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RATE OF DECAY OF CONCENTRATION FUNCTIONS FOR SPREAD OUT MEASURES

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Abstract. Let G be a locally compact unimodular group and μ an adapted spread out probability measure on G. We relate the rate of decay of the concentration functions associated with μ to the growth of a certain subgroup N_μ of G. In particular, we show that when μ is strictly aperiodic (i.e., when $N_\mu = G$) and G satisfies the growth condition $V_G(m) \geq Cm^D$, then for any compact neighborhood $K \subset G$ we have $\sup_{g\in G} \mu^{*n}(gK) \leq C'n^{-D/2}$. This extends recent results of Retzlaff [R2] on discrete groups for adapted probability measures.

1. Introduction

Let G be a locally compact group, μ be a regular probability on G. The concentration functions are defined as follows:

$$f_n(K) = \sup_{g\in G} \mu^{*n}(gK),$$

for every compact $K \subset G$, where μ^{*n} is the n-fold convolution power of μ.

The measure μ is said to be *irreducible* if the support S_μ of μ generates G as a closed semi-group. It is said to be *adapted* if S_μ generates G as a closed group. Hence irreducibility implies adaptedness.

An important group in the analysis of concentration functions is the group N_μ, defined as the smallest closed normal subgroup a coset of which contains S_μ.

It is proved in [DL] that when μ is adapted, the group G/N_μ is monothetic, either compact or isomorphic to $\mathbf{Z}$. In particular, if μ is irreducible, only the first case appears.

The study of concentration functions started with the pioneering work of Paul Levy [L] who studied the case of the real line. The question is whether, under certain assumptions on the group or on the measure, the concentration

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functions go to zero. The main steps of the analysis were obtained by Csiszar [C], Hoffman and Mukherjea [HM] and finally Jaworski, Rosenblatt and Willis [JRW]. Let us recall the important result of Jaworski, Rosenblatt and Willis:

Theorem 1.1 ([JRW]). *Let G be a locally compact non compact group. If μ is an adapted probability on G such that G/N_μ is compact (in particular, if μ is irreducible) then, for any compact $K \subset G$, the concentration functions $(f_n(K))_n$ converge to zero.*

Then Jaworski [J3] completed the results of [JRW] by proving that if the concentration functions fail to go to zero the group G and the measure μ must have a very particular form. Actually a sufficient condition for the convergence to zero is that μ is not carried on a coset of a compact normal subgroup.

In the situation above, one would like to estimate the rate of convergence of the concentration functions under reasonable assumptions on μ. The rate of decay of the concentration functions was investigated by many authors (e.g., [B1], [B2], [VSC], [R2]), mainly under the assumption that μ be irreducible and absolutely continuous with respect to a Haar measure on G. These results led to the conjecture that the concentration function may go to zero with speed $1/\sqrt{n}$, at least.

The case of nonunimodular groups can be treated (for general adapted measures), using the corresponding results on the real line via the use of the modular function (see, e.g., Bougerol [B2]).

We will be concerned with locally compact unimodular groups G and adapted (or irreducible) spread out measures μ. We will follow [R2] (who considered discrete groups) and we will show how to extend his argument to the case of a general unimodular group. The case of locally finite groups remains still open.

We will relate the rate of decay of the concentration functions to the growth of the group. We will work with not necessarily compactly generated groups and hence we need an appropriate notion of growth. We write $|A|$ for the Haar measure of a Borel set A of G. We say that G has polynomial growth of degree (at least) D if there exist a compact neighborhood $K \subset G$ (of the identity) and a constant $C > 0$ such that $|K^n| \geq Cn^D$. Then we will write $V_G(n) \geq Cn^D$. When G is compactly generated this is the standard definition (where K is chosen among the generating compact neighborhoods).

The main tools are Theorem VII.1.1 of [VSC], as well as its proof itself. Let us recall this result:

Theorem 1.2. *Let G be a unimodular compactly generated group. Let U be an open, generating neighborhood of the identity, and let F be a non-negative function such that $\int F = 1$, $\|F\|_\infty < \infty$ and $\inf_U\{F\} \geq \varepsilon > 0$. Then,* *if* $V_G(n) \geq Cn^D$*, we have*

$$\|F^{*k}\|_\infty = O(k^{-D/2}).$$

Recall that a measure μ is said to be *spread out* if it admits a n-fold convolution power μ^{*n} which is non singular with respect to a Haar measure on G.

In this paper, we obtain:

THEOREM 1.3. *Let G be a locally compact unimodular group. Let μ be an adapted (resp. irreducible) spread out probability measure on G. Assume that $V_{N_\mu}(m) \geq Cm^D$ (resp $V_G(m) \geq Cm^D$), for some positive constant C. We have, for every compact neighborhood $K \subset G$,*

$$\sup_{g\in G} \mu^{*n}(gK) = O(n^{-D/2}).$$

Notice that we do not assume that G is compactly generated, nor that the neutral element is in the support of μ, nor that the measure generates G as a semi-group (i.e., μ is irreducible), nor any absolute continuity (in particular we do not assume boundedness of the density if the measure admits one). Unfortunately, the methods used in this paper do not translate to the non-spread out case.

Notice that Varopoulos [V] treated the case of certain absolutely continuous measures whose density does not dominate any neighborhood of the identity. Our theorem was proved by Retzlaff [R2] in the case of discrete groups for adapted probability measures.

Other references on the topic may be found in [VSC].

2. Preliminaries on the open semigroup associated to a measure

The work of Retzlaff [R2], for discrete groups, was simplified by the fact that, in this case, no topology is really involved (measures are all absolutely continuous, their support is open...). To extend his argument we will need to work with an open semi-group made out of open subsets of the supports of the convolution power of the measure μ (under consideration) and to explain that many classical algebraic and topological properties can be described in terms of these sets. For the remainder of this paper all measures will be assumed to be spread out.

If μ is a regular probability measure, following [A] (see also [J2]), for every $n \in \mathbf{N}$ we define $\overset{\circ}{S}_n$ to be the set of all $g \in G$ such that μ^{*n} dominates a multiple of the Haar measure on a neighborhood of g.

Notice that when μ^{*n} admits a continuous density F_n, then $\overset{\circ}{S}_n = \{g \in G : F_n(g) > 0\}$.

For $n < 0$, we define $\overset{\circ}{S}_{-n} := \overset{\circ}{S}_n(\check{\mu}) = \overset{\circ}{S}_n^{\,-1}$, where $\check{\mu}$ is the symmetric measure of μ.

The family $(\overset{\circ}{S}_n)$ is non empty if and only if μ is spread out.

Denote

$$\overset{\circ}{S} = \bigcup_{n \geq 1} \overset{\circ}{S}_n. \tag{1}$$

We will write $\mathrm{gp}(A)$ for the group generated by $A \subset G$. Notice that we do not consider the closed subgroup generated by A. We will denote by S_ν the support of some regular measure ν.

LEMMA 2.1. *Let G be a locally compact group. Let μ be a spread out probability measure on G. Then we have:*

(i) $S_{\mu^{*n}} \overset{\circ}{S}_1 \subset \overset{\circ}{S}_{n+1} \qquad \forall n \geq 1.$

(ii) $\overset{\circ}{S}$ *is an open semi-group.*

Proof. The proof is straightforward and we leave it to the reader. □

We have:

LEMMA 2.2. *Let μ be an adapted spread out measure. Then the subgroup generated by $\overset{\circ}{S}$ is G. If μ is also assumed to be irreducible then $\overset{\circ}{S} = G$.*

Proof. By (i) of Lemma 2.1, we have $S_\mu \overset{\circ}{S} \subset \overset{\circ}{S}$. Hence the first point is obvious.

Assume now that μ is irreducible. Then, by the previous argument, $\overset{\circ}{S}$ is dense in G.

Assume that $\overset{\circ}{S} \neq G$ and let $g \in G - \overset{\circ}{S}$. By Lemma 2.1, $\overset{\circ}{S}$ is a semi-group. Hence $g\overset{\circ}{S}^{-1} \cap \overset{\circ}{S} = \emptyset$, which contradicts the fact that $\overset{\circ}{S}$ and $g\overset{\circ}{S}^{-1}$ are dense and open. □

We now extend Proposition 1.1 of Derriennic and Lin [DL]. The result of [DL] says that for every adapted measure μ on a locally compact group G, we have

$$N_\mu = \overline{\mathrm{gp}\ \Big(\bigcup (S_\mu^n S_\mu^{-n} \cup S_\mu^{-n} S_\mu^n)\Big)}. \tag{2}$$

Notice that when μ is spread out we do not need to take the closure in (2), since open subgroups are closed.

Lemma 2.3. *Let μ be an adapted spread out measure on the locally compact group G. Then*

$$N_\mu = \text{gp}\ \left(\bigcup(\overset{\circ}{S}_n\overset{\circ}{S}_{-n} \cup \overset{\circ}{S}_{-n}\overset{\circ}{S}_n)\right). \tag{3}$$

Moreover, if μ is irreducible,

$$N_\mu = \bigcup \overset{\circ}{S}_n\overset{\circ}{S}_{-n} = \bigcup \overset{\circ}{S}_{-n}\overset{\circ}{S}_n. \tag{4}$$

Proof. Denote by H the right hand side of (3). By (2), H is contained in N_μ. It is clear that H is an open group, so it is closed. Since μ is spread out, there exists $k \geq 1$ such that $\overset{\circ}{S}_k \neq \emptyset$. Then, for every $n \geq 1$, we have, by Lemma 2.1,

$$S_nS_{-n} \subset S_n\overset{\circ}{S}_k\overset{\circ}{S}_{-k}S_{-n} \subset \overset{\circ}{S}_{n+k}\overset{\circ}{S}_{-n-k} \subset H.$$

Also $S_{-n}S_n \subset H$. Hence $H = N_\mu$.

When μ is irreducible, the claimed result follows from Lemma 2.2 and [J2] (Proposition 4.14). □

When μ is irreducible, it is possible to strengthen this result:

Lemma 2.4. *Let μ be an irreducible spread out probability measure on a locally compact group G. Then N_μ is of finite index (say l) in G. For every $k \geq 1$, $\bigcup_{n\geq 1} \overset{\circ}{S}_{kn}$ is a normal subgroup of G. Moreover, $N_\mu = \bigcup_{n\geq 1} \overset{\circ}{S}_{ln}$.*

Proof. Let $k \geq 1$. Define $\Gamma_k := \bigcup_{n\geq 1} \overset{\circ}{S}_{kn}$. Then Γ_k is clearly a semigroup. Let $x \in \Gamma_k$, that is $x \in \overset{\circ}{S}_{kp}$ for some $p \geq 1$. By Lemma 2.2, since μ is irreducible, $x^{-1} \in \overset{\circ}{S}_m$ for some $m \geq 1$. Hence $x^{-1} = (x^{-1})^k x^{k-1} \in \overset{\circ}{S}_{km+k(k-1)p} \subset \Gamma_k$. Hence Γ_k is a subgroup of G.

Let $p \geq 1$. Let $x \in G$. There exist $s, t \geq 1$ such that $x \in \overset{\circ}{S}_s$ and $x^{-1} \in \overset{\circ}{S}_t$. So

$$x\overset{\circ}{S}_{kp}x^{-1} = x\overset{\circ}{S}_{kp}x^{-1}(xx^{-1})^{k-1} \subset \overset{\circ}{S}_{k(p+s+t)} \subset \Gamma_k.$$

Hence Γ_k is normal in G.

Since μ is irreducible and spread out, G/N_μ is compact and discrete, so N_μ has finite index l. In particular $S_{\mu^l} \subset N_\mu$.

Let Λ_l be the subgroup generated by S_{μ^l}. Then $\Gamma_l \subset \Lambda_l \subset N_\mu$. Since Γ_l contains $\mathbf{e}$ and $S_{\mu^l}\Gamma_l \subset \Gamma_l$, then $\Gamma_l = \Lambda_l$.

Fix $y \in S_\mu$. Let $x \in S_\mu$. Then $x^l \in \Gamma_l$ and $(x^{l-1}y)^{-1} \in \Gamma_l$. So $x = y(x^{l-1}y)^{-1}x^l \in y\Gamma_l$. Since, by definition, N_μ is the smallest closed normal subgroup of G a class of which contains S_μ, then $\Gamma_l = N_\mu = \Lambda_l$. So the lemma is true. □

Remarks 2.1. Part of the lemma was proved for discrete groups in Lemma 4 of [R1]. One may use this lemma to prove Theorem 3.10 for irreducible probabilities, using ideas of [R2]. When μ is spread out, N_μ is open, so G/N_μ is discrete. Since μ is irreducible it is not difficult to see that $G/N_\mu \simeq \mathbf{Z}/l\mathbf{Z}$ and so N_μ is of finite index.

We need also a result relating the growth of a group with certain properties of its open subsemigroups. It is essentially contained in Jenkins [Je] (see also Theorem 3.2 of [J1]). There is a (stronger) discrete version in [Ro] (Proposition 2.4) which was used in [R2].

Proposition 2.5. *Let G be a locally compact group. Let A be an open subsemigroup of G such that there exist $a, b \in A$ satisfying $aA \cap bA = \emptyset$. Then G has exponential growth.*

Proof. Let $K \subset A$ be any compact neighborhood. Let $V := aK \cup bK$. By assumption, we have the disjoint union

$$V^{n+1} \supset aKV^n \cup bKV^n \qquad \forall n \geq 0.$$

Hence $|V^{n+1}| \geq 2|V^n| \geq 2^n|V|$, for every $n \geq 0$. So G has exponential growth. □

This result is useful to obtain the following generalization of Lemma 8 of [R2]:

Lemma 2.6. *Let G be a locally compact group and μ be a spread out probability measure such that $\overset{\circ}{S}_n \cap \overset{\circ}{S}_m = \emptyset$ for every $m \neq n$. Then either G has exponential growth or, for every $m \geq 1$, we have $\overset{\circ}{S}_m \overset{\circ}{S}_{-m} \subset \bigcup \overset{\circ}{S}_{-k} \overset{\circ}{S}_k$ and $\overset{\circ}{S}_{-m} \overset{\circ}{S}_m \subset \bigcup \overset{\circ}{S}_k \overset{\circ}{S}_{-k}$.*

Proof. If there exist $a, b \in \overset{\circ}{S}_m$ such that $a\overset{\circ}{S} \cap b\overset{\circ}{S} = \emptyset$ then, by the previous proposition (with $A = \overset{\circ}{S}$), G has exponential growth. Hence, assume that for all $m \geq 1$ and for all $a, b \in \overset{\circ}{S}_m$, $a\overset{\circ}{S} \cap b\overset{\circ}{S} \neq \emptyset$. Let $m \geq 1$ and $a, b \in \overset{\circ}{S}_m$. Then there exists $(s, t) \in \overset{\circ}{S}_k \times \overset{\circ}{S}_l$ such that $as = bt$. Since $as \in \overset{\circ}{S}_{m+k}$ and $bt \in \overset{\circ}{S}_{m+l}$ and since the sets $(\overset{\circ}{S}_p)_p$ are pairwise disjoint, we may have $k = l$. So $a^{-1}b \in \cup \overset{\circ}{S}_k \overset{\circ}{S}_{-k}$ and a, b being arbitrary, $\overset{\circ}{S}_{-m} \overset{\circ}{S}_m \subset \cup \overset{\circ}{S}_k \overset{\circ}{S}_{-k}$. The second inclusion of the lemma can be proven similarly. □

We deduce:

Lemma 2.7. *Let μ be a spread out probability measure on a locally compact group G. Then, at least one of the following occurs:*

(i) *There exists* $m, n \geq 1$, *such that* $\overset{\circ}{S}_{lm} \subset \overset{\circ}{S}_{-ln}\overset{\circ}{S}_{ln} \cap \overset{\circ}{S}_{ln}\overset{\circ}{S}_{-ln}$ *for all* $l \geq 1$.

(ii) $\overset{\circ}{S}_{-m}\overset{\circ}{S}_{m} \subset \bigcup \overset{\circ}{S}_{k}\overset{\circ}{S}_{-k}$ *and* $\overset{\circ}{S}_{m}\overset{\circ}{S}_{-m} \subset \bigcup \overset{\circ}{S}_{-k}\overset{\circ}{S}_{k}$ *for every* $m \geq 1$.

(iii) G *has exponential growth and* $\overset{\circ}{S}_{m} \cap \overset{\circ}{S}_{n} = \emptyset$ *for all* $m \neq n$.

Proof. By the previous lemma, if $\overset{\circ}{S}_{m} \cap \overset{\circ}{S}_{n} = \emptyset$ for all $m \neq n$, then (ii) or (iii) occurs. So assume that $\overset{\circ}{S}_{k} \cap \overset{\circ}{S}_{n} \neq \emptyset$ for some $n \neq k$. Hence, for every $l \geq 1$, $\overset{\circ}{S}_{lk} \cap \overset{\circ}{S}_{ln} \neq \emptyset$. One can assume that $n > k$. We have $\mathbf{e} \in \overset{\circ}{S}_{lk}\overset{\circ}{S}_{-ln}$. Hence

$$\overset{\circ}{S}_{l(n-k)} \subset \overset{\circ}{S}_{l(n-k)}\overset{\circ}{S}_{lk}\overset{\circ}{S}_{-ln} \subset \overset{\circ}{S}_{ln}\overset{\circ}{S}_{-ln}.$$

Similarly, one proves that $\overset{\circ}{S}_{l(n-k)} \subset \overset{\circ}{S}_{-ln}\overset{\circ}{S}_{ln}$. □

3. Rate of decay for adapted spread out probability measures

Lemma 3.1. *Let* μ *be an (adapted) spread out probability measure on the unimodular locally compact group* G. *Assume there exist open sets* A *and* B *such that for some* $n \geq 1$, $A \subset \overset{\circ}{S}_{n}\overset{\circ}{S}_{-n}$, $B \subset \overset{\circ}{S}_{-n}\overset{\circ}{S}_{n}$ *and* $V_{\mathrm{gp}(A)}(m) \geq Cm^D$, $V_{\mathrm{gp}(B)}(m) \geq Cm^D$. *Then there exist* $m \geq 1$, *some positive measures* ν_m, β_m *and some symmetric absolutely continuous probability measures* μ_0 *and* μ_1 *(with respective densities* F_0 *and* F_1*), such that* $\mu^m = \nu_m + \beta_m$, ν_m *is absolutely continuous with bounded continuous density and* $\nu_m * \check{\nu}_m \geq L\mu_0$ *and* $\check{\nu}_m * \nu_m \geq L\mu_1$ *for some* $L > 0$ *and* $\|F_0^{*k}\|_\infty = O(k^{-D/2})$, $\|F_1^{*k}\|_\infty = O(k^{-D/2})$.

Proof. By assumption, there exists a symmetric compact neighborhood $K \subset \mathrm{gp}(A)$ of the identity, such that $|K^m| \geq Cm^D$. Let $g \in K$. There exist $k_{g,1}, \ldots, k_{g,i_g} \in A \cup A^{-1}$ such that $g = k_{g,1} \cdots k_{g,i_g}$. Since $A \cup A^{-1}$ is open, there exist relatively compact (in $A \cup A^{-1}$) open neighborhoods $V_{g,1}, \ldots, V_{g,i_g}$ of $k_{g,1}, \ldots, k_{g,i_g}$, respectively. Hence

$$K \subset \bigcup_{g \in K} V_{g,1} \cdots V_{g,i_g}.$$

By compactness, there exist $g_1, \ldots, g_l \in K$, such that

$$K \subset \bigcup_{j=1}^{l} V_{g_j,1} \cdots V_{g_j,i_{g_j}}.$$

Let K' be a compact neighborhood of the identity with $K' \subset \overset{\circ}{S}_n \overset{\circ}{S}_{-n}$. Define M to be the closure of

$$K' \bigcup \left(\bigcup_{j=1}^{l} \bigcup_{s=1}^{i_{g_j}} (V_{g_j, i_s} \cup V_{g_j, i_s}^{-1}) \right).$$

Then M is a compact (symmetric) neighborhood of the identity and $M \subset \overset{\circ}{S}_n \overset{\circ}{S}_{-n}$. By the Lebesgue decomposition of μ^{*n}, there exist an absolutely continuous positive measure ν_n with density ψ_n and a singular measure β_n such that $\mu^{*n} = \nu_n + \beta_n$. Notice that ψ_n is not trivial since $\overset{\circ}{S}_n \overset{\circ}{S}_{-n}$ is not. Define $\phi_n := \inf\{1, \psi_n\}$ and $\eta_n = \phi_n m_G$ (m_G stands for a Haar measure of G). Then $\eta_n * \check{\eta}_n$ is absolutely continuous with continuous density which is positive on $\overset{\circ}{S}_n \overset{\circ}{S}_{-n}$. Take μ_0 to be the uniform measure on M. The construction of μ_1 is symmetric. The estimation of the norms of F_0 and F_1 follows from Theorem VII.1.1 of [VSC] (i.e., Theorem 1.2). □

Remarks 3.1. When μ is spread out, it is well known (see, e.g., [A]) that $\beta_n(G) \to 0$. Hence, in the preceding lemma we can assume that $\nu_n(G)$ is as large as we please and that the density of ν_n is continuous. In particular, we can choose $n \geq 1$ and then ν_n such that $\nu_n(G) = 1/2$.

The following lemma is implicit in [VSC], see the proof of Lemma VII.4.5. It is stated (and proven) as follows in [R2]:

Lemma 3.2. *Let F, F_0 and F_1 be probability density functions on a locally compact group G such that F_0 and F_1 are symmetric and $F * \check{F} \geq CF_0$ and $\check{F} * F \geq CF_1$, for some positive constant C. Let T, $\check{T}$, T_0 and T_1 the operators of right convolution by F, $\check{F}$, F_0 and F_1 respectively. Then $\|f\|_2^2 - \|T_0^{1/2} f\|_2^2 \leq \frac{1}{C}(\|f\|_2^2 - \|Tf\|_2^2)$ and $\|f\|_2^2 - \|T_1^{1/2} f\|_2^2 \leq \frac{1}{C}(\|f\|_2^2 - \|\check{T}f\|_2^2)$ for all $f \in L^1(G) \cap L^2(G)$.*

Let us recall also Lemma 3 of [R2].

Lemma 3.3. *Let F be a probability density function on a locally compact unimodular group G with $\|F\|_\infty \leq C$ for some constant C. Let T be the operator of right convolution by F. Then $\|Tf\|_2^2 \leq C\|f\|_1^2$ for all $f \in L^1(G) \cap L^2(G)$. Also, if F is symmetric then $T^{1/2}$ is contracting on $L^2(G)$. If F is symmetric and $\|F^{*k}\|_\infty = O(k^{-D/2})$ then $\|(T^{1/2})^k\|_{1\to 2} = O(k^{-D/4})$.*

We need the following generalization of Lemma VI.3.5 of [VSC]. Lemma VI.3.5 of [VSC] asserts that any infinite sequence satisfying (5) satisfies (6) for some $M > 0$. Our lemma claims that one can find a universal constant M such that (6) is true for all sequences satisfying (5) and whose first term is uniformly bounded.

Lemma 3.4. *Let $L, C, n > 0$. There exists a positive constant M, such that for every $l \geq 1$ and every $(t_1, \ldots, t_l) \in (\mathbf{R}^{+*})^l$ with $t_1 \leq L$ and*

$$t_{k+1}^{1+2/n} \leq C(t_k - t_{k+1}) \qquad \forall\ 1 \leq k \leq l-1, \tag{5}$$

we have

$$t_k \leq M k^{-n/2} \qquad \forall\ 1 \leq k \leq l. \tag{6}$$

Proof. Define the function f from $[0, +\infty[$ to $[0, +\infty[$ by $f(x) = x^{1+2/N} + Cx$. Then f is an increasing homeomorphism. Define the sequence $(y_k)_k$ by the recurrence

$$y_1 = L, \qquad y_{k+1} = f^{-1}(Cy_k) \qquad \forall k \geq 1.$$

By Lemma VI.3.5 of [VSC] (see the remark above), there exists $M > 0$ such that (y_k) satisfies (6) (for every $l \geq 1$).

Now, let $n \geq 1$ and let $(t_k)_{1 \leq k \leq l} \in (\mathbf{R}^{+*})^l$ be a sequence satisfying (5) and $t_1 \leq L$. Using the fact that f is increasing one can prove easily that $t_k \leq y_k$ for every $1 \leq k \leq l$. Hence the lemma is proven. □

We deduce the following extension of Lemma VII.2.6 of [VSC]:

Lemma 3.5. *Let $(T_k)_{k \geq 1}$ be a sequence of operators which contract all the spaces $L^p(G)$, $1 \leq p \leq \infty$. Assume that these operators satisfy the family of inequalities*

$$\|T_j f\|_2^{2+4/n} \leq C \left(\|T_j f\|_2^2 - \|T_k T_j f\|_2^2 \right) \|f\|_1^{4/n} \qquad \forall f \in L^1(G) \cap L^2(G) \quad \forall j, k \geq 1,$$

for some $n > 0$. Then there exists $C' > 0$ such that for every $l \geq 1$ and every $(i_1, \ldots, i_l) \in (\mathbf{N}^*)^k$ we have

$$\|T_{i_l} \cdots T_{i_1}\|_{1 \to 2} \leq \left(\frac{C' C n}{l} \right)^{n/4}. \tag{7}$$

Proof. Let $f \in L^1(G) \cap L^2(G)$ with $\|f\|_1 = 1$. Let $l \geq 1$ and let $(i_1, \ldots, i_l) \in (\mathbf{N}^*)^l$. Define, for every $1 \leq k \leq l$, $t_k := \|T_{i_k} \cdots T_{i_1} f\|_2^2$. Then, by assumption, $(t_k)_{1 \leq k \leq l}$ satisfies (5) and $t_1 \leq 1$. So, by Lemma 3.4, there exists $M > 0$ such that for every $f \in L^1(G) \cap L^2(G)$, and every $(i_1, \ldots, i_l) \in (\mathbf{N}^*)^l$,

$$\|T_{i_l} \cdots T_{i_1} f\|_2 \leq \frac{M}{l^{n/4}} \|f\|_1.$$

Hence (7) follows by the density of $L^1(G) \cap L^2(G)$ in $L^1(G)$. □

Lemma 3.6. *Let $E, D > 0$. Then, for every $X, Y, Z, k > 0$ satisfying $X \leq EYk^{-D/2} + kZ$ and $k \leq (Y/Z)^{2/(D+2)} < 2k$ we have $X^{1+2/D} \leq (1 + 2^{D/2}E)^{1+2/D} Y^{2/D} Z$.*

Proof. Let $X, Y, Z, k > 0$ satisfy the inequalities of the lemma. Define $x := X/Z$ and $y := Y/Z$. By assumption we have

$$x \leq Eyk^{-D/2} + k,$$
$$k \leq y^{2/(D+2)},$$
$$k^{-D/2} \leq 2^{D/2} y^{-D/(D+2)}.$$

So $x^{1+2/D} \leq (1 + 2^{D/2}E)^{1+2/D} y^{2/D}$, which proves the lemma. □

Let R be any positive operator which contracts all L^p spaces. For every $p, q \geq 1$ we will write $\|R\|_{p\to q}$ for the norm of R seen as an operator from L^p to L^q.

THEOREM 3.7. *Let G be a locally compact unimodular group. Let R_0 be a symmetric positive operator which contracts $L^2(G)$ and satisfies to $\|R_0^k\|_{1\to 2} \leq C_0 k^{-D/4}$ for some constant C_0. Let (μ_i) be a sequence of absolutely continuous probability measures with densities (F_i) and denote by (T_i) the sequence of operators of right convolution by (μ_i). Assume there exist constants C and C_1 such that for every $i \geq 1$, we have $\|F_i\|_\infty \leq C$, $\|f\|_2^2 - \|R_0 f\|_2^2 \leq C_1(\|f\|_2^2 - \|T_i f\|_2^2)$, for all $f \in L^2(G)$. There exists some constant C' such that for every subsequence $(i_j)_j$ we have*

$$\|T_{i_1} \cdots T_{i_k}\|_{1\to 2} \leq C' k^{-D/4} \qquad \forall k \geq 1.$$

This result is Theorem 2 of [R3]. The proof is slightly different from the proof of Theorem VI.1.2 of [VSC]. We include it for completeness.

Proof. Since R_0 is a positive symmetric contraction of $L^2(G)$, $(I - R_0^2)^{1/2}$ is well defined and commutes with R_0. Let $f \in L^1(G) \cap L^2(G)$. For every $j \geq 1$, we have

$$\begin{aligned}\|R_0^j f\|_2^2 - \|R_0^{j+1} f\|_2^2 &= \|(I - R_0^2)^{1/2} R_0^j f\|_2^2 \\ &\leq \|(I - R_0^2)^{1/2} f\|_2^2 = \|f\|_2^2 - \|R_0 f\|_2^2.\end{aligned}$$

Therefore, for every $k \geq 1$,

$$\begin{aligned}\|f\|_2^2 &= \|R_0^k f\|_2^2 + \sum_{j=0}^{k-1} \left(\|R_0^j f\|_2^2 - \|R_0^{j+1} f\|_2^2 \right) \\ &\leq C_0^2 k^{-D/2} \|f\|_1^2 + k(\|f\|_2^2 - \|R_0 f\|_2^2.\end{aligned}$$

Let $j \geq 1$. Changing f to $T_j f$ in the above and using $\|T_j f\|_1 \leq \|f\|_1$ yields

$$\|T_j f\|_2^2 \leq C_0^2 k^{-D/2} \|f\|_1^2 + k(\|T_j f\|_2^2 - \|R_0 T_j f\|_2^2. \tag{8}$$

Now, using Lemma 3.3 for the second inequality, we have

$$\|T_j f\|_2^2 - \|R_0 T_j f\|_2^2 \leq \|T_j f\|_2^2 \leq C \|f\|_1^2.$$

Hence there exists $n \in \mathbf{N}$ such that

$$2^n \le \left(\frac{C\|f\|_1^2}{\|T_j f\|_2^2 - \|R_0 T_j f\|_2^2} \right)^{2/(D+2)} \le 2^{n+1}.$$

Taking $k = 2^n$ in (8) and $X = \|T_j f\|_2^2$, $Y = C\|f\|_1^2$, $Z = \|T_j f\|_2^2 - \|R_0 T_j f\|_2^2$ and $E = C_0^2/C$ in Lemma 3.6 we obtain

$$\|T_j f\|_2^{2+4/D} \le (1 + 2^{D/2} C_0^2/C)^{1+2/D} (\|T_j f\|_2^2 - \|R_0 T_j f\|_2^2) \|f\|_1^{4/D}.$$

Hence, by the assumptions of the theorem, we obtain

$$\|T_j f\|_2^{2+4/D} \le C_1 (1 + 2^{D/2} C_0^2/C)^{1+2/D} (\|T_j f\|_2^2 - \|T_i T_j f\|_2^2) \|f\|_1^{4/D}.$$

This estimation is satisfied for every $i \ge 1$ and every $f \in L^1(G) \cap L^2(G)$ and the constants involved do not depend on f.

Hence, by Lemma 3.5, there exists a constant $C' > 0$ such that for every $k \ge 1$ and every $(i_1, \ldots, i_k) \in (\mathbf{N}^*)^k$, we have

$$\|T_{i_1} \cdots T_{i_k}\|_{1 \to 2} \le C' k^{-D/4}.$$ □

THEOREM 3.8. *Let G be a unimodular locally compact group and μ be an adapted absolutely continuous probability measure with bounded continuous density F. Assume there exists some symmetric absolutely continuous probability measure μ_0 with density F_0 such that $\mu * \check{\mu} \ge L\mu_0$ for some $L > 0$ and $\|F_0^{*k}\|_\infty = O(k^{-D/2})$. Finally let (ν_k) be a sequence of regular probability measures on G, denote $F_k := F * \nu_k$ and let T_k be the operator of right convolution by F_k. Then there exists $C > 0$, such that $\forall k \ge 1$, $\forall (i_1, \ldots i_k) \in (\mathbf{N}^*)^k$, $\|T_{i_1} \cdots T_{i_k}\|_{1\to 2} \le C k^{-D/4}$.*

Proof. Let R_0, Q and $(Q_k)_k$ denote the operators of right convolution by μ_0, μ and $(\nu_k)_k$. By Lemma 3.3 and since $\|F_0\|_\infty = O(k^{-D/2})$, we have

$$\|R_0^k\|_{1\to 2} \le C_0 k^{-D/4}.$$

By assumption, using Lemma 3.2, we obtain, for every $f \in L^2(G)$,

$$\|f\|_2^2 - \|R_0^{1/2} f\|_2^2 \le C(\|f\|_2^2 - \|Qf\|_2^2) \le C(\|f\|_2^2 - \|Q_k f\|_2^2) \quad \forall k \ge 1,$$

where the last inequality follows from

$$\|Q_k f\|_2^2 = \|T_{\nu_k}(T_\mu f)\|_2^2 \le \|Qf\|_2^2 \qquad \forall f \in L^2(G),\ \forall k \ge 1.$$

Since it is clear that $\sup_{k\ge 1} \|F_k\|_\infty \le \|F\|_\infty$, Theorem 3.7 yields the desired result. □

Theorem 5 of [R2] (for discrete groups) may be extended to unimodular groups and spread out probability measures. We have:

THEOREM 3.9. *Let G be a unimodular locally compact group and μ be an adapted spread out probability measure on G. Assume there exist some open sets A and B with $A \subset \overset{\circ}{S}_n \overset{\circ}{S}_{-n}$, $B \subset \overset{\circ}{S}_{-n} \overset{\circ}{S}_n$ for some $n \geq 1$ and $V_{\mathrm{gp}(A)}(m) \geq Cm^D$, $V_{\mathrm{gp}(B)}(m) \geq Cm^D$. Then, for every compact set K, $\sup_{g\in G} \mu^{*n}(gK) = O(n^{-D/2})$.*

Proof. By Lemma 3.1, there exist an integer $m \geq 1$ and some positive measures ν_m and β_m satisfying the conclusion of Lemma 3.1. By the remark after Lemma 3.1 we may and do assume that $\nu_m(G) = 1/2$. Write $\nu := \nu_m/\nu_m(G)$ and $\beta := \beta_m/\beta_m(G)$. Hence $\mu^m = 1/2(\nu + \beta)$. This reduction is not essential but will simplify computations.

Denote by (F_k) (resp. (G_k)) the densities of the absolutely continuous measures $(\nu * \beta^{*k})$ (resp. $(\beta^{*k} * \nu)$. When $k = 0$, β^{*k} is the Dirac measure at $\mathbf{e}$. Let (T_k) (resp. $(\tilde{T}_k)$) denote the operators of right convolution by (F_k) (resp. by (G_k)). By Theorem 3.8 applied with $(\nu_k) := (\nu * \beta^{*k})$ (resp. $(\nu_k) := (\check{\nu} * \check{\beta}^{*k})$), there exists a constant $C > 0$ such that

$$\|T_{i_1} \cdots T_{i_k}\|_{1\to 2} \leq Ck^{-D/4} \qquad \forall i_1, \dots i_k \geq 1,$$
$$\|\check{\tilde{T}}_{i_1} \cdots \check{\tilde{T}}_{i_k}\|_{1\to 2} \leq Ck^{-D/4} \qquad \forall i_1, \dots i_k \geq 1.$$

From the second inequalities, we obtain

$$\|\tilde{T}_{i_1} \cdots \tilde{T}_{i_k}\|_{2\to\infty} \leq Ck^{-D/4} \qquad \forall i_1, \dots i_k \geq 1.$$

Let $n \geq 2$ and $(i_1, \dots i_{2n}) \in \mathbf{N}^{2n}$ and $f \in L^1(G) \cap L^2(G)$. We have

$$\begin{aligned}\|\tilde{T}_{i_1} \dots \tilde{T}_{i_n} T_{i_{n+1}} \dots T_{i_{2n}} f\|_\infty &\leq \|\tilde{T}_{i_1} \dots \tilde{T}_{i_n}\|_{2\to\infty} \|T_{i_{n+1}} \dots T_{i_{2n}} f\|_2 \\ &\leq Cn^{-D/4} \|T_{i_{n+1}} \dots T_{i_{2n}}\|_{1\to 2} \|f\|_1 \\ &\leq C^2 n^{-D/2} \|f\|_1.\end{aligned}$$

Hence

$$\|\tilde{T}_{i_1} \dots \tilde{T}_{i_n} T_{i_{n+1}} \dots T_{i_{2n}}\|_{1\to\infty} \leq C^2 n^{-D/2}.$$

In particular,

$$\|F_{i_{2n}} * \dots * F_{i_{n+1}} * G_{i_n} * \dots * G_{i_1}\|_\infty \leq C^2 n^{-D/2}.$$

So, for every $n \geq 1$, we obtain

$$\begin{aligned}\|F_{i_{2n}} * \dots * F_{i_1}\|_\infty &= \|F_{i_{2n}} * \dots * F_{i_{n+1}} * G_0 * G_{i_n} * \dots G_{i_2} * \beta^{*i_1}\|_\infty \\ &\leq \|F_{i_{2n}} * \dots * F_{i_{n+1}} * G_0 * G_{i_n} * \dots G_{i_2}\|_\infty \\ &\leq C^2 n^{-D/2}.\end{aligned}$$

Let $n \geq 1$, $g \in G$ and K be a compact of G. We have

$$\begin{aligned}
\mu^{mn}(gK) &= \frac{1}{2^n}(\nu+\beta)^{*n}(gK) \\
&= \frac{1}{2^n}\sum_{j=0}^{n-1} \sum_{i_1,\dots,i_k \geq 0,\ i_1+\dots+i_k=n-j-k} \beta^{*j} * F_{i_1} * \cdots * F_{i_k}(gK) + \frac{1}{2^n}\beta^{*n}(K) \\
&\leq \frac{1}{2^n}\sum_{j=0}^{n-1} \sum_{i_1,\dots,i_k \geq 0,\ i_1+\dots+i_k=n-j-k} \|F_{i_1} * \cdots * F_{i_k}\|_\infty |gK| + \frac{1}{2^n} \\
&\leq \frac{1}{2^n}\sum_{l=0}^{n-1} C_n^l (Cl^{-D/2}) + \frac{1}{2^n} \\
&\leq C_1 n^{-D/2},
\end{aligned}$$

where the last inequality may be deduced from well-known results involving Bernstein's polynomials (see, e.g., Feller [F]). □

We are ready to prove our main results.

THEOREM 3.10. *Let G be a unimodular locally compact group. Let μ be an adapted spread out probability measure on G. If N_μ satisfies $V_{N_\mu}(m) \geq Cm^D$, then $\sup_{g\in G}\mu^{*n}(gK) = O(k^{-D/2})$ for every compact $K \subset G$.*

Proof. We consider the cases arising from Lemma 2.7:

Case I. There exists $m, n \geq 1$, such that $\mathring{S}_{lm} \subset \mathring{S}_{-ln}\mathring{S}_{ln} \cap \mathring{S}_{ln}\mathring{S}_{-ln}$ for all $l \geq 1$.

By Lemma 2.3, we have

$$N_\mu = \mathrm{gp}\left(\bigcup(\mathring{S}_k\mathring{S}_{-k} \cup \mathring{S}_{-k}\mathring{S}_k)\right).$$

Hence, by monotonicity of $(\mathring{S}_k\mathring{S}_{-k})_k$ and $(\mathring{S}_{-k}\mathring{S}_k)_k$ we obtain

$$N_\mu \subset \bigcup_l \mathrm{gp}\left((\mathring{S}_{lm}\mathring{S}_{-lm} \cup \mathring{S}_{-lm}\mathring{S}_{lm})\right).$$

Let $K \subset N_\mu$ be any compact neighborhood such that $|K^n| \geq Cn^D$ for some constant $C > 0$. Then

$$K \subset \bigcup_l \mathrm{gp}\left((\mathring{S}_{lm}\mathring{S}_{-lm} \cup \mathring{S}_{-lm}\mathring{S}_{lm})\right).$$

Since K is compact, there exists l_0 such that

$$K \subset \mathrm{gp}(\mathring{S}_{l_0m}\mathring{S}_{-l_0m} \cup \mathring{S}_{-l_0m}\mathring{S}_{l_0m}).$$

In particular,

$$K \subset \mathrm{gp}(\overset{\circ}{S}_{l_0 m}).$$

Applying Theorem 3.9 with $A = B = \overset{\circ}{S}_{l_0 m}$ leads to the desired result.

Case II. $\overset{\circ}{S}_{-m}\overset{\circ}{S}_m \subset \bigcup \overset{\circ}{S}_k\overset{\circ}{S}_{-k}$ *and* $\overset{\circ}{S}_m\overset{\circ}{S}_{-m} \subset \bigcup \overset{\circ}{S}_{-k}\overset{\circ}{S}_k$ *for every* $m \geq 1$.
By Lemma 2.3, we have

$$N_\mu = \mathrm{gp}\left(\bigcup (\overset{\circ}{S}_k\overset{\circ}{S}_{-k} \cup \overset{\circ}{S}_{-k}\overset{\circ}{S}_k)\right) = \mathrm{gp}\left(\bigcup (\overset{\circ}{S}_k\overset{\circ}{S}_{-k}\right) = \mathrm{gp}\left(\bigcup (\overset{\circ}{S}_{-k}\overset{\circ}{S}_k\right).$$

By assumption, there exists a compact neighborhood K of the identity in N_μ such that $V_{\mathrm{gp}(K)}(m) \geq Cm^D$. We have clearly that

$$K \subset \bigcup_{n\geq 1} \mathrm{gp}\left(\overset{\circ}{S}_n\overset{\circ}{S}_{-n}\right).$$

Since the sets $\mathrm{gp}(\bigcup_{k=1}^n \overset{\circ}{S}_k\overset{\circ}{S}_{-k})$ are open and K is compact and by the monotonicity of $(\bigcup_{k=1}^n \overset{\circ}{S}_k\overset{\circ}{S}_{-k})$, we can find some $n \geq 1$ such that $K \subset \mathrm{gp}(\overset{\circ}{S}_n\overset{\circ}{S}_{-n})$. Taking n larger if necessary we may assume also that $K \subset \mathrm{gp}(\overset{\circ}{S}_{-n}\overset{\circ}{S}_n)$. Then we can apply Theorem 3.9 with $A = \overset{\circ}{S}_n\overset{\circ}{S}_{-n}$ and $B = \overset{\circ}{S}_{-n}\overset{\circ}{S}_n$ to get the desired result.

Case III. G *has exponential growth and* $\overset{\circ}{S}_m \cap \overset{\circ}{S}_n = \emptyset$ *for all* $m \neq n$.
Now define $\nu := 1/2(\delta_{\mathbf{e}} + \mu)$. Then ν is an adapted spread out measure. It is also strictly aperiodic, i.e., $N_\nu = G$. It satisfies clearly the assumption of the case 1 of the theorem. Let $K \subset G$ be a compact set. Since G has exponential growth, for every $D > 0$, there exist $C > 0$, such that

$$\sup_{g\in G} \nu^{*n}(gK) \leq Cn^{-D} \qquad \forall n \geq 1.$$

We have

$$\nu^{*2n}(gK) = \sum_{k=0}^{2n} \frac{2n!}{k!(2n-k)!}\frac{1}{2^{2n}}\mu^{*k}(gK) \geq \frac{2n!}{(n!)^2}\frac{1}{2^{2n}}\mu^{*n}(gK).$$

By the Stirling formula ($n! \sim (n/e)^n\sqrt{2n\pi}$) we obtain

$$\sup_{g\in G} \mu^{*n}(gK) \leq C'n^{D-1},$$

which proves the result. □

Remarks 3.2. Let m be as in Case I. Then it can be proved that G/N_μ is a subgroup of $\mathbf{Z}/m\mathbf{Z}$. In particular, N_μ is of finite index in G and hence by Proposition 1 of [R2], G and N_μ have same growth. Hence, in both Case I and Case III, the rate of decay is actually related to the growth of G itself.

Remarks 3.3. When μ is absolutely continuous with bounded density F, it can be shown that the sup norm of F^{*n} decreases as $n^{-D/2}$. The only difficult part is Case III. The proof is the same as for the discrete case of [R2] using the fact that in Case III, the convolution powers of μ are mutually singular.

We say that the measure μ is *almost aperiodic* if G/N_μ is compact. The following corollary extends Theorem 1 of [R1], which was concerned with discrete groups.

Corollary 3.11. *Let G be a unimodular locally compact group, such that $V_G(m) \geq Cm^D$. Let μ be an almost aperiodic (for instance let μ be irreducible) spread out probability measure on G. Then for every compact $K \subset G$, $\sup_{g\in G} \mu^{*k}(gK) = O(k^{-D/2})$.*

Proof. By assumption, N_μ is of finite index in G. So, as in the remark below Theorem 3.10, G and N_μ have same growth and the result follows from Theorem 3.10. □

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