections. See [23] for a general account.

Let us consider our manifold M with boundary Y and a Laplace type operator $\varDelta$ acting on $C^\infty(M,E)$, where E is a Hermitian vector bundle over M. Imposing the Dirichlet boundary condition for $\varDelta$, we get the operator

$$ \varDelta_d := \ \varDelta : \mathrm{dom}(\varDelta_d) = \{\ F \in H^2(M,E)\ :\ F|_Y = 0\ \} \ \to\ L^2(M,E). $$

We assume that $\varDelta_d$ is invertible. We denote its Poisson operator by $\mathcal{K}_d(\varDelta)$, which provides us with the unique solution of the Dirichlet boundary problem. That is, for a given $f \in C^\infty(Y, E_0)$, the section $F = \mathcal{K}_d(\varDelta)f$ is the unique solution of the Dirichlet problem,

$$\varDelta F = 0 \quad \text{and} \quad F|_Y = f.$$

Suppose that $\varDelta$ has an L^2-invertible extension $\widetilde{\varDelta}$ over a manifold $\widetilde{M}$ of $\dim M$ which contains M as a closed submanifold in its interior. The manifold $\widetilde{M}$ need not be compact and may even have boundary. Then we can define $\varDelta^{-1}$ to be the restriction of $\widetilde{\varDelta}^{-1}$ to M. More precisely, we define

$$\varDelta^{-1} := r_+ \widetilde{\varDelta}^{-1} e_+, \tag{3.1}$$

where $e_+ : L^2(M, E) \to L^2(\widetilde{M}, \widetilde{E})$ is the extension map by 0 from M to $\widetilde{M}$ and $r_+ : H^k(\widetilde{M}, \widetilde{E}) \to H^k(M, E)$ is the restriction map back to M. We also need to introduce the trace map γ,

$$\gamma(\phi) = (\gamma_0(\phi), \gamma_1(\phi)) : C^\infty(M, E) \to C^\infty(Y, E_0) \oplus C^\infty(Y, E_0),$$

where $\gamma_0(\phi) := \phi|_Y$ and $\gamma_1(\phi) := \partial_u \phi|_Y$. Then we can write the inverse of the operator $\varDelta_d$ in terms of $\varDelta^{-1}$:

$$\varDelta_d^{-1} = \varDelta^{-1} - \mathcal{K}_d(\varDelta)\gamma_0 \varDelta^{-1}. \tag{3.2}$$

We now consider a family of Laplace type operators $\varDelta(\lambda)$ depending on a parameter $\lambda \in \mathbb{R}^+$. Then we have:

Proposition 3.1. *For $\lambda \in \mathbb{R}^+$, the following equality holds:*

$$\partial_\lambda \mathcal{K}_d(\varDelta(\lambda)) \;=\; -\varDelta(\lambda)_d^{-1}\, \partial_\lambda \varDelta(\lambda)\, \mathcal{K}_d(\varDelta(\lambda)).$$

Proof. We consider

$$\varDelta(\lambda)\mathcal{K}_d(\varDelta(\lambda)) = 0 \quad , \quad \gamma_0 \mathcal{K}_d(\varDelta(\lambda)) = \mathrm{Id}$$

and take the derivative ∂_λ to both equalities. Then we obtain

$$\varDelta(\lambda)\partial_\lambda \mathcal{K}_d(\varDelta(\lambda)) = -\partial_\lambda \varDelta(\lambda)\mathcal{K}_d(\varDelta(\lambda)) \quad , \quad \gamma_0 \partial_\lambda \mathcal{K}_d(\varDelta(\lambda)) = 0.$$

The second equality means that $\partial_\lambda \mathcal{K}_d(\varDelta(\lambda))$ maps into the domain of $\varDelta_d$. Then the first equality implies the claim. $\square$

We can apply the above constructions to the family of Laplace type operators $\Delta_M(\lambda) := \Delta_M + \lambda$ and $\Delta_Z(\lambda) := \Delta_Z + \lambda$ for $\lambda \in \mathbb{R}^+$ and get the Poisson operator

$$\mathcal{K}_d(\lambda) := \mathcal{K}_d(\Delta_M(\lambda)) \sqcup \mathcal{K}_d(\Delta_Z(\lambda)).$$

The operator $\mathcal{R}(\lambda)$ for $\Delta_X(\lambda) := \Delta_X + \lambda$ is defined by

$$\mathcal{R}(\lambda) := \; D_f \gamma_1 \mathcal{K}_d(\lambda) D_g \;:\; C^\infty(Y, E_0) \;\to\; C^\infty(Y, E_0).$$

Here, the *diagonal map* D_g and *difference map* D_f are defined by

$$D_g(\phi) = (\phi,\phi) : C^\infty(Y,E_0) \to C^\infty(Y,E_0) \oplus C^\infty(Y,E_0),$$
$$D_f(\phi,\psi) = \phi - \psi : C^\infty(Y,E_0) \oplus C^\infty(Y,E_0) \to C^\infty(Y,E_0),$$

and the map γ_1 should be understood as two copies of the previously defined one in the natural way. We remark that for $\lambda \in \mathbb{R}^+$ the shifted Laplace operator $\Delta_X(\lambda) = \Delta_X + \lambda$ is invertible on $L^2(X,E)$ and its pseudodifferential structure is explained thoroughly in [15]. In particular, we can use $\Delta_X(\lambda)$ as an invertible extension to both $\Delta_M(\lambda)$ and $\Delta_Z(\lambda)$. According to Theorem A.2, we have

$$\mathcal{R}(\lambda)^{-1} = \gamma_0(\Delta_X + \lambda)^{-1}\gamma_0^*, \tag{3.3}$$

and $\mathcal{R}(\lambda)$ is a positive self-adjoint elliptic pseudodifferential operator of order 1. Hence we can define its ζ-regularized determinant in the standard way.

4. Comparison with the cylinder

The restriction of the operator $\Delta_X(\lambda)$ to $\hat{Z} := [-1,\infty) \times Y$ defines a family of Laplacians over $\hat{Z}$,

$$\Delta_{\hat{Z}}(\lambda) := -\partial_u^2 + \Delta_Y + \lambda.$$

We impose the Dirichlet boundary condition for $\Delta_{\hat{Z}}(\lambda)$ at $\{-1\} \times Y$ and denote the resulting operator by $\Delta^c(\lambda) := \Delta_{\hat{Z}}(\lambda)_d$. We cut $\hat{Z}$ at $\{0\} \times Y$ into $N = [-1,0] \times Y$ and $Z = [0,\infty) \times Y$ and we impose Dirichlet boundary conditions over the two copies of $\{0\} \times Y$ in order to get the operator

$$\Delta^c(\lambda)_d := \Delta_N(\lambda)_d \sqcup \Delta_Z(\lambda)_d$$

over $N \sqcup Z$. Applying the results in the previous section, we obtain the operator $\mathcal{R}^c(\lambda)$ for $\Delta^c(\lambda)$.

By the natural embedding of $L^2(\hat{Z}, E|_{\hat{Z}})$ into $L^2(X,E)$, we can extend the operators $\Delta^c(\lambda)^{-1}$, $\Delta^c(\lambda)_d^{-1}$ as zero maps over the orthogonal complement of $L^2(\hat{Z}, E|_{\hat{Z}})$. Therefore we can regard the operators $\Delta_X(\lambda)^{-1}$, $\Delta(\lambda)_d^{-1} := \Delta_M(\lambda)_d^{-1} \sqcup \Delta_Z(\lambda)_d^{-1}$, $\Delta^c(\lambda)^{-1}$, and $\Delta^c(\lambda)_d^{-1}$ as defined over the same Hilbert space $L^2(X,E)$. With respect to the decomposition of $L^2(X,E)$ into the sum $L^2(M,E|_M) \oplus L^2(Z,E|_Z)$, the operator $\Delta_X(\lambda)^{-1}$ has the matrix form

$$\Delta_X(\lambda)^{-1} = \begin{pmatrix} \Delta_M(\lambda)^{-1} & r_Z\Delta_X(\lambda)^{-1}e_M \\ r_M\Delta_X(\lambda)^{-1}e_Z & \Delta_Z(\lambda)^{-1} \end{pmatrix},$$

where e_M, e_Z are the extension maps from M, Z and r_M, r_Z are the restriction maps to M, Z defined in Section 3, and where $\Delta_M(\lambda)^{-1} := r_M\Delta_X(\lambda)^{-1}e_M$ and $\Delta_Z(\lambda)^{-1} := r_Z\Delta_X(\lambda)^{-1}e_Z$. We denote the diagonal operator of $\Delta_X(\lambda)^{-1}$ by $\Delta_{X,\mathrm{dig}}(\lambda)^{-1} := \Delta_M(\lambda)^{-1} \sqcup \Delta_Z(\lambda)^{-1}$ and similarly we denote by $\Delta^c_{\mathrm{dig}}(\lambda)^{-1}$ the corresponding diagonal operator for $\Delta^c(\lambda)^{-1}$. The main result of this section is the following proposition.

Proposition 4.1. *The difference*

$$\mathcal{R}(\lambda) - \mathcal{R}^c(\lambda) \;:\; L^2(Y, E_0) \;\to\; L^2(Y, E_0)$$

is a smoothing operator, and the difference

$$(\Delta_{X,\mathrm{dig}}(\lambda)^{-1} - \Delta(\lambda)_d^{-1}) - (\Delta^c_{\mathrm{dig}}(\lambda)^{-1} - \Delta^c(\lambda)_d^{-1}) \;:\; L^2(X, E) \;\to\; L^2(X, E)$$

is of trace class.

Proof. Let $\rho(a, b) : [-1, 1] \to [0, 1]$ equal to 0 for $-a \le u \le a$ and equal to 1 for $b \le |u|$. We use $\rho(a, b)$ to define

$$\phi_1 = 1 - \rho(5/7, 6/7) \;\;,\;\; \psi_1 = 1 - \psi_2 \;\;,$$
$$\phi_2 = \rho(1/7, 2/7) \;\;,\;\; \psi_2 = \rho(3/7, 4/7)$$

and then we extend these functions in the obvious way to define functions over X. Now we define a parametrix $Q(\lambda)$ for the operator $\Delta_X(\lambda)^{-1}$ by

$$Q(\lambda)(x, z) = \phi_1(x)\Delta^c(\lambda)^{-1}(x, z)\psi_1(z) + \phi_2(x)\Delta_X(\lambda)^{-1}(x, z)\psi_2(z).$$

Applying $\Delta_X(\lambda)$ to both sides and using that $\partial_u\phi_1$ and $\partial_u\phi_2$ have supports disjoint to the supports of ψ_1 and ψ_2, respectively, it follows that

$$\Delta_X(\lambda)Q(\lambda) \;=\; \mathrm{Id} + \mathcal{S}(\lambda),$$

where $\mathcal{S}(\lambda)$ is a smoothing operator. This equality allows us to write

$$\Delta_X(\lambda)^{-1} - Q(\lambda) \;=\; -\Delta_X(\lambda)^{-1}\mathcal{S}(\lambda),$$

where $-\Delta_X(\lambda)^{-1}\mathcal{S}(\lambda)$ is a smoothing operator, which then implies that

$$\Delta_X(\lambda)^{-1} - \Delta^c(\lambda)^{-1} = \mathcal{S}'(\lambda) + \mathcal{T}(\lambda), \tag{4.1}$$

where $\mathcal{S}'(\lambda)$ is a smoothing operator and $\mathcal{T}(\lambda)$ is an integral operator whose support does not reach $\{0\} \times Y$. By (3.3), we have

$$\mathcal{R}(\lambda)^{-1} - \mathcal{R}^c(\lambda)^{-1} \;=\; \gamma_0(\Delta_X(\lambda)^{-1} - \Delta^c(\lambda)^{-1})\gamma_0^* \;=\; \gamma_0\mathcal{S}'(\lambda)\gamma_0^*.$$

Hence $\mathcal{R}(\lambda)^{-1} - \mathcal{R}^c(\lambda)^{-1}$ is a smoothing operator, so that $\mathcal{R}(\lambda) - \mathcal{R}^c(\lambda)$ is also a smoothing operator. This completes the proof of the first claim. For the second claim, using the equality in (3.2), we obtain

$$\begin{aligned}\Delta_{X,\mathrm{dig}}(\lambda)^{-1} - &\Delta(\lambda)_d^{-1} - (\Delta^c_{\mathrm{dig}}(\lambda)^{-1} - \Delta^c(\lambda)_d^{-1}) \\ &= \mathcal{K}_d(\lambda)\gamma_0\Delta_{X,\mathrm{dig}}(\lambda)^{-1} - \mathcal{K}^c_d(\lambda)\gamma_0\Delta^c_{\mathrm{dig}}(\lambda)^{-1} \\ &= \big(\mathcal{K}_d(\lambda) - \mathcal{K}^c_d(\lambda)\big)\gamma_0\Delta_{X,\mathrm{dig}}(\lambda)^{-1} \\ &\qquad + \mathcal{K}^c_d(\lambda)\gamma_0\big(\Delta_{X,\mathrm{dig}}(\lambda)^{-1} - \Delta^c_{\mathrm{dig}}(\lambda)^{-1}\big),\end{aligned} \tag{4.2}$$

where $\mathcal{K}^c_d(\lambda)$ is the Poisson operator for the Dirichlet boundary condition of the operator $\Delta^c(\lambda)|_N \sqcup \Delta^c(\lambda)|_Z$. Applying the equality (4.1) again, it is easy to see that $\mathcal{K}_d(\lambda) - \mathcal{K}^c_d(\lambda)$, $\gamma_0(\Delta_{X,\mathrm{dig}}(\lambda)^{-1} - \Delta^c_{\mathrm{dig}}(\lambda)^{-1})$ have regularizing Schwartz kernels. Hence we conclude that the sum of the operators in (4.2) is of trace class. This completes the proof of the second claim. $\square$

From the equality (4.1) and the corresponding equality for $\Delta(\lambda)_d^{-1}-\Delta^c(\lambda)_d^{-1}$ it follows that $\Delta_X(\lambda)^{-1} - \Delta(\lambda)_d^{-1} - \Delta^c(\lambda)^{-1} + \Delta^c(\lambda)_d^{-1}$ has a continuous Schwartz kernel. In particular, we have:

COROLLARY 4.2. *The operator* $\Delta_X(\lambda)^{-1} - \Delta(\lambda)_d^{-1} - \Delta^c(\lambda)^{-1} + \Delta^c(\lambda)_d^{-1}$ *is of trace class over* $L^2(X,E)$.

REMARK 4.3. Looking carefully at the proof of Proposition 4.1, one can see that the proof works for $\lambda = \mu \in \mathbb{C}$ restricted to any sector Γ not intersecting the nonpositive real axis and for $\mu \in \Gamma$, using results from the b-calculus [15, Ch. 6], the operator $\Delta_X(\mu)^{-1} - \Delta(\mu)_d^{-1} - \Delta^c(\mu)^{-1} + \Delta^c(\mu)_d^{-1}$ has a continuous Schwartz kernel that vanishes exponentially along the cylinder and is $O(|\mu|^{-1})$ as $|\mu| \to \infty$ in Γ.

We can now apply the results of Proposition 4.1 and Corollary 4.2 to the variation of ζ-determinants.

PROPOSITION 4.4. *The following equalities hold:*

$$\begin{aligned}(1)\quad & \partial_\lambda \ \log\left(\det{}_\zeta \mathcal{R}(\lambda)\ (\det{}_\zeta \mathcal{R}^c(\lambda))^{-1}\right) \\ &\qquad = \ \mathrm{Tr}(\ \mathcal{R}(\lambda)^{-1}\partial_\lambda \mathcal{R}(\lambda) - \mathcal{R}^c(\lambda)^{-1}\partial_\lambda \mathcal{R}^c(\lambda)\ ),\end{aligned}$$

$$\begin{aligned}(2)\quad & \partial_\lambda\ [\ \log(\ \det{}_{{}^b\zeta}\Delta_X(\lambda)\ (\det{}_{{}^b\zeta}\Delta_Z(\lambda)_d \cdot \det{}_\zeta \Delta_M(\lambda)_d)^{-1}\ ) \\ &\quad - \log(\ \det{}_{{}^b\zeta}\Delta_{\hat{Z}}(\lambda)_d\ (\det{}_{{}^b\zeta}\Delta_Z(\lambda)_d \cdot \det{}_\zeta \Delta_N(\lambda)_d)^{-1}\ )\ ] \\ &\qquad = \ \mathrm{Tr}(\ \Delta_X(\lambda)^{-1} - \Delta(\lambda)_d^{-1} - \Delta^c(\lambda)^{-1} + \Delta^c(\lambda)_d^{-1}\ ).\end{aligned}$$

Proof. The proofs of these two formulas are similar, so we shall focus on the proof of (2). Denote the difference of the logarithms in (2) by $F(\lambda)$. Then according to Singer's formula [24], we have

$$F(\lambda) \ = \ -\int_0^\infty t^{-1}\ {}^b\mathrm{Tr}\left(e^{-t\Delta_X(\lambda)} - e^{-t\Delta(\lambda)_d} - e^{-t\Delta^c(\lambda)} + e^{-t\Delta^c(\lambda)_d}\right)\, dt.$$

Note that since the small time heat asymptotics are determined by local symbols it follows that (cf. the proof of Proposition 4.1) the asymptotic expansion as $t \to 0$ of the integrand is trivial. Using that $\partial_\lambda e^{-t\lambda} = -t\, e^{-t\lambda}$, we obtain

$$\begin{aligned}(4.3)\quad \partial_\lambda F(\lambda) \ &= \ \int_0^\infty {}^b\mathrm{Tr}\left(e^{-t\Delta_X(\lambda)} - e^{-t\Delta(\lambda)_d} - e^{-t\Delta^c(\lambda)} + e^{-t\Delta^c(\lambda)_d}\right)\, dt \\ &= -\int_0^\infty \partial_t\ {}^b\mathrm{Tr}(f(t,\lambda))\, dt,\end{aligned}$$

where

$$f(t,\lambda) = \frac{e^{-t\Delta_X(\lambda)}}{\Delta_X(\lambda)} - \frac{e^{-t\Delta(\lambda)_d}}{\Delta(\lambda)_d} - \frac{e^{-t\Delta^c(\lambda)}}{\Delta^c(\lambda)} + \frac{e^{-t\Delta^c(\lambda)_d}}{\Delta^c(\lambda)_d}.$$

Let $\Upsilon \subset \mathbb{C}$ be the contour $\Upsilon = -\lambda/2 + \{\mu \in \mathbb{C} \mid \arg\lambda = 3\pi/4, 5\pi/4\}$. Then by Cauchy's formula we can write

$$(4.4) \quad f(t,\lambda) = \frac{i}{2\pi}\int_\Upsilon \frac{e^{t\mu}}{\mu}\Big(\Delta_X(\lambda+\mu)^{-1} - \Delta(\lambda+\mu)_d^{-1} \\ - \Delta^c(\lambda+\mu)^{-1} + \Delta^c(\lambda+\mu)_d^{-1}\Big)d\mu.$$

Note that each resolvent on the right is L^2 invertible for all $\mu \notin (-\infty,-\lambda]$, so each resolvent is well-defined on the contour Υ, and that for $\mu \in \Upsilon$, $\operatorname{Re}\mu \leq -\lambda/2$, so $|e^{t\mu}| \leq e^{-t\lambda/2}$ for all $\mu \in \Upsilon$, which implies that the μ integral is well-defined. By Remark 4.3, it follows that the integrand in (4.4) has a continuous Schwartz kernel that vanishes exponentially along the cylinder and is $O(e^{-t\lambda/2}/|\mu|^2)$ as $|\mu| \to \infty$ in Υ. Thus, $f(t,\lambda)$ is a regularizing operator that vanishes exponentially as $t \to \infty$ and is continuous at $t = 0$, whose value we obtain by substituting $t = 0$ in (4.4) and using Cauchy's formula:

$$f(0,\lambda) = \Delta_X(\lambda)^{-1} - \Delta(\lambda)_d^{-1} - \Delta^c(\lambda)^{-1} + \Delta^c(\lambda)_d^{-1}.$$

The Schwartz kernel of this operator restricted to the diagonal is exponentially decreasing along the cylinder, so has no constant term. Therefore the b-trace of $f(0,\lambda)$ equals the usual trace of $f(0,\lambda)$. Now applying the fundamental theorem of calculus to (4.3) completes our proof. □

5. Variation of the ζ-determinant

In this section, we combine the comparison of the ζ-determinants on X with that on the cylinder $\hat{Z}$ and the variation argument of the ζ-determinant found in Proposition 4.4. We begin with the following lemma.

LEMMA 5.1. *For $\lambda \in \mathbb{R}^+$, there is a constant C independent of λ such that*

$$(5.1) \quad \frac{\det_{{}^b\zeta}\Delta_X(\lambda)}{\det_\zeta\Delta_M(\lambda)_d \cdot \det_{{}^b\zeta}\Delta_Z(\lambda)_d} \cdot (\det_\zeta \mathcal{R}(\lambda))^{-1} \\ = C \cdot \frac{\det_{{}^b\zeta}\Delta_{\hat{Z}}(\lambda)_d}{\det_\zeta\Delta_N(\lambda)_d \cdot \det_{{}^b\zeta}\Delta_Z(\lambda)_d} \cdot (\det_\zeta \mathcal{R}^c(\lambda))^{-1}.$$

Proof. We start from the variation of $\log\det_\zeta\mathcal{R}(\lambda)$. By the definition of $\mathcal{R}(\lambda)$ and Proposition 3.1, we have

$$\begin{aligned} \partial_\lambda\mathcal{R}(\lambda) &= -D_f\gamma_1\Delta(\lambda)_d^{-1}\,\partial_\lambda\Delta_{X,\mathrm{dig}}(\lambda)\,\mathcal{K}_d(\lambda)D_g \\ &= -D_f\gamma_1(\Delta_{X,\mathrm{dig}}(\lambda)^{-1} - \mathcal{K}_d(\lambda)\gamma_0\Delta_{X,\mathrm{dig}}(\lambda)^{-1})\mathcal{K}_d(\lambda)D_g, \end{aligned}$$

where we used (3.2) and that $\partial_\lambda\Delta_{X,\mathrm{dig}}(\lambda) = 1$. Now we claim that

$$(5.2) \qquad D_f\gamma_1\Delta_{X,\mathrm{dig}}(\lambda)^{-1}\mathcal{K}_d(\lambda) \;:\; C^\infty(Y,E_0) \;\to\; L^2(Y,E_0)$$

is the zero map. Indeed, by definition, $\mathcal{K}_d(\lambda)$ maps into $H^{1/2}(X,E)$, so that image of $\gamma_1\Delta_{X,\mathrm{dig}}(\lambda)^{-1}\mathcal{K}_d(\lambda)$ lies in $H^1(Y,E_0)$. It follows that the concerned map in (5.2) is the trivial map. Hence,

$$\partial_\lambda \mathcal{R}(\lambda) \;=\; D_f\gamma_1\mathcal{K}_d(\lambda)\gamma_0\Delta_{X,\mathrm{dig}}(\lambda)^{-1}\mathcal{K}_d(\lambda)D_g,$$

so that

$$\mathcal{R}(\lambda)^{-1}\partial_\lambda\mathcal{R}(\lambda) = (D_f\gamma_1\mathcal{K}_d(\lambda)D_g)^{-1}D_f\gamma_1\mathcal{K}_d(\lambda)\gamma_0\Delta_{X,\mathrm{dig}}(\lambda)^{-1}\mathcal{K}_d(\lambda)D_g.$$

We note that D_g has a right inverse on the image of $\gamma_0\Delta_{X,\mathrm{dig}}(\lambda)^{-1}\mathcal{K}_d(\lambda)D_g$ since $\gamma_0\Delta_{X,\mathrm{dig}}(\lambda)^{-1}\mathcal{K}_d(\lambda)D_g \in H^2(Y,E_0)$ as we explained above. Therefore,

$$\begin{aligned}
(5.3)\quad & \mathcal{R}(\lambda)^{-1}\partial_\lambda\mathcal{R}(\lambda)\\
&= (D_f\gamma_1\mathcal{K}_d(\lambda)D_g)^{-1}D_f\gamma_1\mathcal{K}_d(\lambda)\gamma_0\Delta_{X,\mathrm{dig}}(\lambda)^{-1}\mathcal{K}_d(\lambda)D_g\\
&= (D_f\gamma_1\mathcal{K}_d(\lambda)D_g)^{-1}(D_f\gamma_1\mathcal{K}_d(\lambda)D_g)D_g^{-1}\gamma_0\Delta_{X,\mathrm{dig}}(\lambda)^{-1}\mathcal{K}_d(\lambda)D_g\\
&= D_g^{-1}\gamma_0\Delta_{X,\mathrm{dig}}(\lambda)^{-1}\mathcal{K}_d(\lambda)D_g.
\end{aligned}$$

A similar formula holds for $\mathcal{R}^c(\lambda)^{-1}\partial_\lambda\mathcal{R}^c(\lambda)$. Then by Equation (1) in Proposition 4.4 and other equalities proved before, we have

$$\begin{aligned}
\partial_\lambda\ & \log\left(\det{}_\zeta\mathcal{R}(\lambda)(\det{}_\zeta\mathcal{R}^c(\lambda))^{-1}\right)\\
&= \mathrm{Tr}(\mathcal{R}(\lambda)^{-1}\partial_\lambda\mathcal{R}(\lambda) - \mathcal{R}^c(\lambda)^{-1}\partial_\lambda\mathcal{R}^c(\lambda))\\
&= \mathrm{Tr}(\mathcal{K}_d(\lambda)\gamma_0\Delta_{X,\mathrm{dig}}(\lambda)^{-1} - \mathcal{K}_d^c(\lambda)\gamma_0\Delta^c_{\mathrm{dig}}(\lambda)^{-1}) && \text{by (5.3)}\\
&= \mathrm{Tr}(\Delta_{X,\mathrm{dig}}(\lambda)^{-1} - \Delta(\lambda)_d^{-1} - \Delta^c_{\mathrm{dig}}(\lambda)^{-1} + \Delta^c(\lambda)_d^{-1}) && \text{by (3.2)}\\
&= \mathrm{Tr}(\Delta_X(\lambda)^{-1} - \Delta(\lambda)_d^{-1} - \Delta^c(\lambda)^{-1} + \Delta^c(\lambda)_d^{-1}).
\end{aligned}$$

Equation (2) in Proposition 4.4 now completes our proof. $\square$

We can compute the constant C in (5.1) by taking logarithms of both sides and then taking $\lambda\to\infty$. By the proof of Proposition 4.4, we have

$$\begin{aligned}
(5.4)\qquad & \log\det{}_{{}^b\zeta}\Delta_X(\lambda) - \log\det{}_{{}^b\zeta}\Delta_Z(\lambda)_d - \log\det{}_\zeta\Delta_M(\lambda)_d\\
&\qquad - \log\det{}_{{}^b\zeta}\Delta_{\hat{Z}}(\lambda)_d + \log\det{}_{{}^b\zeta}\Delta_Z(\lambda)_d + \log\det{}_\zeta\Delta_N(\lambda)_d\\
&= -\int_0^\infty t^{-1}\,\mathrm{Tr}(e^{-t(\Delta_X+\lambda)} - e^{-t(\Delta_d+\lambda)} - e^{-t(\Delta_{\hat{Z}}+\lambda)} + e^{-t(\Delta_d^c+\lambda)})\,dt.
\end{aligned}$$

The large time portion of the integral $\int_0^\infty\, dt$ on the right-hand side decays exponentially as $\lambda\to\infty$. Hence, the asymptotics of the left-hand side of (5.4) as $\lambda\to\infty$ is determined by the asymptotic expansion of

$$\mathrm{Tr}(e^{-t(\Delta_X+\lambda)} - e^{-t(\Delta_d+\lambda)} - e^{-t(\Delta_{\hat{Z}}+\lambda)} + e^{-t(\Delta_d^c+\lambda)})$$

as $t\to 0$, which, in view of the proof of Proposition 4.4, is trivial. This implies that the asymptotic expansion of the left-hand side of (5.4) is also trivial. We now consider the asymptotic expansions of $\log\det_\zeta\mathcal{R}(\lambda)$ and $\log\det_\zeta\mathcal{R}^c(\lambda)$ as $\lambda\to\infty$. Let us recall the following result from [2].

Proposition 5.2. *For a positive elliptic pseudodifferential operator $P(\lambda)$ with the parameter λ of weight k on an m-dimensional manifold without boundary, there is an asymptotic expansion*

$$\log \det_\zeta P(\lambda) \;\sim\; \sum_{j=-m}^{\infty} a_j \lambda^{-j/k} + \sum_{j=0}^{m} b_j \lambda^{j/k} \log \lambda \quad \text{as} \quad \lambda \to \infty, \tag{5.5}$$

where the coefficients a_j and b_j are determined by local formulas in terms of the symbol of the operator $P(1)$.

Applying Proposition 5.2 to $\log \det_\zeta \mathcal{R}(\lambda)$ and $\log \det_\zeta \mathcal{R}^c(\lambda)$ and denoting the constant terms of these asymptotic expansions by $a_{\mathcal{R}}$, $a^c_{\mathcal{R}}$, we see that

$$\log C \;=\; -(a_{\mathcal{R}} - a^c_{\mathcal{R}}).$$

Since the asymptotics of (5.5) are given in terms of the local symbol asymptotics of $\mathcal{R}(1)$ and $\mathcal{R}^c(1)$, Proposition 4.1 implies that $a_{\mathcal{R}} - a^c_{\mathcal{R}} \;=\; 0$. Hence we can conclude that $C = 1$; in other words, we have:

Proposition 5.3. *For any $\lambda \in \mathbb{R}^+$, the following equality holds:*

$$\begin{aligned} &\frac{\det_{b\zeta} \Delta_X(\lambda)}{\det_\zeta \Delta_M(\lambda)_d \cdot \det_{b\zeta} \Delta_Z(\lambda)_d} \cdot (\det_\zeta \mathcal{R}(\lambda))^{-1} \\ &\qquad = \frac{\det_{b\zeta} \Delta_{\hat{Z}}(\lambda)_d}{\det_\zeta \Delta_N(\lambda)_d \cdot \det_{b\zeta} \Delta_Z(\lambda)_d} \cdot (\det_\zeta \mathcal{R}^c(\lambda))^{-1}. \end{aligned} \tag{5.6}$$

6. Proof of Theorem 1.1

In this section, we prove our main Theorem 1.1 by taking $\lambda \to 0^+$ in (5.6) and combining key results that will be proved in the subsequent sections. First, in Section 7 (see Theorem 7.4), we prove that

$$\lim_{\lambda \to 0^+} \frac{\det_{b\zeta} \Delta_{\hat{Z}}(\lambda)_d}{\det_\zeta \Delta_N(\lambda)_d \cdot \det_{b\zeta} \Delta_Z(\lambda)_d} \cdot (\det_\zeta \mathcal{R}^c(\lambda))^{-1} \;=\; 2^{-\zeta(0,\Delta_Y) - h_Y}. \tag{6.1}$$

Second, in Section 8 (see Theorem 8.1), we prove that as $\lambda \to 0^+$,

$$\log \det_{b\zeta} \Delta_X(\lambda) = \left(h_X + \frac{p}{2} - \frac{h_Y}{4} \right) \log \lambda + \log \det_{b\zeta} \Delta_X + o(1), \tag{6.2}$$

where p is the number of linearly independent extended L^2-solutions of Δ_X. Third, (see Theorem 8.2) we prove that as $\lambda \to 0^+$,

$$\log \det_{b\zeta} \Delta_Z(\lambda)_d = -\frac{h_Y}{4} \log \lambda + \log \det_{b\zeta} \Delta_{Z,d} + o(1). \tag{6.3}$$

Fourth, (see Theorem 8.3) we prove that as $\lambda \to 0^+$,

$$\log \det_\zeta \mathcal{R}(\lambda) = \left(h_X + \frac{p}{2} \right) \log \lambda - \log \det(L + \widetilde{L}) + \log \det_\zeta \mathcal{R} + o(1). \tag{6.4}$$

Finally, to complete the proof of our main theorem, we note that as $\lambda \to 0^+$,

$$\log \det_\zeta(\Delta_{M,d} + \lambda) = \log \det_\zeta(\Delta_{M,d}) + o(1), \tag{6.5}$$

since $\Delta_{M,d}$ has discrete spectrum with no kernel. Combining (6.1), (6.2), (6.3), (6.4), and (6.5), we get our main theorem.

7. Computations over the cylinder

Our goal in this section is to compute the right-hand side in (5.6) above. Let us consider the finite cylinder $N_R := [0, R] \times Y$ and the half infinite cylinder $Z_R := [R, \infty) \times Y$ in Z. By restriction, we obtain Laplace type operators Δ_{N_R} over N_R and Δ_{Z_R} over Z_R of the form $-\partial_u^2 + \Delta_Y$, where Δ_Y denotes the Laplacian over Y as before. We impose Dirichlet boundary conditions at $\{0, R\} \times Y$ for Δ_{N_R} and at $\{R\} \times Y$ for Δ_{Z_R} and denote by $\Delta_{N_R,d}$, $\Delta_{Z_R,d}$ the resulting operators. For the cutting at $\{R\} \times Y$ of Z into $N_R \sqcup Z_R$, we can define the operator $\mathcal{R}_R^c(\lambda)$ for $\Delta_{Z,d} + \lambda$.

We begin by computing $\det_\zeta \Delta_{N_R,d}$ explicitly.

Proposition 7.1. *The following equality holds:*

$$\begin{aligned} \det_\zeta \Delta_{N_R,d} \;=\; & (2R)^{h_Y} \cdot (\det_\zeta \sqrt{\Delta_Y})^{-1} \\ & \cdot e^{R(\Gamma(s)^{-1}\zeta(s-1/2,\Delta_Y))'(0)} \cdot \prod_{l=h_Y+1}^{\infty} (1 - e^{-2R\sqrt{\mu_l}}), \end{aligned} \tag{7.1}$$

where $\{\mu_l\}$ are the eigenvalues of Δ_Y.

Proof. Since the spectrum of $\Delta_{N_R,d}$ is $\{\ \mu_l + \frac{\pi^2 k^2}{R^2} \mid l, k \in \mathbb{N}\ \}$, we have

$$\zeta(s, \Delta_{N_R,d}) \;=\; \sum_{l=h_Y+1}^{\infty} \sum_{k=1}^{\infty} \Big(\mu_l + \frac{\pi^2 k^2}{R^2}\Big)^{-s} \;+\; h_Y (R/\pi)^{2s} \zeta(2s),$$

where $\zeta(s)$ is the Riemann-zeta function. We can rewrite the first term on the right as

$$\frac{1}{\Gamma(s)} \sum_{l=h_Y+1}^{\infty} \mu_l^{-s} \int_0^\infty \frac{1}{2}\bigg(\sum_{k\in\mathbb{Z}} \exp\bigg(-\Big(1+\Big(\frac{\pi k}{R\sqrt{\mu_l}}\Big)^2\Big)x\bigg) - e^{-x}\bigg) x^{s-1}\, dx.$$

Applying the Poisson summation formula

$$\sum_{k\in\mathbb{Z}} e^{-a^2k^2} \;=\; \sum_{k\in\mathbb{Z}} \frac{\sqrt{\pi}}{a} e^{-\pi^2k^2/a^2} \;=\; \frac{\sqrt{\pi}}{a} + 2\sum_{k\in\mathbb{N}} \frac{\sqrt{\pi}}{a} e^{-\pi^2k^2/a^2},$$

where a is a positive real number, we can rewrite this term as

$$\frac{1}{\Gamma(s)}\sum_{l=h_Y+1}^{\infty}\mu_l^{-s}\int_0^\infty\Big(\sum_{k\in\mathbb{N}}\frac{R\sqrt{\mu_l}}{\sqrt{\pi x}}\exp\Big(-\frac{(R\sqrt{\mu_l}k)^2}{x}\Big)e^{-x}\Big)x^{s-1}\,dx$$
$$+\frac{1}{\Gamma(s)}\sum_{l=h_Y+1}^{\infty}\mu_l^{-s}\frac{1}{2}\int_0^\infty\frac{R\sqrt{\mu_l}}{\sqrt{\pi x}}e^{-x}x^{s-1}\,dx-\frac{1}{2}\zeta(s,\Delta_Y).$$

Now observe that the function

$$\int_0^\infty\Big(\sum_{k\in\mathbb{N}}\frac{R\sqrt{\mu_l}}{\sqrt{\pi x}}\exp\Big(-\frac{(R\sqrt{\mu_l}k)^2}{x}\Big)e^{-x}\Big)x^{s-1}\,dx$$

is regular at $s=0$, and

$$\sum_{l=h_Y+1}^{\infty}\mu_l^{-s}\frac{1}{2}\int_0^\infty\frac{R\sqrt{\mu_l}}{\sqrt{\pi x}}e^{-x}x^{s-1}\,dx \;=\; \frac{1}{2}\cdot\frac{R}{\sqrt{\pi}}\Gamma\Big(s-\frac{1}{2}\Big)\zeta\Big(s-\frac{1}{2},\Delta_Y\Big).$$

Therefore, we conclude that

$$\begin{aligned}\zeta'(0,\Delta_{N_R,d}) \;=\;& \frac{R}{\sqrt{\pi}}\sum_{l=h_Y+1}^{\infty}\sqrt{\mu_l}\int_0^\infty\Big(\sum_{k\in\mathbb{N}}\exp\Big(-\frac{(R\sqrt{\mu_l}k)^2}{x}\Big)e^{-x}\Big)x^{-3/2}\,dx\\ &+\frac{R}{2\sqrt{\pi}}\Gamma\Big(-\frac{1}{2}\Big)\ \Big(\Gamma(s)^{-1}\zeta\Big(s-\frac{1}{2},\Delta_Y\Big)\Big)'(0)\\ &-\frac{1}{2}\zeta'(0,\Delta_Y)+h_Y(2\log(R/\pi)\ \zeta(0)+2\zeta'(0)).\end{aligned}$$

In other words, we have

$$\begin{aligned}\zeta'(0,\Delta_{N_R,d}) \;=\;& \sum_{l=h_Y+1}^{\infty}\sum_{k\in\mathbb{N}}\frac{e^{-2R\sqrt{\mu_l}k}}{k}-R\Big(\Gamma(s)^{-1}\zeta\Big(s-\frac{1}{2},\Delta_Y\Big)\Big)'(0)\\ &-\frac{1}{2}\zeta'(0,\Delta_Y)-h_Y\log(2R).\end{aligned}$$

This completes the proof. $\square$

Proposition 7.2. *For $\lambda\in[0,\infty)$, we have*

$$\frac{\det_{{}^b\zeta}(\Delta_{Z,d}+\lambda)}{\det_{{}^b\zeta}(\Delta_{Z_R,d}+\lambda)} \;=\; \exp\Big(\ R\Big(\Gamma(s)^{-1}\zeta\Big(s-\frac{1}{2},\Delta_Y+\lambda\Big)\Big)'(0)\ \Big). \tag{7.2}$$

Proof. Using the well-known formulas for $e^{-t(\Delta_{Z,d}+\lambda)}$ and $e^{-t(\Delta_{Z_R,d}+\lambda)}$ (cf. Equation (2.4)), it is straightforward to prove that for any $\lambda\in[0,\infty)$,

$${}^b\mathrm{Tr}(e^{-t(\Delta_{Z,d}+\lambda)})-{}^b\mathrm{Tr}(e^{-t(\Delta_{Z_R,d}+\lambda)}) \;=\; R(4\pi t)^{-1/2}\,\mathrm{Tr}(e^{-t(\Delta_Y+\lambda)}).$$

For $\lambda\neq0$, this gives us

$${}^b\zeta(s,\Delta_{Z,d}+\lambda)-{}^b\zeta(s,\Delta_{Z_R,d}+\lambda) \;=\; \frac{R}{2\sqrt{\pi}}\frac{1}{\Gamma(s)}\int_0^\infty t^{s-\frac{3}{2}}\,\mathrm{Tr}(e^{-t(\Delta_Y+\lambda)})\,dt,$$

from which we can get the claimed equality for $\lambda > 0$. When $\lambda = 0$, ${}^b\zeta(s, \Delta_{Z,d}) - {}^b\zeta(s, \Delta_{Z_R,d})$ is the sum of the meromorphic extensions of the following functions which are a priori defined by

$$\frac{R}{2\sqrt{\pi}} \frac{1}{\Gamma(s)} \int_0^1 t^{s-\frac{3}{2}} \operatorname{Tr}(e^{-t\Delta_Y})\ dt$$

for $\Re s \gg 0$, and

$$\frac{R}{2\sqrt{\pi}} \frac{1}{\Gamma(s)} \int_1^\infty t^{s-\frac{3}{2}} \operatorname{Tr}(e^{-t\Delta_Y})\ dt$$

for $\Re s \ll 0$. Hence we get the claimed equality even when $\lambda = 0$. $\square$

Finally, we compute the ζ-determinant of $\mathcal{R}^c_R(\lambda)$.

Proposition 7.3. *The following equality holds:*

$$\det{}_\zeta \mathcal{R}^c_R(\lambda) = \begin{cases} 2^{\zeta(0,\Delta_Y+\lambda)} \cdot \det{}_\zeta \sqrt{\Delta_Y + \lambda} \cdot \det{}_F (1 - e^{-2R\sqrt{\Delta_Y+\lambda}})^{-1} & \text{if } \lambda \neq 0, \\ R^{-h_Y} 2^{\zeta(0,\Delta_Y)} \cdot \det{}_\zeta \sqrt{\Delta_Y} \cdot \det{}_F (1 - e^{-2R\sqrt{\Delta^*_Y}})^{-1} & \text{if } \lambda = 0, \end{cases}$$

where $\det_F(\cdot)$ *denotes the Fredholm determinant and* Δ^*_Y *denotes the restriction of* Δ_Y *to* $(\ker \Delta_Y)^\perp$. *In particular, the function* $\lambda \mapsto \det_\zeta \mathcal{R}^c_R(\lambda)$ *is continuous at* $\lambda = 0$.

Proof. By elementary computations, we find that

$$\mathcal{R}^c_R(\lambda) = \begin{cases} 2\sqrt{\Delta_Y + \lambda}\,(1 - e^{-2R\sqrt{\Delta_Y+\lambda}})^{-1} & \text{if } \lambda \neq 0, \\ 2\sqrt{\Delta_Y}\,(1 - e^{-2R\sqrt{\Delta_Y}})^{-1} P_1 + R^{-1} P_0 & \text{if } \lambda = 0, \end{cases}$$

where P_1, P_0 are the orthogonal projections onto $(\ker \Delta_Y)^\perp$ and $\ker \Delta_Y$, respectively. The claim follows from this explicit formula for $\mathcal{R}^c_R(\lambda)$. $\square$

Theorem 7.4. *The following equality holds:*

$$\lim_{\lambda \to 0^+} \frac{\det_{{}^b\zeta} \Delta_{\hat{Z}}(\lambda)_d}{\det_\zeta \Delta_N(\lambda)_d \cdot \det_{{}^b\zeta} \Delta_Z(\lambda)_d} \cdot (\det{}_\zeta \mathcal{R}^c(\lambda))^{-1} = 2^{-\zeta(0,\Delta_Y)-h_Y}.$$

Proof. Setting $R = 1$, making the change of variables $u \mapsto u - 1$ (which changes Z to $\hat{Z}$), then using Propositions 7.1, 7.2, and 7.3 proves our theorem. We note that $\det_\zeta \Delta_N(\lambda)_d$ is continuous at $\lambda = 0$ since $\Delta_N(\lambda)_d$ has no kernel for any $\lambda \in [0, \infty)$. $\square$

8. Asymptotics as $\lambda \to 0^+$

In this section, we prove our main results concerning the asymptotic expansions of our various determinants as $\lambda \to 0^+$.

THEOREM 8.1. *We have the following asymptotic relation: As $\lambda \to 0^+$,*

$$\log \det{}_{{}^b\zeta} \Delta_X(\lambda) = \Big(h_X + \frac{p}{2} - \frac{h_Y}{4}\Big) \log \lambda + \log \det{}_{{}^b\zeta} \Delta_X + o(1).$$

Proof. From the definition of the b-zeta function, we can write ${}^b\zeta(s, \Delta_X(\lambda)) = {}^b\zeta_1(s,\lambda) + {}^b\zeta_2(s,\lambda) + b_0\lambda^{-s}$, where $b_0 = h_X + \frac{p}{2} - \frac{h_Y}{4}$,

$${}^b\zeta_1(s,\lambda) = \frac{1}{\Gamma(s)} \int_0^1 t^{s-1}\, e^{-t\lambda} \big({}^b\mathrm{Tr}\, e^{-t\Delta_X} - b_0\big)\, dt$$

is meromorphically extended from $\Re s \gg 0$, and

$${}^b\zeta_2(s,\lambda) = \frac{1}{\Gamma(s)} \int_1^\infty t^{s-1}\, e^{-t\lambda} \big({}^b\mathrm{Tr}\, e^{-t\Delta_X} - b_0\big)\, dt.$$

Using that ${}^b\mathrm{Tr}\, e^{-t\Delta_X} - b_0 = O(t^{-1/2})$ as $t \to \infty$ by (2.3), it follows that ${}^b\zeta_1(s,\lambda) + {}^b\zeta_2(s,\lambda)$ converges uniformly to ${}^b\zeta(s, \Delta_X)$ as $\lambda \to 0^+$ for $s \in \mathbb{C}$ near zero. Taking derivatives of these functions at $s = 0$, we get our result. □

THEOREM 8.2. *We have the following asymptotic relation: As $\lambda \to 0^+$,*

$$\log \det{}_{{}^b\zeta} \Delta_Z(\lambda)_d = -\frac{h_Y}{4} \log \lambda + \log \det{}_{{}^b\zeta} \Delta_{Z,d} + o(1).$$

Proof. By (2.5) we can write

$${}^b\mathrm{Tr}\, e^{-t\Delta_Z(\lambda)_d} = -\frac{e^{-t\lambda}}{4}\, \mathrm{Tr}\, e^{-t\Delta_Y} = -\frac{1}{4}\Big(h_Y\, e^{-t\lambda} + \sum_k e^{-t(\lambda+\mu_k)}\Big),$$

where $\{\mu_k\}$ are the positive eigenvalues of Δ_Y. This equality implies that

$${}^b\zeta(s, \Delta_Z(\lambda)_d) = -\frac{1}{4}\Big(h_Y \cdot \lambda^{-s} + \sum_k (\lambda+\mu_k)^{-s}\Big).$$

Taking the derivative with respect to s, multiplying by -1, and using that the μ_k's are positive, proves our theorem. □

The following theorem is the last ingredient needed to prove our main theorem, Theorem 1.1.

THEOREM 8.3. *As $\lambda \to 0^+$, we have*

$$\log \det{}_\zeta \mathcal{R}(\lambda) = \Big(h_X + \frac{p}{2}\Big) \log \lambda - \log \det(L + \widetilde{L}) + \log \det{}_\zeta \mathcal{R} + o(1),$$

where L and $\widetilde{L}$ are defined in (1.1).

Proof. If $P : L^2(Y, E_0) \longrightarrow V$ denotes the orthogonal projection onto V, then

$$\zeta(s, \mathcal{R}(\lambda)) = \mathrm{Tr}(\mathcal{R}(\lambda)^{-s}) = \mathrm{Tr}(P\mathcal{R}(\lambda)^{-s}) + \mathrm{Tr}(P^\perp \mathcal{R}(\lambda)^{-s}).$$

Theorem A.4 implies that as $\lambda \to 0^+$,

$$-\frac{d}{ds}\Big|_{s=0} \operatorname{Tr}(P^{\perp}\mathcal{R}(\lambda)^{-s}) = \log\det{}_\zeta \mathcal{R} + o(1).$$

Since $\operatorname{Tr}(P\mathcal{R}(\lambda)^{-s}) = \operatorname{Tr}(P\mathcal{R}(\lambda)^{-s}P)$, it follows that

$$-\frac{d}{ds}\Big|_{s=0} \operatorname{Tr}(P\mathcal{R}(\lambda)^{-s}) = -\log\det\Big(P\mathcal{R}(\lambda)^{-1}P\Big).$$

Thus, we are left to prove that as $\lambda \to 0^+$,

$$\log\det\Big(P\mathcal{R}(\lambda)^{-1}P\Big) = -\Big(h_X + \frac{p}{2}\Big)\log\lambda + \log\det(L + \widetilde{L}) + o(1),$$

or after exponentiating and setting $\lambda = \nu^2$, it remains to prove that

$$\det\Big(P\mathcal{R}(\nu^2)^{-1}P\Big) = \nu^{-2h_X - p}\Big(\det(L+\widetilde{L}) + o(1)\Big). \tag{8.1}$$

To prove this we replace λ with ν^2 in the formula (A.3) to get

$$P\mathcal{R}(\nu^2)^{-1}P = \frac{1}{\nu^2}\sum v_j \otimes v_j^* + \frac{1}{\nu}\sum V_j \otimes V_j^* + E(\nu),$$

where $E(\nu) = P\gamma_0 Q(\nu^2)\gamma_0^* P$ is an operator that depends continuously on $\nu \in [0,\infty)$. This implies that

$$\det\Big(P\mathcal{R}(\nu^2)^{-1}P\Big) = \nu^{-2h_X - 2p}\det\Big(L + \nu\widetilde{L} + \nu^2 E(\nu)\Big).$$

To compute the determinant on the right, consider the linear map $S_\nu : V \to V$ defined by $v_j \mapsto v_j$ and $V_j \mapsto \nu^{-1}V_j$. Clearly, $S_\nu L = L$, $S_\nu(\nu\widetilde{L}) = \widetilde{L}$, and $S_\nu(\nu^2 E(\nu)) = O(\nu)$. Thus,

$$L + \nu\widetilde{L} + \nu^2 E(\nu) = S_\nu^{-1}\big(L + \widetilde{L} + O(\nu)\big),$$

which, together with the fact that $\det S_\nu^{-1} = \nu^p$, immediately gives (8.1). $\square$

Appendix A. Some analytic properties of $\mathcal{R}(\lambda)$ for $\lambda \in [0,\infty)$

Throughout this section we let λ denote a parameter in $[0,\infty)$. For any $\lambda \in [0,\infty)$ and $\varphi \in C^\infty(Y, E_0)$, there is a unique smooth bounded solution $\Phi = \Phi(\lambda) = (\Phi_1, \Phi_2) \in C^\infty(M,E) \oplus C^\infty(Z,E)$ that is continuous at Y with value φ such that $(\Delta_X + \lambda)\Phi = 0$ off of Y. (See below for the uniqueness on Z.) The usual theory of elliptic boundary value problems shows that $\Phi_1(\lambda)$ is a continuous (even smooth) function of $\lambda \in [0,\infty)$. To see that $\Phi_2(\lambda)$ is continuous in λ, let $\{\varphi_j\}$ be a basis of Δ_Y with the eigenvalues $\{\mu_j\}$ for each j. If $\varphi = \sum_j a_j \varphi_j$, then one can check that the unique solution $\Phi_2(\lambda)$ is given explicitly by

$$\Phi_2(\lambda) = \sum_j a_j e^{-\sqrt{\mu_j + \lambda}\,u}\varphi_j.$$

Hence, $\Phi_2(\lambda)$ is a continuous function of λ and this formula shows that $\partial_u \Phi_2(\lambda)$ is also a continuous function of λ too. Then

$$\mathcal{R}(\lambda)\,\varphi := \partial_u \Phi_1(\lambda)\Big|_Y - \partial_u \Phi_2(\lambda)\Big|_Y.$$

This discussion implies the following lemma.

Lemma A.1. *$\mathcal{R}(\lambda)$ is a continuous function of $\lambda \in [0,\infty)$.*

We now find a formula for $\mathcal{R}(\lambda)$. Fix $\lambda \in [0,\infty)$. On a neighbourhood of Y, define the function $\Psi(u,y)$ by the formula

$$\partial_u \Phi = -\mathcal{R}(\lambda)\varphi\, H(u) + \Psi(u,y),$$

where $H(u)$ is the Heaviside function. Then $\Psi(0^-,y) = \partial_u \Phi_1$ and

$$\Psi(0^+,y) = \partial_u \Phi_2 + \mathcal{R}(\lambda)\varphi = \partial_u \Phi_2 + (\partial_u \Phi_1 - \partial_u \Phi_2) = \partial_u \Phi_1,$$

hence $\Psi(u,y)$ is continuous at Y. Since $\Psi(u,y)$ is smooth up to Y from each side, it follows that $\partial_u \Psi$ has at most a jump discontinuity at Y. On the collar, $\Delta_X + \lambda = -\partial_u^2 + \Delta_Y + \lambda$; therefore

$$(\Delta_X + \lambda)\Phi = -\partial_u^2 \Phi + (\Delta_Y + \lambda)\Phi = \mathcal{R}(\lambda)\varphi \otimes \delta_Y - \partial_u \Psi + (\Delta_Y + \lambda)\Phi, \tag{A.1}$$

where δ_Y is the delta distribution concentrated at Y. Off of the hypersurface Y, we know that $(\Delta_X+\lambda)\Phi = 0$, thus (A.1) implies that $-\partial_u \Psi + (\Delta_Y+\lambda)\Phi = 0$ off of Y. Since $-\partial_u \Psi$ has at most a jump discontinuity at Y and $(\Delta_Y+\lambda)\Phi$ is smooth at Y, it follows that $-\partial_u \Psi + (\Delta_Y + \lambda)\Phi$ must be identically zero since it is zero off of Y with at most a jump discontinuity at Y. Therefore, (A.1) implies that

$$(\Delta_X + \lambda)\Phi = \mathcal{R}(\lambda)\varphi \otimes \delta_Y = \gamma_0^*\, \mathcal{R}(\lambda)\varphi.$$

If $\lambda > 0$, then we can multiply both sides by $(\Delta_X + \lambda)^{-1}$ and then restrict to Y, obtaining

$$\varphi = \gamma_0 (\Delta_X + \lambda)^{-1} \gamma_0^*\, \mathcal{R}(\lambda)\varphi, \tag{A.2}$$

where $\gamma_0^* = (\cdot \otimes \delta_Y)$ is the adjoint to γ_0. This formula is the basis for establishing the following theorem.

Theorem A.2. *For $\lambda > 0$, $\mathcal{R}(\lambda)$ is a positive definite first order elliptic classical pseudodifferential operator such that*

$$\mathcal{R}(\lambda)^{-1} = \gamma_0 (\Delta_X + \lambda)^{-1} \gamma_0^*.$$

Proof. Fix $\lambda > 0$. We begin by proving that $A(\lambda) = \gamma_0(\Delta_X + \lambda)^{-1}\gamma_0^*$ is an elliptic classical pseudodifferential operator of order -1. Note that $A(\lambda)$ is self-adjoint and therefore its kernel and cokernel have the same dimension. This fact, together with the formula (A.2), immediately imply that $A(\lambda)$ must in fact be invertible with inverse $\mathcal{R}(\lambda)$. Hence, $\mathcal{R}(\lambda) = A(\lambda)^{-1}$ is a classical elliptic operator of order 1. At the end of this proof we show that $\mathcal{R}(\lambda)$ is positive definite. To see that $A(\lambda)$ is a classical elliptic operator of order -1 we choose a partition of unity of Y in order to work in local coordinates. In a coordinate patch $[-1,1]_u \times \mathbb{R}^{n-1}$ on X, where $n = \dim X$ and where the factor $\mathbb{R}^{n-1}$ is a coordinate patch on Y, we can write

$$(\Delta_X + \lambda)^{-1}(\gamma_0^*\varphi) = \int_{\mathbb{R}^n} e^{i(u,y)\cdot(\tau,\eta)}\, a(u,y,\tau,\eta;\lambda)\, \widehat{(\gamma_0^*\varphi)}(\tau,\eta)\, đ\tau\, đ\eta,$$

where $a(u,y,\tau,\eta;\lambda) = \sigma((\Delta_X+\lambda)^{-1})$ is an elliptic classical symbol of order -2 and $đ$ is d divided by as many 2π's as there are variables. Since $\widehat{(\gamma_0^*\varphi)}(\tau,\eta) = \widehat{\varphi}(\eta)$, we see that

$$A(\lambda)\varphi = \gamma_0(\Delta_X + \lambda)^{-1}(\gamma_0^*\varphi) = \int_{\mathbb{R}^{n-1}} e^{iy\cdot\eta}\, b(y,\eta;\lambda)\, \widehat{\varphi}(\eta)\, đ\eta,$$

where $b(y,\eta;\lambda) = \int_{\mathbb{R}} a(0,y,\tau,\eta;\lambda)\, đ\tau$. One can check that $b(y,\eta;\lambda)$ is an elliptic classical symbol of order -1 in η. This proves that $A(\lambda)$ is an elliptic classical pseudodifferential operator of order -1.

To prove that $\mathcal{R}(\lambda)$ is positive definite, let $\chi(u) \in C_c^\infty((-1,1))$ be such that $\int \chi(u)\, du = 1$ and for $j = 1,2,\ldots$, define $\psi_j(u,y) := j\,\chi(ju)\,\varphi(y)$. Then it is straightforward to check that

$$\langle \mathcal{R}(\lambda)^{-1}\varphi, \varphi\rangle = \langle \gamma_0(\Delta_X + \lambda)^{-1}\gamma_0^*\varphi, \varphi\rangle = \lim_{j\to\infty} \langle (\Delta_X + \lambda)^{-1}\psi_j, \psi_j\rangle.$$

Since the operator $(\Delta_X + \lambda)^{-1}$ is positive definite, the last limit involves only nonnegative real numbers; therefore the limit $\langle \mathcal{R}(\lambda)^{-1}\varphi, \varphi\rangle$ is nonnegative. Of course, $\langle \mathcal{R}(\lambda)^{-1}\varphi, \varphi\rangle$ cannot be zero unless $\varphi = 0$, so $\mathcal{R}(\lambda)^{-1}$ is positive definite. Thus, $\mathcal{R}(\lambda)$ is also positive definite. □

We now analyze $\mathcal{R} = \mathcal{R}(0)$. To do so, we use the formula

$$\mathcal{R}(\lambda)^{-1} = \gamma_0(\Delta_X + \lambda)^{-1}\gamma_0^*,$$

where γ_0 is the restriction map to the boundary Y and $\gamma_0^* = (\cdot \otimes \delta_Y)$. According to Proposition 6.28 in [15] we have

$$(\Delta_X + \lambda)^{-1} = \frac{1}{\lambda}\sum_j u_j \otimes u_j^* + \frac{1}{\sqrt{\lambda}}\sum_j U_j \otimes U_j^* + Q(\lambda),$$

where $Q(\lambda)$ is a b-pseudodifferential operator of order -2 depending continuously on $\lambda \in [0,\infty)$, $\{u_j\}$ is an orthonormal basis for the kernel of Δ_X on $L^2(X,E)$, and $\{U_j\}$ is a basis of the extended L^2-solutions, the bounded solutions $\Delta_X U_j = 0$ such that at ∞ on the cylinder, $\{U_j(\infty)\}$ is an orthonormal set in $L^2(Y,E_0)$. Thus,

$$\mathcal{R}(\lambda)^{-1} = \frac{1}{\lambda}\sum_j \gamma_0 u_j \otimes u_j^*\gamma_0^* + \frac{1}{\sqrt{\lambda}}\sum_j \gamma_0 U_j \otimes U_j^*\gamma_0^* + \gamma_0 Q(\lambda)\,\gamma_0^*.$$

If $v_j = u_j|_Y$ and $V_j = U_j|_Y$ are the restrictions of u_j and U_j to Y, respectively, then setting $L = \sum_j v_j \otimes v_j^*$ and $\widetilde{L} = \sum_j V_j \otimes V_j^*$, we have

$$\mathcal{R}(\lambda)^{-1} = \frac{1}{\lambda} L + \frac{1}{\sqrt{\lambda}} \widetilde{L} + \gamma_0 Q(\lambda)\, \gamma_0^*. \tag{A.3}$$

In the following lemma we collect various facts concerning the sections $\{v_j, V_j\}$ and the operator $\gamma_0 Q(\lambda)\, \gamma_0^*$.

LEMMA A.3. *The sections $\{v_j, V_j\}$ are linearly independent in $L^2(Y, E_0)$ and the kernel of $\mathcal{R} = \mathcal{R}(0)$ is exactly the subspace $V = \operatorname{span}\{v_j, V_j\} \subset L^2(Y, E_0)$. The operator $\gamma_0 Q(\lambda)\, \gamma_0^*$ is a self-adjoint classical elliptic pseudo-differential operator of order -1 depending continuously on $\lambda \in [0, \infty)$.*

Proof. Suppose there is a linear relation

$$\sum a_j\, v_j + b_j\, V_j = 0, \quad a_j, b_j \in \mathbb{C}. \tag{A.4}$$

Consider the section

$$\Phi = \sum a_j\, u_j + b_j\, U_j \in C^\infty(X, E).$$

Then Φ is an extended L^2-solution of Δ_X and by (A.4), $\Phi|_Y = 0$; therefore by the uniqueness of the solutions to the Dirichlet problems on M and on Z, Φ must be identically zero on all of X. This implies that $a_j = b_j = 0$ for each j.

The fact that $V \subset \ker \mathcal{R}$ follows almost from the definition of $\mathcal{R}$, so assume that $\mathcal{R}\varphi = 0$. By definition of $\mathcal{R}$ we can choose a smooth bounded function $\Phi = (\Phi_1, \Phi_2) \in C^\infty(M, E) \oplus C^\infty(Z, E)$ such that $\Phi_1|_Y = \Phi_2|_Y = \varphi$, $\Delta_X \Phi = 0$ off of Y, and

$$0 = \mathcal{R}\varphi = \partial_u \Phi_1\Big|_{u=0} - \partial_u \Phi_2\Big|_{u=0}.$$

This implies that Φ defines a continuously differentiable bounded function on all of X such that $\Delta_X \Phi = 0$. By elliptic regularity, Φ must actually be smooth on all of X and hence $\Phi \in \operatorname{span}\{u_j, U_j\}$. Therefore, $\varphi = \Phi|_Y \in V$.

Finally, since near Y the operator $Q(0)$ has the structure of a usual classical pseudodifferential operator of order -2 on a closed manifold [15], the argument in Theorem A.2 can be used to show that $\gamma_0 Q(\lambda)\, \gamma_0^*$ is an elliptic classical pseudodifferential operator on Y of order -1. □

THEOREM A.4. *The operator $\mathcal{R} = \mathcal{R}(0)$ is a nonnegative self-adjoint first order elliptic classical pseudodifferential operator such that*

$$\mathcal{R} = \begin{cases} 0 & \text{on } V, \\ A^{-1} & \text{on } V^\perp, \end{cases}$$

where $A = P^\perp \gamma_0 Q(0) \gamma_0^* P^\perp$.

Proof. If $A(\lambda) = \gamma_0(\Delta_X + \lambda)^{-1}\gamma_0^*$, then we know that $\mathcal{R}(\lambda)^{-1} = A(\lambda)$, so in particular, given a smooth section $\varphi \in V^\perp$, for $\lambda > 0$ we have $\varphi = \mathcal{R}(\lambda)A(\lambda)\varphi$. Since $\varphi \in V^\perp$, we have $L\varphi = 0 = \widetilde{L}\varphi$, so according to (A.3),

$$A(\lambda)\varphi = \left(\frac{1}{\lambda}\, L + \frac{1}{\sqrt{\lambda}}\, \widetilde{L} + \gamma Q(\lambda)\, \gamma_0^*\right) \varphi = \gamma_0 Q(\lambda)\gamma_0^*\varphi,$$

which is continuous at $\lambda = 0$. Therefore, taking $\lambda \to 0^+$ in the equation $\varphi = \mathcal{R}(\lambda)A(\lambda)\varphi$ and using that $\mathcal{R}(\lambda)$ is continuous at $\lambda = 0$, $\mathcal{R}P = 0$, and $\varphi = P^\perp\varphi$, we get

$$\begin{aligned}(\mathrm{Id} - P)\varphi = \varphi &= \mathcal{R}(0)\, A(0)\varphi \\ &= \mathcal{R}(P + P^\perp)A(0)\varphi = \mathcal{R}\, P^\perp A(0)P^\perp\varphi = \mathcal{R}\, A\varphi,\end{aligned}$$

where $A = P^\perp A(0)P^\perp = P^\perp\gamma_0 Q(0)\gamma_0^* P^\perp$ is, by Lemma A.3, a self-adjoint elliptic classical pseudodifferential operator on Y of order -1. Since A vanishes on V, the equation $(\mathrm{Id} - P)\varphi = \mathcal{R}\, A\varphi$ holds for any smooth section φ, so

$$\mathrm{Id} - P = \mathcal{R}\, A \tag{A.5}$$

on all smooth sections on Y. Since A is a self-adjoint elliptic operator of order -1, it has a Green's operator, an elliptic self-adjoint operator B of order 1 on Y such that

$$AB = \mathrm{Id} - K = BA, \qquad K = \sum_j \psi_j \otimes \psi_j^*,$$

where $\{\psi_j\} \subset C^\infty(Y, E_0)$ is an orthonormal basis for the kernel of A. Hence,

$$(\mathrm{Id} - P)B = \mathcal{R}\, AB = \mathcal{R}\Big(\mathrm{Id} - \sum_j \psi_j \otimes \psi_j^*\Big) = \mathcal{R} - \sum_j (\mathcal{R}\psi_j) \otimes \psi_j^*,$$

which implies that

$$\mathcal{R} = B - PB + \sum_j (\mathcal{R}\psi_j) \otimes \psi_j^*.$$

Since P is a finite rank smoothing operator, the operator to the right of B in this equation is also a finite rank smoothing operator. Thus, $\mathcal{R}$ differs from B by a smoothing operator, so $\mathcal{R}$ is a first order elliptic pseudodifferential operator. Since $\mathcal{R}(\lambda)$ is positive definite for $\lambda > 0$ and is continuous as a function of λ, taking $\lambda \to 0^+$ shows that $\mathcal{R} = \mathcal{R}(0)$ is nonnegative. Since the kernel of $\mathcal{R}$ is exactly V, the equation (A.5) implies that A must in fact be the Green's operator of $\mathcal{R}$. This completes the proof of our theorem. $\square$

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Paul Loya, Department of Mathematics, Binghamton University, Vestal Parkway East, Binghamton, NY 13902, USA

E-mail address: paul@math.binghamton.edu

Jinsung Park, Mathematisches Institut, Universität Bonn, Beringstrasse 1, D-53115 Bonn, Germany

E-mail address: jpark@math.uni-bonn.de