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WEIGHTED L^2 ESTIMATES FOR MAXIMAL OPERATORS ASSOCIATED TO DISPERSIVE EQUATIONS

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ABSTRACT. Let $Tf(x,t) = e^{2\pi it\phi(D)}f(x)$ be the solution of the general dispersive equation with phase ϕ and initial data f in the Sobolev space H^s. We prove a weighted L^2 estimate for the global maximal operator T^{**} defined by taking the supremum over the time variable $t \in \mathbb{R}$ so that $\|T^{**}f\|_{L^2(w\,dx)} \le C\|f\|_{H^s}$. The exponent s depends on the phase function ϕ, whose gradient may vanish or have singularities.

1. Introduction

The general dispersive equation with initial data f in the Schwartz class $\mathcal{S}(\mathbb{R}^n)$ $(n \ge 2)$ is

$$iu_t(x) = -2\pi\phi(D)u(x), \quad u(x,0) = f(x) \quad \text{on} \quad \mathbb{R}^n \times \mathbb{R},$$

where $D = \frac{1}{2\pi i}\nabla$ and ϕ is a measurable phase function. The formal solution of this equation is $u(x,t) = \int e^{2\pi i(x\cdot\xi + t\phi(\xi))}\widehat{f}(\xi)\,d\xi$, where $\widehat{f}(\xi) = \int e^{-2\pi ix\cdot\xi}f(x)\,dx$. Let us define an operator T by $Tf(x,t) = u(x,t)$. The corresponding maximal operators are

$$T_N^* f(x) := \sup_{-N\le t\le N}|Tf(x,t)|, \quad T^{**}f(x) := \sup_{t\in\mathbb{R}}|Tf(x,t)|.$$

The purpose of this paper is to study the mapping properties of T^{**} from the inhomogeneous Sobolev space H^s to $L^2(wdx)$ for some nonnegative integrable function w, i.e., study bounds of the form

$$\|T^{**}f\|_{L^2(wdx)} \le C\|f\|_{H^s}, \tag{1.1}$$

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where the constant C is independent of f. Here, the Sobolev space H^s is defined by the norm

$$\|f\|_{H^s} \equiv \|f\|_{L^2} + \left(\sum_{j\in\mathbb{Z}} 2^{2sj}\|\Delta_j f\|_{L^2}^2\right)^{1/2} \sim \left(\int (1+|\xi|)^{2s}|\widehat{f}(\xi)|^2\,d\xi\right)^{1/2},$$

where $\widehat{\Delta_j f}(\xi) = \varphi_j(\xi)\widehat{f}(\xi)$ for some Littlewood-Paley function φ_j (see [1]).

The homogeneity and regularity of the phase ϕ play a crucial role in obtaining a small value for the exponent s. If $\det(\frac{\partial^2\phi}{\partial\xi_i\partial\xi_j}) \neq 0$, then it is expected that one can take $s = 1/4$ for some functions w. In fact, if f is a function of a single variable, or a radial function multiplied by spherical harmonics, then s can attain the critical value $1/4$ with the weight $|x|^{-1/2}$ (see Theorem 1.1 in [4]). For other results related to T_N^* and T^{**}, see [2], [3], [5], [6], [9]–[13], and [4], [7], [14], [15]. In particular, P. Sjölin [11] showed that the maximal operator T^{**} defined with $\phi = |\xi|^a$ $(a > 1)$ does not satisfy the global L^2 boundedness. Thus, in order to obtain a global L^2 estimate, we have to employ an appropriate weight w. L. Vega [15] studied global estimates with phase $\phi(\xi) = |\xi|^a$ $(a > 1)$ and weight $w(x) = (1+|x|)^{-b_1}$ $(b_1 > 1)$. Later, H. P. Heinig and S. Wang [7] used homogeneous phase functions whose gradients may have zeros and weights of the form $w(x) = |x|^{-b_1}(1+|x|)^{-b_2}$ $(1 < b_1 < n,\ \ b_1 + b_2 > n)$. They obtained the estimate (1.1) for $s > b_1/2$.

In this paper, we consider a wider class of phase functions, whose gradient may not exist, or may have zeros and singularities. To be specific, we assume that their zeros and singularities on the unit sphere are of *regular type*. We define regular zeros as in [7]:

Definition 1.1. Let ψ be a continuous function on S^{n-1} such that $\psi(\xi_0') = 0$ for some point $\xi_0' \in S^{n-1}$. If $\angle(\xi', \xi_0')$ is the angle between ξ_0' and ξ', then ξ_0' is called a regular zero of order α^*, provided for all $\alpha < \alpha^*$,

$$\lim_{\xi'\to\xi_0'} \frac{|\psi(\xi')|}{(\angle(\xi',\xi_0'))^\alpha} = 0.$$

Similarly, we define regular singularities:

Definition 1.2. Let ψ be a continuous function except for a finite subset S of unit sphere. If $\xi_0' \in S$, then ξ_0' is called a regular type singularity of order β^*, if for all $\beta > \beta^*$,

$$\lim_{\xi'\to\xi_0'} (\angle(\xi',\xi_0'))^\beta |\psi(\xi')| = 0.$$

When $\alpha^* = 0$ (resp. $\beta^* = 0$), we mean that ψ has no zero (resp. singularity).

Now let us assume:

(A1) ϕ has finitely many regular singularities $\xi'_1, \dots, \xi'_l \in S^{n-1}$, and each ξ'_i is of order $\beta^*_i \geq 0$. (Note that by homogeneity ϕ and $\nabla\phi$ have the same singularities of the same order.)

(A2) ϕ is continuous on the complement of $S = \{\xi'_1, \dots, \xi'_l\}$ and differentiable except for a finite subset F on S^{n-1}, which has the property that for all $\xi' \in F$ there exist positive constants c_1 and c_2 such that

$$\inf_{\angle(\eta',\xi')\leq c_1} |\nabla\phi(\eta')| \geq c_2.$$

(A3) $\nabla\phi$ has a finite regular zero set $Z = \{\eta'_1, \dots, \eta'_m\} \subset S^{n-1} \setminus (S \cup F)$, and each η'_j is of order $\alpha^*_j \geq 0$.

(A4) ϕ is differentiable and $\nabla\phi(\xi) \neq 0$ on a subset E of $\mathbb{R}^n$.

(A5) For any $(\xi_1, \dots, \xi_{k-1}, \xi_{k+1}, \dots, \xi_n) \in \mathbb{R}^{n-1}$ and any $r \in \mathbb{R}$, the equation

$$\phi(\xi_1, \dots, \xi_{k-1}, x, \xi_{k+1}, \dots, \xi_n) = r \ (0 \leq k \leq n-1)$$

has at most N_0 solutions.

The assumptions (A4)–(A5) were first proposed by C. E. Kenig, G. Ponce and L. Vega [8].

Let us consider a weight w satisfying

$$\begin{cases} w(x) = \mathcal{O}(\frac{1}{|x|}) \quad \text{as} \quad x \to 0, \\ w(x) = \mathcal{O}(\frac{1}{|x|^n(\log|x|)^\delta}) \quad \text{as} \quad x \to \infty. \end{cases} \tag{1.2}$$

for some $\delta > 1$.

Now we are ready to state our main result.

THEOREM 1.3. *Suppose ϕ is homogeneous of degree $a \in \mathbb{R}$ and satisfies* (A1)–(A3), *and* (A4)–(A5) *with E the complement of any fixed neighborhood of L which is the union of straight lines of direction $\xi' \in S \cup F \cup Z$ from the origin. Let $\lambda_0 = \max(\alpha^*_1, \dots, \alpha^*_l)$, $\lambda_s = (\beta^*_1, \dots, \beta^*_m)$. If $\lambda_0 = \lambda_s = 0$, then a is assumed to be a positive number. Let w be a nonnegative function satisfying* (1.2) *and locally bounded on $\mathbb{R}^n \setminus \{0\}$. Then for any $f \in \mathcal{S}(\mathcal{R}^n)$ the inequality* (1.1) *holds for*

(1) $s > \frac{1}{2} + \frac{(\lambda_0+\lambda_s)n}{4(n-1)}$, *if* $a \leq \frac{(\lambda_0-\lambda_s)n}{2(n-1)}$,

(2) $s > \frac{1}{2} - \frac{a}{2} + \frac{\lambda_0 n}{2(n-1)}$, *if* $\frac{(\lambda_0-\lambda_s)n}{2(n-1)} < a$,

and fails for $s < 1/2$.

If $a \geq \lambda_0 n/(n-1)$ and $a \neq 0$, then we recover the results of H. Heinig and S. Wang [7], and L. Vega [15]. However, we could not obtain sharp necessary conditions for the maximal inequalities in the presence of singularities and zeros. Our future concern is to find such conditions.

Examples. If

$$\phi = \frac{(\xi_1+\xi_2)^{m_1} + (\xi_2+\xi_3)^{m_1} + \cdots + (\xi_{n-1}+\xi_n)^{m_1}}{(\xi_1-\xi_2)^{m_2} + (\xi_2-\xi_3)^{m_2} + \cdots + (\xi_{n-1}-\xi_n)^{m_2}},$$

where m_1 and m_2 are positive integers, then ϕ has singularities at $(\pm\frac{1}{\sqrt{n}}, \dots, \pm\frac{1}{\sqrt{n}})$ of order m_2 and $\nabla\phi$ has zeros at $(\pm\frac{1}{\sqrt{n}}, \mp\frac{1}{\sqrt{n}}, \dots, \mp(-1)^n\frac{1}{\sqrt{n}})$ of order m_1-1. If $\phi = |\xi_1|^{a_1}|\xi_2|^{a_2}\dots|\xi_n|^{a_n}$ $(0 < a_i \le 1)$, then ϕ is continuous. But $\nabla\phi$ does not exist at $\pm e_i$, where e_i is a unit normal vector whose i-th component is 1, and $\nabla\phi$ has no zero. If $\phi = |\xi_1|^a|\xi_2|^b$ $(0 < a < 1,\, b > 1)$, then ϕ is continuous and $\nabla\phi$ has zeros at $(\pm1, 0)$ of order $b-1$, but $\nabla\phi$ does not exist at $(0, \pm1)$. If $\phi = \xi_1^{m_1}\xi_2^{m_2}(\xi_1-\xi_2)^{m_3}$, where m_1, m_2, m_3 are nonnegative integers, then $\nabla\phi$ has zeros at $(\pm1,0)$, $(0,\pm1)$, $(\pm\frac{1}{\sqrt{2}}, \pm\frac{1}{\sqrt{2}})$ of order m_1-1, m_2-1, and m_3-1, respectively.

If not specified, we use C to denote a positive constant that may not be the same at each occurrence, and use $A \lesssim B$ and $A \sim B$ to denote $|A| \le CB$ and $C^{-1}B \le |A| \le CB$, respectively.

2. Preliminary lemmas

In this section, we study local L^2 estimates for a maximal operator with a fixed phase function whose gradient is non-vanishing. For any open subset E of $\mathbb{R}^n$ let us define an operator T_E^{**} by

$$T_E^{**}f(x) = \sup_{t\in\mathbb{R}} \left| \int_E e^{2\pi i(x\cdot\xi + t\phi(\xi))} \widehat{f}(\xi)\, d\xi \right|.$$

Throughout this section, we assume that ϕ satisfies the assumptions (A4)–(A5) and that $\nabla\phi \ne 0$ in E. Let us define subsets D_i $(i = 1, \dots, n)$ by

$$D_i = \left\{ \xi \in E : |\partial_i\phi(\xi)| \ge \frac{1}{2n}|\nabla\phi(\xi)| \right\}.$$

Since $\nabla\phi \ne 0$, by the implicit function theorem we can find subsets $E_{i,k}$ of D_i and C^1 functions $\Psi_{i,k}$ such that

(i) $\Psi_{i,k}$ is a diffeomorphism with

$$\Psi_{i,k}(\xi_1, \dots, \xi_i, \dots, \xi_n) = (\xi_1, \dots, \phi(\xi), \dots, \xi_n) = \omega,$$

(ii) $|\det(D\Psi_{i,k})(\xi)| = \left|\frac{\partial\phi}{\partial\xi_i}(\xi)\right| \ge \frac{1}{2n}|\nabla\phi(\xi)| > 0$ for all $\xi \in E_{i,k}$,

(iii) $|E \setminus \bigcup_{i,k} E_{i,k}| = 0$ and the $E_{i,k}$'s are mutually disjoint,

(iv) $\sum_k \chi_{\Psi_{i,k}(E_{i,k})} \le N_0$.

Denoting by B_j the dyadic ball $E \cap B(0, 2^j)$ for $j \in \mathbb{Z}$, let us set

$$M_{0,j} := \sup_{\xi \in B_{j+1} \setminus B_j} |\nabla \phi(\xi)|,$$

$$M_{1,j} := \sup_{\xi \in B_{j+1} \setminus B_j} \frac{1}{|\nabla \phi(\xi)|},$$

$$M_{2,j} := \sup_{\xi \in B_{j+1} \setminus B_j} \frac{|\phi(\xi)|}{|\nabla \phi(\xi)|}.$$

If $B_{j+1} = \emptyset$, then let us set $M_{0,j} = M_{1,j} = 0$. We have the following result:

LEMMA 2.1. *If $R > 0$ and $f \in \mathcal{S}(\mathbb{R}^n)$, then the inequality*

$$\|T_E^{**} f\|_{L^2(B(0,R))} \lesssim R^{1/2} \sum_{j \in \mathbb{Z}} M_{0,j}^{1/4} M_{1,j}^{1/4} M_{2,j}^{1/2} \|\Delta_j f\|_{L^2} \tag{2.1}$$

holds for some Littlewood-Paley function φ_j with $\widehat{\Delta_j f} = \varphi_j \widehat{f}$.

Proof. We first prove the lemma when $|\phi| \geq \lambda > 0$. It is enough to show that for any large N

$$\|T_{E,N}^{*} f\|_{L^2(B(0,R))} \leq C R^{1/2} \sum_{j \in \mathbb{Z}} M_{0,j}^{1/4} M_{1,j}^{1/4} M_{2,j}^{1/2} \|\Delta_j f\|_{L^2}, \tag{2.2}$$

where $T_{E,N}^{*} f(x) := \sup_{|t|<N} |T_E f(x,t)|$, and the constant C does not depend on N and λ.

Let us choose smooth functions ψ, φ with compact support in $\mathbb{R}^n$ such that

$$\psi(x) = \begin{cases} 1 & \text{if } |x| < 1, \\ 0, & \text{if } |x| > 2, \end{cases}$$

$$\varphi(x) = \begin{cases} 1 & \text{if } 1/2 < |x| < 1, \\ 0 & \text{if } |x| < 1/4, \text{ or } |x| > 2, \end{cases}$$

$$\sum_{j \in \mathbb{Z}} \varphi\left(\frac{\cdot}{2^j}\right) \equiv \sum_{j \in \mathbb{Z}} \varphi_j(\cdot) = 1 \text{ on } \mathbb{R}^n \setminus \{0\}.$$

Also choose a one dimensional smooth function η such that $\eta(t) = 1$ if $|t| < 1$, and $\eta(t) = 0$ if $|t| > 2$. For each $j \in \mathbb{Z}$ we define an operator A_j by

$$A_j f(x,t) = \psi\left(\frac{x}{R}\right) \eta(t) \int_E e^{2\pi i (x \cdot \xi + N t \phi(\xi))} \widehat{f}(\xi) \varphi_j(\xi) d\xi. \tag{2.3}$$

For the interval $I = [-1, 1]$ let us first observe that

$$T_{E,N}^{*} f(x) \leq \sum_j \sup_I |A_j f(x,t)|. \tag{2.4}$$

We then estimate the maximal function of $A_j f$ using the following simple lemma:

LEMMA 2.2. *For any $F(t) \in C^1(J)$, $J = [a,b]$, we have*

$$\sup_J |F(t)| \le \frac{1}{|J|}\int_J |F(t)|dt + \int_J |F'(t)|dt.$$

Let us divide I into subintervals I_l with $|I_l| \sim \delta, 0 < \delta < 1$. Applying Lemma 2.2 to these subintervals I_l and using Schwartz's inequality, we get for each $x \in B(0,R)$

$$\begin{aligned}\sup_I |A_j f(x,t)|^2 &\le \sum_l \sup_{I_l} |A_j f(x,t)|^2 \\ &\le 2\sum_l \left(\frac{1}{\delta}\int_{I_l} |A_j f(x,t)|^2 dt + \delta \int_{I_l} |\frac{d}{dt}A_j f(x,t)|^2 dt\right) \\ &= 2\left(\frac{1}{\delta}\int |A_j f(x,t)|^2 dt + \delta \int |\frac{d}{dt}A_j f(x,t)|^2 dt\right)\end{aligned}$$

and hence

$$\begin{aligned}&\int_{B(0,R)} \sup_I |A_j f(x,t)|^2 dx \\ &\quad\lesssim \frac{1}{\delta}\iint |A_j f(x,t)|^2 dxdt + \delta \iint |\frac{d}{dt}A_j f(x,t)|^2 dxdt \\ &\quad= \iint \widehat{\Delta_j f}(\xi)\overline{\widehat{\Delta_j f}}(\xi')\left(\frac{1}{\delta}K_j^1(\xi,\xi') + \delta K_j^2(\xi,\xi')\right) d\xi d\xi',\end{aligned}$$

where

$$K_j^1(\xi,\xi') = \chi_j\chi_j' \iint e^{2\pi i[x\cdot(\xi-\xi')+Nt(\phi(\xi)-\phi(\xi'))]}\psi^2(\frac{x}{R})\eta^2(t)dxdt, \tag{2.5}$$

$$K_j^2(\xi,\xi') = -4\pi^2 N^2\phi(\xi)\phi(\xi')K_j^1(\xi,\xi') + \tilde{K}_j^1(\xi,\xi'), \tag{2.6}$$

$$\tilde{K}_j^1 = \chi_j\chi_j' \iint e^{2\pi i[x\cdot(\xi-\xi')+Nt(\phi(\xi)-\phi(\xi'))]}\psi^2(\frac{x}{R})(\eta^2)'(t)dxdt, \tag{2.7}$$

and $\chi_j = \chi_{B_{j+1}\setminus B_j}(\xi)$, $\chi_j' = \chi_{B_{j+1}\setminus B_j}(\xi')$.

Now we estimate the above kernels. By the Fourier transform decay of smooth functions, we have, for any positive μ and ν,

$$|K_j^1(\xi,\xi')| \lesssim R^n\chi_j\chi_j'(1+R|\xi-\xi'|)^{-\mu}(1+N|\phi(\xi)-\phi(\xi')|)^{-\nu}\ .$$

Fixing $\xi' \in B_{j+1}\setminus B_j$, we divide the region of integration into two parts, the conic neighborhood D of ξ' with angle 1 and its complement. Now, let us write

$$\begin{aligned}\int |K_j^1(\xi,\xi')|d\xi &\lesssim R^n\left(\int_{(B_{j+1}\setminus B_j)\cap D^c} + \int_{(B_{j+1}\setminus B_j)\cap D}\right) \\ &\lesssim R^n \sum_{i,k}\left(\int_{(B_{j+1}\setminus B_j)\cap D^c\cap E_{i,k}} + \int_{(B_{j+1}\setminus B_j)\cap D\cap E_{i,k}}\right).\end{aligned}$$

Let $I_j^{i,k}(\mu_1,\nu_1)+II_j^{i,k}(\mu_2,\nu_2)$ denote the summand in parentheses in this inequality. Then, in the complement of D, noting that $|\xi-\xi'|\gtrsim 2^j$, we have

$$\begin{aligned} R^{\mu_1}2^{\mu_1 j}I_j^{i,k} &\lesssim \int_{(B_{j+1}\setminus B_j)\cap E_{i,k}}(1+N|\phi(\xi)-\phi(\xi')|)^{-\nu_1}d\xi \\ &= \int_{\Psi_{i,k}((B_{j+1}\setminus B_j)\cap E_{i,k})}(1+N|\omega_i-\phi(\xi')|)^{-\nu_1}|\partial_i\phi(\Psi_{i,k}^{-1}(\omega))|^{-1}d\omega \\ &\lesssim M_{1,j}\int_{\Psi_{i,k}((B_{j+1}\setminus B_j)\cap E_{i,k})}(1+N|\omega_i-\phi(\xi')|)^{-\nu_1}d\omega\ . \end{aligned}$$

For large ν_1 it follows by direct integration that

$$\sum_k I_j^{i,k}\lesssim R^{-\mu_1}2^{-\mu_1 j}M_{1,j}\,2^{(n-1)j}\,N^{-1}. \tag{2.8}$$

For the second part, we get

$$\begin{aligned} II_j^{i,k} &\lesssim \int_{(B_{j+1}\setminus B_j)\cap D\cap E_{i,k}}(1+R|\xi-\xi'|)^{-\mu_2}(1+N|\phi(\xi)-\phi(\xi')|)^{-\nu_2}d\xi \\ &\lesssim \int_{(B_{j+1}\setminus B_j)\cap D\cap E_{i,k}}\prod_{1\le l\le n}(1+R|\xi_l-\xi_l'|)^{-\mu_2/n}(1+N|\phi(\xi)-\phi(\xi')|)^{-\nu_2}d\xi \\ &\lesssim \int_{\Psi_{i,k}((B_{j+1}\setminus B_j)\cap E_{i,k})}\prod_{l\ne i}(1+R|\omega_l-\xi_l'|)^{-\mu_2/n}(1+N|\omega_i-\phi(\xi')|)^{-\nu_1}\frac{d\omega}{|\nabla\phi(\xi)|}. \end{aligned}$$

Thus for large μ_2 and ν_2 we have $\sum_k II_j^{i,k}\lesssim R^{-(n-1)}N^{-1}M_{1,j}$. If we choose $\mu_1=n-1$ in (2.8), then

$$\sup_{\xi'\in\mathbb{R}^n}\int|K_j^1(\xi,\xi')|d\xi\lesssim RN^{-1}M_{1,j}.$$

By symmetry, we also have

$$\sup_{\xi\in\mathbb{R}^n}\int|K_j^1(\xi,\xi')|d\xi'\lesssim RN^{-1}M_{1,j}.$$

We estimate $\int|K_j^2|d\xi$ similarly. If $N>1/\lambda$, then by (2.6),

$$\sup_{\xi'}\int|K_j^2(\xi,\xi')|d\xi\lesssim RNM_{0,j}M_{2,j}^2+RN^{-1}M_{1,j}\lesssim RNM_{0,j}M_{2,j}^2.$$

Using Schur's lemma and the identity $f=\sum_k\Delta_k f$, we obtain

$$\int\sup_I|A_jf(x,t)|^2dx\lesssim\left(\frac{1}{\delta}RN^{-1}M_{1,j}+\delta RNM_{0,j}M_{2,j}^2\right)\|\Delta_jf\|_{L^2}^2\ . \tag{2.9}$$

If we choose $\delta=N^{-1}\left(M_{1,j}/M_{0,j}\right)^{1/2}M_{2,j}^{-1}$, then

$$\delta\le\sup_{B_{j+1}\setminus B_j}\frac{1}{N|\phi(\xi)|}\le\frac{1}{N\lambda}<1.$$

This implies that there exists a constant C, independent of j and λ, such that

$$\int \sup_I |A_j f(x,t)|^2 dx \le CR\, M_{0,j}^{1/2}\, M_{1,j}^{1/2} M_{2,j} \|\Delta_j f\|_{L^2}^2\ . \tag{2.10}$$

This and (2.4) yield (2.2) and thus the result for the case when $|\phi| \ge \lambda$.

For the general case, we observe that

$$T_E^{**} f(x) \le T_+^{**} f(x) + T_-^{**} f(x),$$

where

$$T_+^{**} f(x) = \sup_{t\in\mathbb{R}} \left| \int_{E\cap\{\phi\ge 0\}} e^{2\pi i (x\cdot\xi + t(\phi(\xi)+\lambda))} \widehat{f}(\xi) d\xi \right|,$$
$$T_-^{**} f(x) = \sup_{t\in\mathbb{R}} \left| \int_{E\cap\{\phi< 0\}} e^{2\pi i (x\cdot\xi + t(\phi(\xi)-\lambda))} \widehat{f}(\xi) d\xi \right|$$

for any positive constant λ. Applying the previous estimate to the phase functions $\phi \pm \lambda$ and $\Delta_j f$, we get

$$\begin{aligned}\|T_+^{**}\Delta_j f\|_{L^2(B(0,R))} &+ \|T_-^{**}\Delta_j f\|_{L^2(B(0,R))} \\ &\le CR^{1/2} \sum_{j-2\le k\le j+2} M_{0,k}^{1/4} M_{1,k}^{1/4} \left(\sup_{B_{k+1}\setminus B_k} \frac{|\phi(\xi)|+\lambda}{|\nabla\phi(\xi)|} \right)^{1/2} \|\Delta_k \Delta_j f\|_{L^2},\end{aligned}$$

where the constant C does not depend on λ. Letting $\lambda \to 0$ and summing over j, we obtain the desired result. □

Now we introduce a local version of Lemma 2.5 in [7]:

Lemma 2.3. *If $R > 0$ and $s_0 > 1/2$, then*

$$\|T_E^{**} f\|_{L^2(B(0,R))} \lesssim R^{1/2} \left(\int_E \frac{(1+|\phi(\xi)|)^{2s_0}}{|\nabla\phi(\xi)|} |\widehat{f}(\xi)|^2\, d\xi \right)^{1/2}.$$

Proof. Lemma 2.3 follows easily from Lemma 2.5 in [7] using the weight $w_R = \psi(\frac{x_1}{R}) \times \cdots \times \psi(\frac{x_n}{R})$ for some function $\psi \in C_0^\infty(\mathbb{R})$, supported in $[-3,3]$ and satisfying $\psi = 1$ on $[-2,2]$, instead of the weight $w(x) = |x|^{-b_1}(1+|x|)^{-b_2}$ used in [7]. □

3. Proof of the main result

For simplicity, let us assume that ϕ has one singularity η_1', and that $\nabla\phi$ does not exist at one point ξ_2' and has one zero at ξ_3' on the unit sphere. Let us define a subset C such that

$$C = \bigcup_{\substack{j\geq 1,\\ i=1,2,3}} C_j^i \quad \text{of} \quad \mathbb{R}^n,$$

$$C_j^i = \left\{\xi : |\xi| \sim 2^j,\ \angle(\xi', \xi_i') \lesssim \frac{1}{j^{2/(n-1)}} 2^{-nj/(n-1)},\ \xi = |\xi|\xi'\right\}.$$

Then we have

$$|C| \sim \sum_{\substack{j\geq 1,\\ i=1,2,3}} |C_j^i| \lesssim \sum_{j\geq 1} \frac{1}{j^2} \lesssim 1.$$

Next, we split the function Tf into two parts:

$$Tf(x,t) = \int_{B_0\cup C} + \int_{B_0^c\cap C^c} = T_{B_0\cup C}(x,t)f + T_{B_0^c\cap C^c}f(x,t).$$

Since, by Hölder inequality,

$$T^{**}_{B_0\cup C}(x,t)f \lesssim (|B_0| + |C|)^{1/2}\|f\|_{L^2} \lesssim \|f\|_{L^2},$$

and since w is integrable, we have

$$\|T^{**}_{B_0\cup C}(x,t)f\|_{L^2(wdx)} \lesssim \|f\|_{L^2}. \tag{3.1}$$

Applying Lemmas 2.1 and 2.3 with $E = B_0 \cap C^c$, we can estimate the second integral and obtain

$$\begin{aligned} &\|T^{**}_{B_0^c\cap C^c} f\|\|_{L^2(B(0,R))} \\ &\qquad \lesssim \begin{cases} R^{1/2}\sum_{j\geq 1} M_{0,j}^{1/4} M_{1,j}^{1/4} M_{2,j}^{1/2}\|\Delta_j f\|_{L^2}, \\ R^{1/2}\left(\int_{B_0^c\cap C^c} \frac{(1+|\phi(\xi)|)^{2s_o}}{|\nabla\phi(\xi)|}|\widehat{f}(\xi)|^2\, d\xi\right)^{1/2} \end{cases}. \end{aligned} \tag{3.2}$$

From the homogeneity of ϕ and the regularity at ξ_i', we deduce that $M_{2,j} \lesssim 2^j$,

$$|\nabla\phi(\xi)| \lesssim |\xi|^{a-1}(\angle(\xi', \xi_i'))^{-\lambda_s} \lesssim j^{\frac{2\lambda_s}{(n-1)}} 2^{(a-1+\lambda_s\frac{n}{n-1})j},$$

$$|\nabla\phi(\xi)| \gtrsim |\xi|^{a-1}(\angle(\xi', \xi_i'))^{\lambda_0} \gtrsim \frac{1}{j^{\frac{2\lambda_0}{(n-2)}}} 2^{(a-1-\lambda_0\frac{n}{n-1})j}$$

on $(B_{j+1}\setminus B_j)\cap C^c$. This implies that

$$M_{0,j}^{1/4} M_{1,j}^{1/4} M_{2,j}^{1/2} \leq C(\varepsilon) 2^{(\frac{1}{2}+\frac{(\lambda_0+\lambda_s)n}{4(n-1)}+\varepsilon)j} \tag{3.3}$$

for some small ε depending on λ_0, λ_s such that $\varepsilon = 0$ if $\lambda_0 = \lambda_s = 0$, where $C(\varepsilon)$ satisfies $C(0) < \infty$, $\lim_{\varepsilon\to 0} C(\varepsilon) = \infty$. Similarly, on $(B_{j+1}\setminus B_j)\cap C^c$ we have

$$\begin{aligned} \frac{(1+|\phi(\xi)|)^{2s_0}}{|\nabla\phi(\xi)|} &\leq \left(\frac{1}{|\nabla\phi(\xi)|} + \frac{|\phi(\xi)|}{|\nabla\phi(\xi)|}\right)(1+|\phi(\xi)|)^{2s_0-1} \\ &\leq C(\varepsilon)(2^{(1-a+\frac{\lambda_0 n}{n-1}+\varepsilon)j} + 2^j)(1+2^{(a+\frac{\lambda_s n}{n-1}+\varepsilon)j})^{2s_0-1}. \end{aligned} \tag{3.4}$$

If s is a number satisfying the conditions of the theorem, then from (3.3) and (3.4) it follows that

$$\|T^{**}_{B_0^c\cap C^c} f\|_{L^2(B(0,R))} \lesssim R^{1/2}\|f\|_{H^s}. \tag{3.5}$$

Now we are ready to prove the theorem. Employing a suitable nonnegative dyadic function $\tilde\psi$, let us write

$$\|T^{**}_{B_0^c\cap C^c} f\|^2_{L^2(w\,dx)} \sim \sum_{k\in\mathbb{Z}} \int (T^{**}_{B_0^c\cap C^c} f)^2 \tilde\psi(\frac{x}{2^k})\, w(x)\, dx.$$

Then it follows from (3.5) that for large fixed K

$$\begin{aligned}\sum_{k\ge K} \int (T^{**}_{B_0^c\cap C^c} f)^2 \tilde\psi(\frac{x}{2^k})\, w(x)\, dx &\lesssim \sum_{k\ge K} 2^{-nk}k^{-\delta} \int_{|x|\sim 2^k} (T^{**}_{B_0^c\cap C^c} f)^2 \tilde\psi(\frac{x}{2^k})\, dx\\ &\lesssim \sum_{k\ge K} 2^{-(n-1)k}k^{-\delta}\|f\|^2_{H^s} \lesssim \|f\|^2_{H^s},\end{aligned}$$

where s is as in Theorem 1.3. Since $w(x)\le C$ for all $x\in B_K\setminus B_{-K}$, from (3.5) we also get

$$\sum_{-K\le k\le K} \int (T^{**}_{B_0^c\cap C^c} f)^2 \tilde\psi(\frac{x}{2^k}) w(x)\, dx \lesssim \|f\|^2_{H^s}.$$

If $k\le -K$, then we proceed as follows. Observing that

$$\sum_{k\le -K} \int (T^{**}_{B_0^c\cap C^c} f)^2 \tilde\psi(\frac{x}{2^k}) w(x) dx \lesssim \sum_{k\le -K} 2^{-k} \int (T^{**}_{B_0^c\cap C^c} f)^2 \tilde\psi(\frac{x}{2^k})\, dx \equiv \Gamma,$$

we estimate Γ. We write

$$\Gamma^{1/2} \lesssim \sum_{j\ge 1}\sum_{k\le -K,} 2^{-k/2}\left(\int |T^{**}_{B_0^c\cap C^c}(\Delta_j f)|^2 \tilde\psi(\frac{x}{2^k}) dx\right)^{1/2} \equiv \sum_{j\ge 1}\Gamma_j^{1/2}. \tag{3.6}$$

For each $j\ge 1$, we divide $\Gamma_j^{1/2}$ into two parts,

$$\Gamma_j^{1/2} = \sum_{-j\le k\le -K} (\cdot) + \sum_{k\le -j} (\cdot).$$

For the first sum, we get from (3.5)

$$\sum_{-j\le k\le -K} (\cdot) \lesssim \sum_{-j\le k\le -K} 2^{-k/2}2^{k/2}2^{sj/2}\|\Delta_j f\|_{L^2} \lesssim 2^{(s+\varepsilon)j/2}\|\Delta_j f\|_{L^2}.$$

Since $T^{**}(\Delta_j f)(x) \lesssim 2^{nj/2}\|\Delta_j f\|_{L^2}$, we obtain for the second sum

$$\sum_{k\le -j} (\cdot) \lesssim \sum_{k\le -j} 2^{-k/2}2^{(nk/2)+(nj/2)}\|\Delta_j f\|_{L^2} \lesssim 2^{j/2}\|\Delta_j f\|_{L^2}.$$

Substituting these estimates into (3.6), we obtain the desired result.

For the proof of the remaining part, suppose that (1.1) holds for some s, and let us choose a Schwartz function f such that $\widehat{f} \geq 0$, $\operatorname{supp}(\widehat{f}) \subset B(0,1)^c$ and $\int \widehat{f}(\xi)d\xi = C > 0$. With this function f, we have $Tf(0,0) = \int \widehat{f}(\xi)d\xi = C$. Hence, for sufficiently small $r > 0$, $|Tf(x,0)| \geq C/2$ for all $x \in B(0,2r) \setminus B(0,r)$. If we use the notation $f_N(x) := f(Nx)$, then by the dilation invariance of T^{**}, we have for all large N

$$\begin{aligned} N^{2s-n} &\gtrsim \|f_N\|_{H^s}^2 \gtrsim \|T^{**}f_N\|_{L^2(w\,dx)}^2 \gtrsim \int (T^{**}f(Nx))^2 w(x)\,dx \\ &\gtrsim N^{-n}\int (T^{**}f(x))^2 w(\frac{x}{N})\,dx \gtrsim N^{1-n}\|T^{**}f\|_{L^2(\{|x|\sim r\})}^2 \gtrsim N^{1-n}. \end{aligned}$$

This implies that (1.1) fails for $s < 1/2$ and thus completes the proof of the theorem. □

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