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SHARP L^p ESTIMATES FOR SOME OSCILLATORY INTEGRAL OPERATORS IN $\mathbb{R}^1$

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ABSTRACT. We give sharp endpoint estimates for the decay rates of L^p operator norms of oscillatory integral operators with some real homogeneous polynomial phases.

1. Introduction

In this paper we consider oscillatory integral operators T_λ in $\mathbb{R}$ defined by

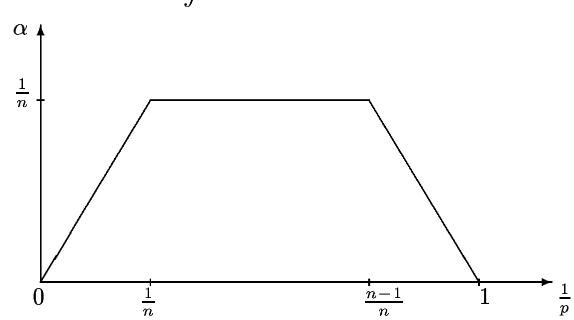
$$T_\lambda f(x) = \int e^{i\lambda S(x,y)} f(y)\chi(x,y)dy,$$

where $x, y \in \mathbb{R}$, S is a real homogeneous polynomial of the form

$$S(x,y) = \sum_{i=0}^{n} a_i x^{n-i} y^i \tag{1.1}$$

with $a_1 \neq 0$ and $a_{n-1} \neq 0$, and χ is a smooth cut-off function supported in a small neighborhood of the origin. These operators are related to averaging operators $\mathcal{R}$ in the plane defined by

$$\mathcal{R}f(x,t) = \int f(y, t + S(x,y))\chi(x,t,y)dy.$$



Figure

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Phong and Stein [PS] obtained L^p regularity and $L^p - L^q$ estimates for $\mathcal{R}$, but not endpoint estimates, when S is a homogeneous polynomial. Strong endpoint results of L^p regularity for $\mathcal{R}$ are not known. It is known that such estimates break down in translation invariant cases [Ch]. However there have been strong endpoint results for $L^p - L^q$ estimates of $\mathcal{R}$ and decay rate estimates of the L^p operator norm of T_λ. Some endpoint $L^p - L^q$ estimates have been obtained in [B], [BOS]. When S is smooth and T_λ has two-sided Whitney fold, Greenleaf and Seeger [GS] obtained endpoint estimates for the decay rate of the L^p operator norm of T_λ. In this paper, we shall give endpoint estimates for decay rate of the L^p operator norm of T_λ when S is of the form (1.1). More precisely, we shall prove:

THEOREM 1.1. *If S is of the form* (1.1) *and $n \geq 2$, then T_λ is bounded on $L^n(\mathbb{R})$ and $L^{n/(n-1)}(\mathbb{R})$ with operator norm $O(|\lambda|^{-1/n})$ as $\lambda \to \infty$.*

REMARK 1.2. (1) If $n = 1$, then $S(x, y) = a_0 x + a_1 y$ and one cannot expect any decay for $\|T_\lambda\|_{L^1 \to L^1}$. Actually in this case $T_\lambda f$ can be written as

$$T_\lambda f(x) = e^{ia_0 \lambda x} \int e^{ia_1 \lambda y} f(y) \chi(x, y) dy.$$

If we set $f(y) = e^{-ia_1 \lambda} \chi_{[0,\epsilon]}$ with ϵ small, then it is easy to see that $\|T_\lambda\|_{L^1 \to L^1} = O(1)$. If $n = 2$, the L^2 estimate in [PS] implies Theorem 1.1. Therefore we are interested in the case $n \geq 3$.

(2) Without loss of generality we may assume that $a_n = 0$ in (1.1). If we set

$$\widetilde{S}(x, y) = \sum_{i=0}^{n-1} a_i x^{n-i} y^i,$$
$$\widetilde{T}_\lambda g(x) = \int e^{i\lambda \widetilde{S}(x,y)} g(y) \chi(x, y) dy,$$

and $\tilde{f}(y) = f(y) e^{i\lambda a_n y^n}$, then it is immediate from the definition that $T_\lambda f = \widetilde{T}_\lambda \tilde{f}$. By using the fact $\|f\|_p = \|\tilde{f}\|_p$, we can easily see that $\|T_\lambda\|_{L^p \to L^p} = \|\widetilde{T}_\lambda\|_{L^p \to L^p}$. Therefore we assume that $a_n = 0$ in (1.1) throughout this paper.

(3) This result is sharp because the region in the figure gives the optimal relation between $1/p$ and α, where α is the maximal decay rate of the L^p operator norm of T_λ. See Remark 2.6 below.

To prove Theorem 1.1 we shall consider oscillatory integral operators with factors, $1/|S''_{xy}|^{-1/(n-2)}$ and $|S''_{xy}|^{1/2}$, and use complex interpolation. For the first operator we shall obtain $H^1 - L^1$ boundedness without any decay rate and for the second operator we use the $L^2 \to L^2$ bounds of Phong and Stein [PS]. To get an $H^1 - L^1$ bound we develop the method of Pan [P], but since

$1/|S''_{xy}(x,y)|^{-1/(n-2)}$ is not a singular kernel, we use the standard H^1 space rather than a modified one.

DEFINITION 1.3. (1) Let I be a bounded interval with center x_I. An atom is a function a satisfying

$$\text{supp}(a) \subset I, \tag{1.2}$$

$$|a(x)| \le \frac{1}{|I|}, \tag{1.3}$$

$$\int_I a(y)dy = 0. \tag{1.4}$$

(2) The space H^1 is the subspace of L^1 of functions f which can be written as $f = \sum_j \alpha_j a_j$, where the a_j's are atoms and $\alpha_j \in \mathbb{C}$ with $\sum_j |\alpha_j| < \infty$ and the norm $\|\cdot\|_{H^1}$ is defined by

$$\|f\|_{H^1} = \inf \sum_j |\alpha_j|,$$

where the infimum is taken over all decompositions $f = \sum_j \alpha_j a_j$.

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2. Proof of Theorem 1.1

When $S''_{xy}(x,y) = C(y-bx)^{n-2}$, the argument of Greenleaf and Seeger in [GS] can be directly applied. Therefore it suffices to deal with the complementary case. In what follows we assume that $n \ge 4$ and that $S''_{xy}(x,y) = 0$ has at least two distinct real roots or one complex root. Now we consider an analytic family of operators $T_{\lambda,\alpha}$ defined by

$$T_{\lambda,\alpha} f(x) = \int_{\mathbb{R}} e^{i\lambda S(x,y)} |S''_{xy}(x,y)|^{\alpha} \chi(x,y) f(y) dy. \tag{2.1}$$

When $\Re\alpha = 1/2$, we know that $T_{\lambda,\alpha}$ is bounded on $L^2(\mathbb{R})$ with a norm $O((1+|\Im\alpha|)\lambda^{-1/2})$ as $\lambda \to \infty$ [PS]. Therefore, by using complex interpolation and the duality argument, the $H^1 - L^1$ boundedness of $T_{\lambda,\alpha}$ with $\Re\alpha = -1/(n-1)$ implies Theorem 1.1. The remaining part of this section is devoted to proving the following lemma.

LEMMA 2.1. *If S is a homogeneous polynomial of the form* (1.1) *and S is not of the form $S(x,y) = a(y-bx)^n$, then $T_{\lambda,\alpha}$ is bounded from $H^1(\mathbb{R})$ to $L^1(\mathbb{R})$ with operator norm, $O((1+|\Im\alpha|))$, when $\Re\alpha = -1/(n-2)$.*

Proof. Throughout the proof, we shall assume $\alpha = -1/(n-2)$. When α is a complex number with $\Re\alpha = -1/(n-2)$, the factor $(1+|\Im\alpha|)$ will arise only when we apply the mean value theorem in (2.6) and (2.7) below. We shall need the following lemmas.

LEMMA 2.2. *If S is as in Lemma 2.1, then $T_{\lambda,\alpha}$ is bounded on $L^p(\mathbb{R})$ for $1 < p < \infty$.*

Proof. By homogeneity, $|S''_{xy}(x,y)| = |x|^{n-2}|S''_{xy}(1,y/x)|$. Thus by using a change of variables and Minkowski's inequality, we obtain

$$\begin{aligned}\|T_{\lambda,\alpha}f\|_{L^p} &\le \left[\int\left|\int \frac{f(y)}{|S''_{xy}(x,y)|^{1/(n-2)}}dy\right|^p dx\right]^{1/p}\\ &\le \left[\int\left|\int \frac{f(xy)}{|S''_{xy}(1,y)|^{1/(n-2)}}dy\right|^p dx\right]^{1/p}\\ &\le \|f\|_{L^p}\int \frac{y^{-1/p}}{|S''_{xy}(1,y)|^{1/(n-2)}}dy \le C\|f\|_{L^p}. \qquad \square\end{aligned}$$

LEMMA 2.3. *Let $\phi(x)$ be a real valued polynomial of degree k and ψ be a smooth cut-off function. Then*

$$\left|\int e^{i\phi(x)}\psi(x)dx\right| \le C|b_k|^{-1/k}(\|\psi\|_{L^\infty} + \|\nabla\psi\|_{L^1}),$$

where b_k is the coefficient of x^k in ϕ.

See Stein [St] for the proof of Lemma 2.3.

LEMMA 2.4. *Suppose $\phi(x)$ is same as in Lemma 2.3 and $\epsilon < 1/k$. Then*

$$\int_{|x|\le 1} |\phi(x)|^{-\epsilon}dx \le A_\epsilon\left(\sum_{j=0}^{k}|b_j|\right)^{-\epsilon},$$

where b_j is the coefficient of x^j in ϕ.

See Ricci and Stein [RS] for the proof of Lemma 2.4.

Proof of Lemma 2.1 continued. By the atomic decomposition, it suffices to prove that for any atom a as in (1.2), (1.3), and (1.4)

$$\int_{\mathbb{R}} |T_{\lambda,\alpha}a(x)|dx \le C, \tag{2.2}$$

where C is a constant which is independent of a. We choose an atom a supported in $I = [-\delta + x_I, \delta + x_I]$ and define T^P as

$$T^P f(x) = \int e^{iP(x,y)} K(x,y) f(y) dy,$$

where P is any homogeneous polynomial of degree n and

$$K(x,y) = |S''_{xy}(x,y)|^{-1/(n-2)} \chi(x,y).$$

It suffices to prove that

$$\int |T^P a(x)| dx \leq C, \tag{2.3}$$

where C is a constant independent of a and the coefficients of P. We note that for this proof P is unrelated to S, but in our application of (2.3) $\lambda S = P$. For the sake of convenience we assume that $x_I > 0$. We set

$$P(x,y) = \sum_{j=0}^{l} b_j x^{n-j} y^j, \tag{2.4}$$

where $b_l \neq 0$ and factorize S''_{xy} as

$$S''_{xy}(x,y) = \prod_{j=1}^{s} (x - \beta_j y)^{m_j} \prod_{i=1}^{r} Q_j(x,y), \tag{2.5}$$

where the β_j's are real with $|\beta_1| < \cdots < |\beta_s|$ and the Q_j's are irreducible quadratic polynomials. We may assume that $\beta_s > 0$ and $\beta_s = \max_{1 \leq i \leq s} |\beta_i|$. To prove (2.3) we use the induction on $l \leq n-1$ (see Remark 1.2 above), the degree of y in P. First we show:

Lemma 2.5. *If $P(x,y) = b_0 x^n$, that is, $l = 0$, then* (2.3) *is true.*

Proof. If $l = 0$ in (2.4), we can pull out $e^{ib_0 x^n}$ to see that $T^P f(x) = e^{ib_0 x^n} T^0 f(x)$. We consider two cases: $x_I \leq 2\delta$ and $x_I \geq 2\delta$.

Case I. $x_I \leq 2\delta$.

We define $M = 4 \max\{\beta_s, 1\}$ and split the integral on the left-hand side of (2.2) as follows:

$$\begin{aligned}\int_{\mathbb{R}} |T^0 a(x)| dx &= \int_{|x| \leq M\delta} |T^0 a(x)| dx + \int_{|x| \geq M\delta} |T^0 a(x)| dx \\ &= I_1 + I_2.\end{aligned}$$

Using Lemma 2.2 and Hölder's inequality we have

$$I_1 = \int_{|x| \leq M\delta} |T^0 a(x)| dx \leq (2M\delta)^{1/2} \|T^0 a\|_{L^2} \leq M^{1/2}.$$

To treat I_2, we observe that since $-\delta + x_I \le y \le \delta + x_I$ and $x_I \le 2\delta$, $-\delta \le y \le 3\delta$ and that if $|x| > M\delta$, then

$$|K(x,y) - K(x,0)| \le C\frac{|y|}{|x|^2}. \tag{2.6}$$

We then have

$$\begin{aligned} I_2 &= \int_{|x|\ge M\delta} \left| \int K(x,y)a(y)dy \right| dx \\ &= \int_{|x|\ge M\delta} \left| \int (K(x,y) - K(x,0))a(y)dy \right| dx \\ &\le C \int_{|x|\ge M\delta} \frac{1}{|x|^2} \int_{|y-x_I|\le\delta} |y||a(y)|dydx \le C. \end{aligned}$$

Case II. $x_I \ge 2\delta$.

We again split up the integral in (2.2):

$$\begin{aligned} \int_{\mathbb{R}} |T^0 a(x)|dx &= \int_{|x|\le Mx_I} |T^0 a(x)|dx \\ &\quad + \int_{|x|\ge Mx_I} |T^0 a(x)|dx = I_3 + I_4. \end{aligned}$$

To show that I_3 is bounded, it suffices to prove that the integral of K in x over the interval $[-Mx_I, Mx_I]$ is bounded by a constant which is independent of x_I and δ. Since $x_I \ge 2\delta$ and $x_I - \delta \le y \le x_I + \delta$, $x_I/2 \le y \le 3x_I/2$. Therefore

$$\begin{aligned} \int_{-Mx_I}^{Mx_I} K(x,y)dx &\le C \int_{-Mx_I}^{Mx_I} \frac{|S''_{xy}(x/y,1)|^{-1/(n-2)}}{y} dx \\ &\le C \int_{-2M}^{2M} |S''_{xy}(x,1)|^{-1/(n-2)} dx \le C. \end{aligned}$$

If $|x| \ge Mx_I$, then

$$|K(x,y) - K(x,x_I)| \le \frac{C|y-x_I|}{|x|^2}. \tag{2.7}$$

For I_4 we get

$$\begin{aligned} I_4 &= \int_{|x|\ge Mx_I} \left| \int K(x,y)a(y)dy \right| dx \\ &= \int_{|x|\ge Mx_I} \left| \int (K(x,y) - K(x,x_I))a(y)dy \right| dx \\ &\le C \int_{|x|\ge Mx_I} \frac{1}{|x|^2} \int_{|y-x_I|\le\delta} |y-x_I||a(y)|dydx \le C. \end{aligned}$$

This completes the proof of Lemma 2.5. □

We turn to the proof of Lemma 2.1. We assume that (2.2) is true if the degree of P in y is less than l and treat the case where the degree is l. As in the proof of Lemma 2.5 we consider two cases: $x_I \le 2\delta$, $x_I \ge 2\delta$.

Case I. $x_I \le 2\delta$.

We split the integral on the left-hand side of (2.2) as follows:

$$\begin{aligned}\int_{\mathbb{R}} |T^P a(x)|dx &= \int_{|x|\le M\delta} |T^P a(x)|dx + \int_{|x|\ge M\delta} |T^P a(x)|dx \\ &= I_5 + I_6.\end{aligned}$$

The treatment of I_5 is same to that of I_1. We split I_6 as

$$I_6 = \int_{M\delta\le|x|\le r} |T^P a(x)|dx + \int_{|x|>\max\{M\delta,r\}} |T^P a(x)|dx = I_7 + I_8.$$

To obtain estimates for I_7 and I_8 we observe that

$$K(x,y) \le \frac{C}{|x|} \tag{2.8}$$

and that (2.6) holds. Now, letting $Q(x,y) := \sum_{j=0}^{l-1} b_j x^{n-j} y^j$, we obtain

$$\begin{aligned} I_7 &\le \int_{M\delta\le|x|<r} \left| \int (e^{iP(x,y)} - e^{iQ(x,y)}) K(x,y)a(y)dy \right| dx \\ &\quad + \int_{M\delta\le|x|<r} \left| \int e^{iQ(x,y)} K(x,y)a(y)dy \right| dx \\ &\le C + C\int_{|x|<r} |b_l||x|^{n-l-1}dx \le C + C|b_l|r^{n-l} \end{aligned}$$

by the induction hypothesis. If we set $r = |b_l|^{-1/(n-l)}$, then I_7 is bounded by a constant. We split I_8 as

$$\begin{aligned} I_8 &\le \int_{|x|>\max\{M\delta,r\}} \int |K(x,y) - K(x,0)||a(y)|dydx \\ &\quad + \int_{|x|>\max\{M\delta,r\}} |K(x,0)| \left| \int e^{i\lambda P(x,y)} a(y)dy \right| dx = I_9 + I_{10}. \end{aligned}$$

We use (2.6) to obtain

$$I_9 \le \int_{|x|>M\delta} \frac{1}{|x|^2} \int_{x_I-\delta}^{x_I+\delta} |y||a(y)|dydx \le C.$$

Now it remains to prove that I_{10} is bounded by a constant independent of a and the coefficients of P. Let

$$R_j = \{x \in \mathbb{R} : 2^j \le |x| < 2^{j+1}\},$$

for $j \geq 0$, and let χ_j be the characteristic function of R_j and φ be a smooth cut-off function such that $\varphi(x) = 1$ for $|x| \leq 1$ and $\varphi(x) = 0$ for $|x| \geq 2$. We define T_j^P by

$$T_j^P f(x) = \chi_j(x) \int e^{iP(x,y)} f(y)dy.$$

The kernel L_j of $T_j^P T_j^{P*}$ is of the form

$$L_j(x,z) = \chi_j(x)\chi_j(z) \int e^{i(P(x,y)-P(z,y))} |\varphi(y)|^2 dy.$$

We write

$$P(x,y) - P(z,y) = b_l(x^{n-l} - z^{n-l})y^l + Q_1(x,y,z),$$

where Q is a polynomial in which the degree of y is less than l. Lemma 2.3 and Lemma 2.4 imply

$$\begin{aligned}\sup_z \int |2^j L_j(2^j x, 2^j z)|dx &\leq C2^j \sup_z \left(|b_l|2^{(n-l)j} + \left|b_l z 2^{(n-l)j}\right|\right)^{-1/(Nl)} \\ &\leq C2^j |b_l|^{-1/(Nl)} 2^{-j(n-l)/(Nl)}\end{aligned}$$

This estimate together with a similar estimate for $\sup_x \int |2^j L_j(2^j x, 2^j z)|dz$ yields

$$\|T_j^P\|_{L^2\to L^2} \leq C2^{j/2}|b_l|^{-1/(2Nl)} 2^{-j(n-l)/(2Nl)}.$$

Now for I_{10} we obtain

$$\begin{aligned}I_{10} &\leq C \int_{|x|>\max\{M\delta, r\}} \frac{1}{|x|} \left| \int e^{iP(x,y)} a(y)dy \right| dx \\ &\leq C \sum_{j\geq j_0} \int_{2^j \leq |x| \leq 2^{j+1}} \frac{1}{|x|} |T_j^P(a)(x)|dx \\ &\leq C \sum_{j\geq j_0} \left(\int_{2^j\leq|x|\leq 2^{j+1}} \frac{1}{|x|^2} dx \right)^{1/2} \|T_j(a)\|_{L^2} \\ &\leq C \sum_{j\geq j_0} 2^{-j/2} 2^{j/2} |b_l|^{-1/(2Nl)} 2^{-j(n-l)/(2Nl)} \leq C\end{aligned}$$

because $2^{j_0+1} \geq |b_l|^{-1/(n-l)}$.

Case II. $x_I \geq 2\delta$.

In this case we use x_I to split the integral in (2.2) as

$$\begin{aligned}\int_{\mathbb{R}} |T^P a(x)|dx &= \int_{|x|\leq Mx_I} |T^P a(x)|dx \\ &\quad + \int_{|x|\geq Mx_I} |T^P a(x)|dx = I_{11} + I_{12}.\end{aligned}$$

The treatment of I_{11} is same as that of I_3. Thus it remains to show that I_{12} is bounded by a constant independent of a. To do this, we observe that since $x_I/2 \leq y \leq 3x_I/2$ and $|x| \geq Mx_I$,

$$|K(x, x_I)| \leq \frac{C}{|x|}, \tag{2.9}$$

and (2.7) holds. Now it is easy to check that the procedure used in dealing with I_6 can be applied to get the desired results. □

REMARK 2.6. (1) Now we shall give examples which show that Theorem 1.1 cannot be improved. Suppose that T_λ is bounded on L^p with operator norm $O(\lambda^{-\alpha})$. We define f_λ^1 and g_λ^1 by

$$f_\lambda^1(y) = \begin{cases} e^{-i\lambda S(0,y)} & \text{if} \quad c_1\lambda^{-1/n} \leq y \leq c_2\lambda^{-1/n}, \\ 0 & \text{otherwise}, \end{cases}$$

and

$$g_\lambda^1(x) = \begin{cases} e^{-i\lambda S(x,0)} & \text{if} \quad c_1\lambda^{-1/n} \leq x \leq c_2\lambda^{-1/n}, \\ 0 & \text{otherwise}. \end{cases}$$

In the above definitions of f_λ^1 and g_λ^1, the values $e^{-i\lambda S(0,y)}$ and $e^{-i\lambda S(x,0)}$ can be replaced with 1 because we assume that $S(x, 0)$ and $S(0, y)$ are monomials of degree n. We use these values to stress that pure x and y powers in $S(x, y)$ do not affect the decay of the operator norm of T_λ. If x and y are in the supports of g_λ^1 and f_λ^1, respectively, then

$$|S(x, y) - S(x, 0) - S(0, y)| = \left|\sum_{i=1}^{n-1} a_i x^{n-i} y^i\right| \leq \sum_{i=1}^{n-1} |a_i| c_2^n \lambda^{-1}.$$

If we choose $c_2 > c_1 > 0$ small enough to have

$$\lambda|S(x, y) - S(x, 0) - S(0, y)| \leq \frac{\pi}{4} \tag{2.10}$$

in the support of f_λ^1 and g_λ^1, then we obtain

$$\begin{aligned} \left|\int (T_\lambda f_\lambda^1)(x) g_\lambda^1(x) dx\right| &= \left| \iint\limits_{c_1\lambda^{-1/n} \leq x, y \leq c_2\lambda^{-1/n}} e^{i\lambda(S(x,y)-S(x,0)-S(0,y))} dx dy \right| \\ &\geq C\lambda^{-2/n}. \end{aligned}$$

Since $\|f\|_{L^p} \approx \lambda^{-1/np}$ and $\|g\|_{L^{p'}} \approx \lambda^{-1/np'}$, where p' is the Hölder conjugate of p, we have

$$\|T_\lambda\|_{L^p \to L^p} \geq O(\lambda^{-1/n}),$$

and this implies that $\alpha \le 1/n$. Next, we define f_λ^2 and g_λ^2 by

$$f_\lambda^2(y) = \begin{cases} e^{-i\lambda S(0,y)} & \text{if} \quad \lambda^{-1} \le y \le 2\lambda^{-1}, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$g_\lambda^2(x) = \begin{cases} e^{-i\lambda S(x,0)} & \text{if} \quad c_1 \le x \le c_2, \\ 0 & \text{otherwise.} \end{cases}$$

If x and y are in the supports of g_λ^2 and f_λ^2, respectively, then

$$|S(x,y) - S(x,0) - S(0,y)| = \left| \sum_{i=1}^{n-1} a_i x^{n-i} y^i \right| \le \sum_{i=1}^{n-1} |a_i| c_2^{n-i} \lambda^{-i}.$$

If we take $c_2 > c_1 > 0$ sufficiently small so that (2.10) holds in the supports of f_λ^2 and g_λ^2, then we obtain the relation $\alpha \le 1 - 1/p$. By exchanging the roles of f_λ^2 and g_λ^2, we also have $\alpha \le 1/p$. Therefore $(1/p, \alpha)$ must be in the region $\mathcal{A}$ defined by

$$\mathcal{A} = \{(a,b) \in [0,1] \times \mathbb{R} \mid b \le 1/n,\ b \le a,\ \text{and}\ b \le 1 - a\},$$

which is the same region as in the figure. Therefore Theorem 1.1 is a sharp result.

(2) The complex interpolation of Theorem 1.1 with [PS] yields sharp L^p estimates for damped oscillatory integral operators T_λ^γ defined by

$$T_\lambda^\gamma f(x) = \int e^{i\lambda S(x,y)} |S''_{xy}(x,y)|^\gamma \chi(x,y) f(y) dy,$$

where $0 \le \gamma \le 1/2$. It would be interesting to understand mapping properties of oscillatory integral operators with weights $|g|^\gamma$ which are not related to S''_{xy}. Some work in this direction has been done by M. Pramanik [Pr].

References

[B] J.-G. Bak, *An L^p-L^q estimate for Radon transforms associated to polynomials*, Duke Math. J. **101** (2000), 259–269. MR 1738178 (2001b:42012)

[BOS] J.-G. Bak, D. M. Oberlin, and A. Seeger, *Two endpoint bounds for generalized Radon transforms in the plane*, Rev. Mat. Iberoamericana **18** (2002), 231–247. MR 1924693 (2003h:44002)

[Ch] M. Christ, *Failure of an endpoint estimate for integrals along curves*, Fourier analysis and partial differential equations (Miraflores de la Sierra, 1992), Stud. Adv. Math., CRC, Boca Raton, FL, 1995, pp. 163–168. MR 1330238 (97e:44007)

[GS] A. Greenleaf and A. Seeger, *On oscillatory integral operators with folding canonical relations*, Studia Math. **132** (1999), 125–139. MR 1669698 (2000g:58040)

[P] Y. Pan, *Hardy spaces and oscillatory singular integrals*, Rev. Mat. Iberoamericana **7** (1991), 55–64. MR 1109480 (92e:42009)

[PS] D. H. Phong and E. M. Stein, *Models of degenerate Fourier integral operators and Radon transforms*, Ann. of Math. (2) **140** (1994), 703–722. MR 1307901 (96c:35206)

[Pr] M. Pramanik, *Convergence of two-dimensional weighted integrals*, Trans. Amer. Math. Soc. **354** (2002), 1651–1665 (electronic). MR 1873022 (2003a:41033)

[RS] F. Ricci and E. M. Stein, *Harmonic analysis on nilpotent groups and singular integrals. I. Oscillatory integrals*, J. Funct. Anal. **73** (1987), 179–194. MR 890662 (88g:42023)

[S1] A. Seeger, *Degenerate Fourier integral operators in the plane*, Duke Math. J. **71** (1993), 685–745. MR 1240601 (94h:35292)

[S2] ———, *Radon transforms and finite type conditions*, J. Amer. Math. Soc. **11** (1998), 869–897. MR 1623430 (99f:58202)

[St] E. M. Stein, *Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals*, Princeton Mathematical Series, vol. 43, Princeton University Press, Princeton, NJ, 1993. MR 1232192 (95c:42002)

[Y] C. W. Yang, L^p *improving estimates for some classes of Radon transforms*, Trans. Amer. Math. Soc., to appear.

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