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HENSTOCK-KURZWEIL FOURIER TRANSFORMS

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Abstract. The Fourier transform is considered as a Henstock–Kurzweil integral. Sufficient conditions are given for the existence of the Fourier transform and necessary and sufficient conditions are given for it to be continuous. The Riemann–Lebesgue lemma fails: Henstock–Kurzweil Fourier transforms can have arbitrarily large point-wise growth. Convolution and inversion theorems are established. An appendix gives sufficient conditions for interchanging repeated Henstock–Kurzweil integrals and gives an estimate on the integral of a product.

1. Introduction

If $f:\mathbb{R}\to\mathbb{R}$ then its Fourier transform at $s\in\mathbb{R}$ is defined as $\widehat{f}(s)=\int_{-\infty}^{\infty}e^{-isx}f(x)\,dx$. The inverse transform is $\check{f}(s)=(2\pi)^{-1}\int_{-\infty}^{\infty}e^{isx}f(x)\,dx$. In this paper we consider Fourier transforms as Henstock–Kurzweil integrals. This is an integral equivalent to the Denjoy and Perron integrals but with a definition in terms of Riemann sums. We let $\mathcal{HK}_A$ be the Henstock–Kurzweil integrable functions over a set $A\subset\mathbb{R}$, dropping the subscript when $A=\mathbb{R}$. (The symbol $\subset$ allows set equality.) Then $\mathcal{HK}$ properly contains the union of L^1 and the Cauchy-Lebesgue integrable functions (i.e., improper Lebesgue integrals). The main points of $\mathcal{HK}$ integration that we use can be found in [1] and [10]. Several of our results depend on being able to reverse the order of repeated integrals. In the Lebesgue theory this is usually justified with Fubini's Theorem. For $\mathcal{HK}$ integrals, necessary and sufficient conditions were given in [12]. Lemma 25 in the Appendix gives sufficient conditions that are readily applicable to the cases at hand. Also in the Appendix are some conditions for convergence of rapidly oscillatory integrals (Lemma 23) and an estimate of the integral of a product (Lemma 24).

There is a substantial body of theory relating to Fourier transforms when they are considered as Lebesgue integrals. Necessary and sufficient for existence of $\widehat{f}$ on $\mathbb{R}$ is that $f\in L^1$. This is because the multipliers for L^1 are the

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(essentially) bounded measurable functions and $|e^{\pm isx}| \leq 1$. The multipliers for $\mathcal{HK}$ are the functions of (essentially) bounded variation. As $x \mapsto e^{-isx}$ is not of bounded variation, except for $s = 0$, we do not have an elegant existence theorem for $\mathcal{HK}$ Fourier integrals. Various existence conditions are given in Proposition 2. Example 3(f) gives a function whose Fourier transform diverges on a countable set. For L^1 convergence, $\widehat{f}$ is uniformly continuous with limit 0 at infinity (the Riemann–Lebesgue lemma). We show below (Example 3(e)) that the Riemann–Lebesgue lemma fails dramatically in $\mathcal{HK}$: $\widehat{f}$ can have arbitrarily large point-wise growth. And, $\widehat{f}$ need not be continuous. Continuity of $\widehat{f}$ is equivalent to quasi-uniform convergence (Theorem 5). Some sufficient conditions for continuity of $\widehat{f}$ appear in Proposition 6. Although $\widehat{f}$ need not be continuous, when it exists at the endpoints of a compact interval, it exists almost everywhere on that interval and is integrable over that interval; see Proposition 7. As in the L^1 theory, we have linearity, symmetry, conjugation, translation, modulation, dilation, etc.; see formulas (2)–(9) in [5, p. 117] and [2, p. 9]. We draw attention to the differentiation of Fourier transforms (Proposition 8) and transforms of derivatives (Proposition 9). One of the most important properties of Fourier transforms is their interaction with convolutions. Propositions 10, 11, 13, 14 and 15 contain various results on existence of convolutions; estimates using the variation, L^1 norm and Alexiewicz norm; and the transform and inverse transform of convolutions. Proposition 16 gives a Parseval relation. An inversion theorem is obtained using a summability kernel (Theorem 18). A uniqueness theorem follows as a corollary. The paper concludes with an example of a function f for which $\widehat{f}$ exists on $\mathbb{R}$ but $\check{\widehat{f}}$ exists nowhere.

As Henstock–Kurzweil integrals allow conditional convergence, they make an ideal setting for the Fourier transform. We remark that many of the Fourier integrals appearing in tables such as [5] diverge as Lebesgue integrals but converge as improper Riemann integrals. Thus, they exist as $\mathcal{HK}$ integrals.

We use the following notation. Let $A \subset \mathbb{R}$ and f be a real-valued function on A. The functions of bounded variation over A are denoted $\mathcal{BV}_A$ and the variation of function f over set A is $V_A f$. We say a set is in $\mathcal{BV}$ if its characteristic function is in $\mathcal{BV}$. All our results are stated for real-valued functions but the extension to complex-valued functions is immediate. Note that for complex-valued functions, the variation of the real part and the variation of the imaginary part are added. The Alexiewicz norm of $f \in \mathcal{HK}_A$ is $\|f\|_A = \sup_{I \subset A} |\int_I f|$, the supremum being taken over all intervals $I \subset A$. For each of these definitions, the label A is omitted when $A = \mathbb{R}$ or it is obvious which set is A. Whereas indefinite Lebesgue integrals are absolutely continuous (AC), indefinite Henstock–Kurzweil integrals are ACG_*; see [9] for the definition of ACG_* and the related space AC_*. Finally, a convergence theorem that we use throughout is:

Theorem 1. *Let f and g_n $(n \in \mathbb{N})$ be real-valued functions on $[a,b]$. If $f \in \mathcal{HK}$, $Vg_n \leq M$ for all $n \in \mathbb{N}$, and $g_n \to g$ as $n \to \infty$ then $\int_a^b fg_n \to \int_a^b fg$ as $n \to \infty$.*

The theorem holds for $[a,b] \subset \overline{\mathbb{R}}$, where $\overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$ is the extended real line. For a proof see [11].

2. Basic properties

We first tackle the problem of existence. If $f:\mathbb{R} \to \mathbb{R}$ then $\widehat{f}$ exists as a Lebesgue integral on $\mathbb{R}$ if and only if $f \in L^1$. This follows from the fact that $|e^{\pm isx}| \leq 1$ for all $s, x \in \mathbb{R}$ and the multipliers for L^1 are the bounded measurable functions. No such simple necessary and sufficient conditions are known for existence of $\mathcal{HK}$ Fourier integrals. However, we do have the following results.

Proposition 2. *Let $f:\mathbb{R} \to \mathbb{R}$.*

(a) *In order for $\widehat{f}$ to exist at some $s \in \mathbb{R}$ it is necessary that $f \in \mathcal{HK}_{\mathrm{loc}}$.*

(b) *If $f \in \mathcal{HK}_{\mathrm{loc}}$ then $\widehat{f}$ exists on $\mathbb{R}$ if $|f|$ is integrable in a neighbourhood of infinity or if f is of bounded variation in a neighbourhood of infinity with limit 0 at infinity.*

(c) *Let $f \in \mathcal{HK}$. Define $F_1(x) = \int_x^\infty f$ and $F_2(x) = \int_{-\infty}^x f$. Then $\widehat{f}$ exists at $s \in \mathbb{R}$ if and only if both the integrals $\int_0^\infty e^{-isx} F_1(x)\,dx$ and $\int_{-\infty}^0 e^{-isx} F_2(x)\,dx$ exist.*

Proof. (a) For each $s \in \mathbb{R}$, the function $x \mapsto e^{isx}$ is of bounded variation on any compact interval.

(b) This follows from the Chartier–Dirichlet convergence test; see [1].

(c) Let $T > 0$. Integrate by parts to obtain

$$\int_0^T e^{-isx} f(x)\,dx = F_1(0) - F_1(T)e^{-isT} - is\int_0^T e^{-isx} F_1(x)\,dx.$$

Since F_1 is continuous with limit 0 at infinity, $\int_0^\infty e^{-isx} f(x)\,dx$ exists if and only if $\int_0^\infty e^{-isx} F_1(x)\,dx$ exists. The other part of the proof is similar. □

Although F_1 is continuous with limit 0 at infinity, it need not be of bounded variation. So, $f \in \mathcal{HK}$ does not imply the existence of $\widehat{f}$; see Example 3(c) below. Notice that part (b) (with $\mathcal{HK}_{\mathrm{loc}}$ replaced by L^1_{loc}) and part (c) are false for L^1 convergence of $\widehat{f}$.

Titchmarsh [15] gives several sufficient conditions for existence of conditionally convergent Fourier integrals (§1.10–1.12). However, these all require that $f \in L^1_{\mathrm{loc}}$.

When $f \in L^1$ and $s, h \in \mathbb{R}$, $\widehat{f}(s+h) = \int_{-\infty}^{\infty} e^{-i(s+h)x} f(x)\,dx$. By dominated convergence this tends to $\widehat{f}(s)$ as $s \to h$. So, $\widehat{f}$ is uniformly continuous on $\mathbb{R}$. When $\widehat{f}$ exists in $\mathcal{HK}$ in a neighbourhood of s, the function $x \mapsto e^{-isx} f(x)$ is in $\mathcal{HK}$ but the factor e^{-ihx} is not of bounded variation on $\mathbb{R}$ except for $h = 0$. In general we cannot take the limit $h \to 0$ under the integral sign and $\widehat{f}$ need not be continuous. And, for $f \in L^1$ and $s \neq 0$, the change of variables $x \mapsto x + \pi/s$ gives $\widehat{f}(s) = (1/2) \int_{-\infty}^{\infty} e^{-isx} \left[f(x) - f(x + \pi/s)\right] dx$. Writing $f_y(x) = f(x + y)$ for $x, y \in \mathbb{R}$, we have $|\widehat{f}(s)| \leq (1/2)\|f - f_{\pi/s}\|_1$. Continuity of f in the L^1 norm now yields the Riemann–Lebesgue lemma: $\widehat{f}(s) \to 0$ as $|s| \to \infty$. It is true that if $f \in \mathcal{HK}$ then f is continuous in the Alexiewicz norm [14]. However, since the variation of $x \mapsto e^{-isx}$ is not uniformly bounded as $|s| \to \infty$, existence of $\widehat{f}$ does not let us conclude that $\widehat{f}$ tends to 0 at infinity.

The following examples show some of the differences between L^1 and $\mathcal{HK}$ Fourier transforms.

Example 3. The transforms (a)–(d) appear in [5]. Convergence in (a) is by Lemma 23, (b) is similar, after integrating by parts, and (c) and (f) are Frullani integrals.

(a) If $f(x) = \mathrm{sgn}(x)|x|^{-1/2}$ then f is not in $\mathcal{HK}$ or in any L^p space ($1 \leq p \leq \infty$) and yet $\widehat{f}(s) = \sqrt{2\pi}\,\mathrm{sgn}(s)|s|^{-1/2}$ for $s \neq 0$. Notice that, even though f is odd, $\widehat{f}$ does not exist at 0 since $\mathcal{HK}$ convergence does not allow principal value integrals.

(b) Let $g(x) = e^{ix^2}$. Then $\widehat{g}(s) = \sqrt{\pi}\, e^{i(\pi - s^2)/4}$. In this example, $\widehat{g}$ is not of bounded variation at infinity, nor does $\widehat{g}$ tend to 0 at infinity, nor is $\widehat{g}$ uniformly continuous on $\mathbb{R}$. The same can of course be said for g.

(c) Let $h(x) = \sin(ax)/|x|$. Then $\widehat{h}(s) = i \log|(s-a)/(s+a)|$ for $s \neq a$.

(d) Let $k(x) = x/(x^2+1)$. Then $\widehat{k}(s) = -i\pi\,\mathrm{sgn}(s) e^{-|s|}$ for $s \neq 0$. Note that $\widehat{k}$ does not exist at 0, even though its principal value is 0.

(e) Fourier transforms in $\mathcal{HK}$ can have arbitrarily large point-wise growth. Given any sequence $\{a_n\}$ of positive real numbers, there is a continuous function f on $\mathbb{R}$ such that $\widehat{f}$ exists on $\mathbb{R}$ and $\widehat{f}(n) \geq a_n$ for all $n \geq 1$ [13].

(f) Let $\{a_n\}$ and $\{b_n\}$ be sequences in $\mathbb{R}$. Define $f(x) = \sum_{n=1}^{\infty} a_n \frac{\sin(b_n x)}{|x|}$ for $x \neq 0$ and $f(0) = 0$. Assume that $a_n > 0$, $\sum a_n < \infty$ and $\sum a_n |b_n| < \infty$. Then f is continuous on $\mathbb{R}$, except at the origin, where it has a finite jump discontinuity. Suppose s is not in the closure of $\{-b_n, b_n\}_{n \in \mathbb{N}}$. Then

$$\widehat{f}(s) = \sum_{n=1}^{\infty} a_n \int_{-\infty}^{\infty} e^{-isx} \sin(b_n x)\,\frac{dx}{|x|} \tag{1}$$

$$= i \sum_{n=1}^{\infty} a_n \int_0^{\infty} \left(\cos\left[(s+b_n)x\right] - \cos\left[(s-b_n)x\right]\right) \frac{dx}{x} \tag{2}$$

$$= i \sum_{n=1}^{\infty} a_n \log\left|\frac{s-b_n}{s+b_n}\right|. \tag{3}$$

The reversal of summation and integration in (1) is justified using Corollary 7 in [12]. Hence, $\widehat{f}$ exists on $\mathbb{R}$, except perhaps on the closure of $\{-b_n, b_n\}_{n\in\mathbb{N}}$. Note that $\widehat{f}(0) = 0$.

We will now show $\widehat{f}$ diverges at each b_k with $a_k b_k \neq 0$. Let $T_1, T_2 > 0$ and consider

$$\begin{aligned}
&\int_{-T_1}^{T_2} e^{-ib_k x} \sum_{n=1}^{\infty} a_n \sin(b_n x) \frac{dx}{|x|} \\
&= \sum_{n=1}^{\infty} a_n \int_{-T_1}^{T_2} e^{-ib_k x} \sin(b_n x) \frac{dx}{|x|} \qquad (4)\\
&= \sum_{n=1}^{\infty} a_n \int_{-T_1}^{T_2} \frac{\sin[(b_k+b_n)x] - \sin[(b_k-b_n)x]}{2} - i\sin(b_k x)\sin(b_n x)\frac{dx}{|x|}. \qquad (5)
\end{aligned}$$

In (4), convergence of $\sum a_n |b_n|$ permits reversal of summation and integration. The real part of (5) converges for all $k \geq 1$, uniformly for $T_1, T_2 \geq 0$. Hence, the real part of $\widehat{f}$ exists on $\mathbb{R}$. The kth summand of the imaginary part of (5) is

$$-a_k \int_{-T_1}^{T_2} \sin^2(b_k x) \frac{dx}{|x|} = -a_k \int_{-T_1|b_k|}^{T_2|b_k|} \sin^2 x \frac{dx}{|x|}.$$

This diverges as $T_1, T_2 \to \infty$. Hence, $\widehat{f}(b_k)$ does not exist.

If $\{-b_n, b_n\}_{n\in\mathbb{N}}$ has no limit points then we have an example of a function whose Fourier transform exists everywhere except on a countable set.

Now suppose $s \notin \{-b_n, b_n\}_{n\in\mathbb{N}}$ but s is a limit point of $\{-b_n, b_n\}_{n\in\mathbb{N}}$. As noticed above, the real part of $\widehat{f}(s)$ exists. And,

$$\left| \int_{-1}^{1} \sin(sx)\sin(b_n x)\frac{dx}{|x|} \right| \leq 2|s|.$$

So, $\widehat{f}(s)$ exists if and only if

$$\lim_{T\to\infty} \sum_{n=1}^{\infty} a_n \int_1^T \left(\cos[(s+b_n)x] - \cos[(s-b_n)x]\right) \frac{dx}{x}$$

exists. Suppose $s \neq 0$ and $T > 1$. If $|s-b_n|T > 1$ and $|s-b_n| < 1$ then

$$\begin{aligned}
\left| \int_1^T \cos(|s-b_n|x)\,\frac{dx}{x} \right| &= \left| \int_{|s-b_n|}^{|s-b_n|T} \cos x\,\frac{dx}{x} \right| \qquad (6)\\
&= \left| \int_{|s-b_n|}^{1} \cos x\,\frac{dx}{x} + \int_1^{|s-b_n|T} \cos x\,\frac{dx}{x} \right| \\
&\leq \log(1/|s-b_n|) + c.
\end{aligned}$$

The constant c is equal to the supremum of $|\int_1^t \cos x\,dx/x|$ over $t>1$. When $|s-b_n|T\le 1$, we have

$$\begin{aligned}\left|\int_1^T \cos(|s-b_n|x)\,\frac{dx}{x}\right| &= \left|\int_{|s-b_n|}^{|s-b_n|T} \cos x\,\frac{dx}{x}\right|\\ &\le \log T\\ &\le \log\left(1/|s-b_n|\right).\end{aligned}$$

The case for $|s+b_n|T$ is similar. It follows that the condition

$$\sum_{n=1}^{\infty} a_n \left|\log\left|\frac{s-b_n}{s+b_n}\right|\right| < \infty \tag{7}$$

is sufficient for existence of $\widehat{f}(s)$.

If $1/T<|s-b_n|<1$ then, as in (6),

$$\int_1^T \cos(|s-b_n|x)\,\frac{dx}{x} \ge \cos(1)\log\left(1/|s-b_n|\right) - c.$$

Therefore,

$$\begin{aligned}&\sum_{1/T<|s-b_n|<1} a_n \int_1^T \cos(|s-b_n|x)\,\frac{dx}{x}\\ &\qquad \ge \sum_{1/T<|s-b_n|<1} a_n\left[\cos(1)\log\left(1/|s-b_n|\right) - c\right].\end{aligned}$$

Let $T\to\infty$. Then condition (7) is also necessary for existence of $\widehat{f}(s)$. Hence, it is possible for $\widehat{f}$ to exist at a finite number of limit points of $\{-b_n, b_n\}_{n\in\mathbb{N}}$.

Finally, enumerate the rational numbers in $[0,1]$ by $b_1=0, b_2=1/1, b_3=1/2, b_4=1/3, b_5=2/3, b_6=3/3, b_7=1/4$, etc. Let $A_m>0$ be such that $\sum mA_m<\infty$. Put $a_1=0$ and define $a_n=A_m$ for the m consecutive values of n such that $b_n=l/m$ for some $1\le l\le m$. Let $s\in[-1,1]\setminus\mathbb{Q}$ and let $\bar{s}$ be the distance to the nearest rational number. Then

$$\sum_{n=1}^{\infty} a_n\left|\log\left|\frac{s-b_n}{s+b_n}\right|\right| = \sum_{m=1}^{\infty} A_m \sum_{l=1}^{m}\left|\log\left|\frac{s-l/m}{s+l/m}\right|\right| \le \log(2/\bar{s})\sum_{m=1}^{\infty} mA_m.$$

This furnishes an example of a function whose Fourier transform exists on $\mathbb{R}$ except for the rational numbers in $[-1,1]$.

Examples 3(a), (c), (d) and (f) show that $\widehat{f}$ need not be continuous. However, continuity of $\widehat{f}$ is equivalent to quasi-uniform continuity.

Definition 4 (Quasi-uniform continuity). Let $f:\mathbb{R}^2\to\mathbb{R}$. If $F(x):=\int_{-\infty}^{\infty} f(x,y)\,dy$ exists in a neighbourhood of $x_0\in\mathbb{R}$ then F is quasi-uniformly continuous at x_0 if for all $\epsilon > 0$ and $M > 0$ there exist $m = m(x_0, \epsilon, M) \geq M$ and $\delta = \delta(x_0, \epsilon, M) > 0$ such that if $|x - x_0| < \delta$ then $|\int_{|y|>m} f(x, y)\, dy| < \epsilon$.

This is a modification of a similar definition for series, originally introduced by Dini; see [3, p. 140].

THEOREM 5. *Let $f : \mathbb{R} \to \mathbb{R}$. Then, $\widehat{f}$ is continuous at $s_0 \in \mathbb{R}$ if and only if $\widehat{f}$ is quasi-uniformly continuous at s_0.*

Proof. For $m > 0$, let $F_m(s) = \int_{-m}^{m} e^{-isx} f(x)\, dx$. Let $h \in \mathbb{R}$. Then, $F_m(s+h) - F_m(s) = \int_{-m}^{m} [e^{-ihx} - 1] e^{-isx} f(x)\, dx$. Note that either assumption implies $x \mapsto e^{-isx} f(x)$ is in $\mathcal{HK}_{\text{loc}}$ for each $s \in \mathbb{R}$. And, $V_{[-m,m]}[x \mapsto e^{-ihx} - 1] \leq 4m|h|$. Taking the limit $h \to 0$ inside the above integral now shows F_m is continuous on $\mathbb{R}$ for each $m > 0$.

Suppose $\widehat{f}$ is quasi-uniformly continuous at $s_0 \in \mathbb{R}$. Given $\epsilon > 0$, take $M > 0$ such that $|\int_{|x|>t} e^{is_0 x} f(x)\, dx| < \epsilon$ for all $t > M$. From quasi-uniform continuity, we have $m > M$ and $\delta > 0$. Then, for $|s - s_0| < \delta$,

$$
\begin{aligned}
&\left| \int_{|x|>m} \left[e^{-isx} - e^{-is_0 x} \right] f(x)\, dx \right| \\
&\qquad \leq \left| \int_{|x|>m} e^{-isx} f(x)\, dx \right| + \left| \int_{|x|>m} e^{-is_0 x} f(x)\, dx \right| \\
&\qquad \leq 2\epsilon.
\end{aligned}
$$

It follows that $\widehat{f}$ is continuous at s_0.

Suppose $\widehat{f}$ is continuous at s_0 and we are given $\epsilon > 0$ and $M > 0$. Since $\widehat{f}$ exists at s_0, there is $N = N(s_0, \epsilon) > 0$ such that $|\int_{|x|>m} e^{-is_0 x} f(x)\, dx| < \epsilon$ whenever $m > N$. Continuity of $\widehat{f}$ at s_0 implies the existence of $\xi = \xi(s_0, \epsilon) > 0$ such that $|\widehat{f}(s) - \widehat{f}(s_0)| < \epsilon$ when $|s - s_0| < \xi$. And, F_m is continuous on $\mathbb{R}$. Hence, there exists $\eta = \eta(s_0, \epsilon, m) > 0$ such that when $|s - s_0| < \eta$ we have $|F_m(s) - F_m(s_0)| < \epsilon$. Let $m = \max(M, N)$ and $\delta = \min(\xi, \eta)$. Then for $|s - s_0| < \delta$ we have

$$
\begin{aligned}
&\left| \int_{|x|>m} e^{-isx} f(x)\, dx \right| \\
&\qquad \leq \left| \int_{|x|>m} e^{is_0 x} f(x)\, dx \right| + \left| \widehat{f}(s) - \widehat{f}(s_0) \right| + |F_m(s) - F_m(s_0)| \\
&\qquad < 3\epsilon.
\end{aligned}
$$

And, $\widehat{f}$ is quasi-uniformly continuous at s_0. □

We now present two sufficient conditions for a Fourier transform to be continuous. The first is in the spirit of the Chartier–Dirichlet convergence test and the second is in the spirit of the Abel convergence test. For simplicity, the results are stated for functions on $[0, \infty)$. The general case follows easily.

PROPOSITION 6. *Let g and h be real-valued functions on $[0, \infty)$ where $g \in \mathcal{BV}$ and $h \in \mathcal{HK}_{\mathrm{loc}}$. Define $f = gh$.*

(a) *Suppose there are positive constants M, δ and K such that, if $|s - s_0| < \delta$ and $M_1, M_2 > M$ then $|\int_{M_1}^{M_2} e^{-isx} h(x)\, dx| < K$. If $g(x) \to 0$ as $x \to \infty$ then $\widehat{f}$ is continuous at s_0.*

(b) *Let $H_s(x) = \int_0^x e^{-ist} f(t)\, dt$. If $\widehat{h}$ is continuous at s_0 and there are $\delta, K > 0$ such that for all $x > 0$ and $|s - s_0| < \delta$ we have $|H_s(x)| \le K$ then $\widehat{f}$ is continuous at s_0.*

Proof. Write $\phi_s(x) = e^{-isx} h(x)$. With no loss of generality, $g(\infty) = 0$.

For (a), let $|s - s_0| < \delta$ and $M_1, M_2 > M$. Using Lemma 24,

$$
\begin{aligned}
\left| \int_{M_1}^{M_2} e^{-isx} f(x)\, dx \right| &\le \left| \int_{M_1}^{M_2} \phi_s(x)\, dx \right| \inf_{[M_1, M_2]} |g| + \|\phi_s\|_{[M_1, M_2]} V_{[M_1, M_2]} g \\
&\le K \left[\inf_{[M_1, M_2]} |g| + V_{[M, \infty]} g \right] \qquad (8) \\
&\to 0 \quad \text{as } M \to \infty.
\end{aligned}
$$

Therefore, $\widehat{f}$ exists in a neighbourhood of s_0. Taking the limit $M_2 \to \infty$ in (8) shows that $\widehat{f}$ is quasi-uniformly continuous and hence continuous.

For (b), since $g \in \mathcal{BV}$ we have $\lim_{x \to \infty} g(x) = c \in \mathbb{R}$. Writing $f = h(g - c) + ch$ we need only consider $|\int_M^\infty \phi_s(g - c)| \le \|\phi_s\|_{[M, \infty)} V_{[M, \infty)}(g - c)$. By our assumption, $\|\phi_s\|_{[M, \infty)} \le 2K$ for $|s - s_0| < \delta$. And, $V_{[M, \infty)}(g - c) \to 0$ as $M \to \infty$. $\square$

Although $\widehat{f}$ need not be continuous, when it exists at the endpoints of a compact interval it is integrable over the interval.

PROPOSITION 7. *Let $[a, b]$ be a compact interval. If $\widehat{f}$ exists at a and b then $\widehat{f}$ exists almost everywhere on (a, b), $\widehat{f}$ is integrable over (a, b) and $\int_a^b \widehat{f} = i \int_{-\infty}^{\infty} f(x)[e^{-ibx} - e^{-iax}]\, dx/x$.*

Proof. The integral $I := i \int_{-\infty}^{\infty} f(x) \left[e^{-ibx} - e^{-iax} \right] \frac{dx}{x}$ exists since $x \mapsto f(x) e^{-ibx}/x$ and $x \mapsto f(x) e^{-iax}/x$ are integrable over $\mathbb{R} \setminus (-1, 1)$ and $x \mapsto \left[e^{-ibx} - e^{-iax} \right]/x$ is of bounded variation on $[-1, 1]$. And,

$$
I = \int_{-\infty}^{\infty} f(x) e^{-ibx} \int_a^b e^{-i(s-b)x} ds\, dx = \int_a^b \int_{-\infty}^{\infty} f(x) e^{-isx} dx\, ds = \int_a^b \widehat{f}.
$$

Hence, $\widehat{f}$ exists almost everywhere on (a,b) and is integrable over (a,b). Lemma 25(a) justifies the reversal of x and s integration. $\square$

The usual algebraic properties of linearity, symmetry, conjugation, translation, modulation, dilation, etc., familiar from the L^1 theory, continue to hold for $\mathcal{HK}$ Fourier transforms; see formulas (2)–(9) in [5, p. 117] and [2, p. 9]. The proofs are elementary. There are also differentiation results analogous to the L^1 case (pages 117 and 17, respectively, of the previous references).

Proposition 8 (Frequency differentiation). *Suppose $\widehat{f}$ exists on the compact interval $[\alpha,\beta]$. Define $g(x) = xf(x)$ and suppose $g \in \mathcal{HK}$. Then $\hat{f}\,' = -i\widehat{g}$ almost everywhere on (α,β). In particular, $\hat{f}\,'(s) = -i\widehat{g}(s)$ for all $s \in (\alpha,\beta)$ such that $\frac{d}{ds}\int_\alpha^s \widehat{g} = \widehat{g}(s)$.*

Proof. The necessary and sufficient condition that allows differentiation under the integral, $\hat{f}\,'(s) = -i\int_{-\infty}^{\infty} e^{-isx}xf(x)\,dx$, for almost all $s \in (\alpha,\beta)$ is that

$$\int_{-\infty}^{\infty}\int_a^b e^{-isx}xf(x)\,ds\,dx = \int_a^b\int_{-\infty}^{\infty} e^{-isx}xf(x)\,dx\,ds \tag{9}$$

for all $[a,b] \subset [\alpha,\beta]$; see [12, Theorem 4]. We have $g \in \mathcal{HK}$, $|e^{-isx}| \le 1$ and $V_I[x \mapsto e^{-isx}] \le 2|I||s|$ for a compact interval $I \subset \mathbb{R}$. The left member of (9) is $i[\widehat{f}(b) - \widehat{f}(a)]$. Hence, by Lemma 25(a), (9) holds, and $\hat{f}\,'(s) = -i\widehat{g}(s)$ for almost all $s \in (\alpha,\beta)$. Examining the proof of [12, Theorem 4], we see that we get equality $\hat{f}\,'(s) = -i\widehat{g}(s)$ when $\frac{d}{ds}\int_\alpha^s \widehat{g} = \widehat{g}(s)$. $\square$

There are similar results for n-fold differentiation when the function $x \mapsto x^n f(x)$ is in $\mathcal{HK}$ for a positive integer n.

Proposition 9 (Time differentiation).

(a) *If $f \in ACG_*(\mathbb{R})$ and $f(x) \to 0$ as $|x| \to \infty$ then for each $s \neq 0$, both $\widehat{f}(s)$ and $\widehat{f'}(s)$ fail to exist or $\widehat{f'}(s) = is\widehat{f}(s)$.*

(b) *Suppose $f \in ACG_*(\mathbb{R})$ and $f, f' \in \mathcal{HK}$. Then for each $s \neq 0$, either both $\widehat{f}(s)$ and $\widehat{f'}(s)$ fail to exist or $\widehat{f'}(s) = is\widehat{f}(s)$.*

Proof. (a) Let $M_1, M_2 > 0$. Integrate by parts to get

$$\int_{-M_1}^{M_2} e^{-isx}f'(x)\,dx = e^{-isM_2}f(M_2) - e^{isM_1}f(-M_1) + is\int_{-M_1}^{M_2} e^{-isx}f(x)\,dx.$$

Now take the limits $M_1, M_2 \to \infty$.

(b) Consider $\int_x^\infty f' = \int_M^\infty f' + f(M) - f(x)$ for $x, M \in \mathbb{R}$. Since $f' \in \mathcal{HK}$, the limits as $|x| \to \infty$ exist. Hence, f has a limit at infinity. But, $f \in \mathcal{HK}$ so this limit must be 0 and we have reduction to case (a). $\square$

3. Convolution

If f and g are real-valued functions on $\mathbb{R}$ then their convolution is $f*g(x) = \int_{-\infty}^{\infty} f(x-t)g(t)\,dt$. The following proposition gives the basic properties of convolution.

Proposition 10. *Let f and g be real-valued functions on $\mathbb{R}$. Define $f_x\colon \mathbb{R}\to\mathbb{R}$ by $f_x(y) = f(x+y)$ for $x, y \in \mathbb{R}$. For an interval $I = [\alpha,\beta] \subset \mathbb{R}$ and $y \in \mathbb{R}$, define $I - y = [\alpha - y, \beta - y]$.*

(a) *If $f * g$ exists at $x \in \mathbb{R}$ then $f * g(x) = g * f(x)$.*
(b) *If $f \in \mathcal{HK}$, $g \in \mathcal{BV}$ and $h \in L^1$ then $(f * g) * h = f * (g * h)$ on $\mathbb{R}$.*
(c) *Let $f \in \mathcal{HK}$. Suppose that for each compact interval $I \subset \mathbb{R}$ there are constants K_I and M_I such that $|g| * |h|(z) \leq K_I$ for all $z \in I$ and the function $y \mapsto h(y)V_{I-y}g$ is in L^1. If $f * (g * h)$ exists at $x \in \mathbb{R}$ then $(f * g) * h(x) = f * (g * h)(x)$.*
(d) *$(f*g)_x = f_x*g = f*g_x$ wherever any one of these convolutions exists.*
(e) $\mathrm{supp}(f * g) \subset \{x + y : x \in \mathrm{supp}(f), y \in \mathrm{supp}(g)\}$.

Proof. For (a), (d) and (e), the L^1 proofs hold without change; see [6, Proposition 8.6]. To prove (b), write

$$\begin{aligned}
(f*g)*h(x) &= \int_{-\infty}^{\infty} f*g(x-y)h(y)\,dy \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x-y-z)g(z)\,dz\,h(y)\,dy \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x-z)g(z-y)h(y)\,dy\,dz \\
&= f*(g*h)(x).
\end{aligned}$$

Lemma 25(b) allows us to change the order of y and z integration. The proof of (c) is similar but now we use Lemma 25(a). □

The next proposition gives some sufficient conditions for existence of the convolution and some point-wise estimates.

Proposition 11.

(a) *Let $f \in \mathcal{HK}$ and $g \in \mathcal{BV}$. Then $f * g$ exists on $\mathbb{R}$ and $|f * g(x)| \leq \|f\|[\inf|g| + Vg]$ for all $x \in \mathbb{R}$.*
(b) *Let $f \in \mathcal{HK}_{\mathrm{loc}}$ and $g \in \mathcal{BV}$ with the support of g in the compact interval $[a,b]$. Then $f*g$ exists on $\mathbb{R}$ and $|f*g(x)| \leq |\int_{x-b}^{x-a} f|\inf_{[a,b]}|g| + \|f\|_{[x-a,x-b]}V_{[a,b]}g$.*

Proof. (a) Using Lemma 24,

$$\begin{aligned}|f*g(x)| &= \left|\int_{-\infty}^{\infty} f(x-t)g(t)\,dt\right| \\ &\le \left|\int_{-\infty}^{\infty} f\right| \inf|g| + \|f\| V g \\ &\le \|f\| \left[\inf|g| + Vg\right].\end{aligned}$$

(b) Now,

$$\begin{aligned}|f*g(x)| &= \left|\int_a^b f(x-t)g(t)\,dt\right| \\ &\le \left|\int_{x-b}^{x-a} f\right| \inf_{[a,b]}|g| + \|f\|_{[x-b,x-a]} V_{[a,b]} g.\end{aligned}$$

□

These conditions are sufficient but not necessary for existence of the convolution. Also, if $f, g \in \mathcal{HK}$ then $f*g$ need not exist at any point.

EXAMPLE 12.

(a) Let $f(x) = \log|x|\sin(x)$ and $g(x) = |x|^{-\alpha}$, where $0 < \alpha < 1$. Then f and g do not have compact support and are not in $\mathcal{HK}$, $\mathcal{BV}$ or L^p $(1 \le p \le \infty)$. And yet $f*g$ exists on $\mathbb{R}$.
(b) Let $f(x) = \sin(x)/|x|^{1/2}$ and $g(x) = (\sin(x)+\cos(x))/|x|^{1/2}$. Then $f, g \in \mathcal{HK}$ but $f*g$ exists nowhere.

When $f \in \mathcal{HK}$ and $g \in L^1 \cap \mathcal{BV}$ then $f*g$ exists on $\mathbb{R}$ and we can estimate it in the Alexiewicz norm.

PROPOSITION 13. *Let $f \in \mathcal{HK}$ and $g \in L^1 \cap \mathcal{BV}$. Then $f*g$ exists on $\mathbb{R}$ and $\|f*g\| \le \|f\|\|g\|_1$.*

Proof. Existence comes from Proposition 11. Let $-\infty \le a < b \le \infty$. Using Lemma 25(a), we can interchange the repeated integrals,

$$\int_a^b f*g\,dx = \int_a^b \int_{-\infty}^{\infty} f(x-t)g(t)\,dt\,dx \tag{10}$$

$$= \int_{-\infty}^{\infty} g(t) \int_a^b f(x-t)\,dx\,dt. \tag{11}$$

And,

$$\begin{aligned}\left|\int_a^b f*g\,dx\right| &\le \|g\|_1 \sup_{t\in\mathbb{R}}\left|\int_a^b f(x-t)\,dx\right|\\ &= \|g\|_1 \sup_{t\in\mathbb{R}}\left|\int_{a-t}^{b-t} f\right|\\ &\le \|f\|\|g\|_1. \qquad\square\end{aligned}$$

Under suitable conditions on f and g, we have the usual interactions between convolution and Fourier transformation and inversion.

PROPOSITION 14. *If $\widehat{f}$ exists at $s\in\mathbb{R}$ and $g\in L^1\cap\mathcal{BV}$ then $\widehat{f*g}(s)=\widehat{f}(s)\,\widehat{g}(s)$.*

Proof. We have

$$\begin{aligned}\widehat{f*g}(s) &= \int_{-\infty}^{\infty} e^{-isx}\int_{-\infty}^{\infty}\left[e^{-ist}f(t)\right]\left[e^{ist}g(x-t)\right]\,dt\,dx\\ &= \int_{-\infty}^{\infty} e^{-ist}f(t)\int_{-\infty}^{\infty} e^{-is(x-t)}g(x-t)\,dx\,dt\\ &= \widehat{f}(s)\,\widehat{g}(s).\end{aligned}$$

The interchange of integrals is validated by Lemma 25(a), since

$$\begin{aligned}\int_{-\infty}^{\infty} V_{[a,b]}\left[t\mapsto e^{-is(x-t)}g(x-t)\right]\,dx &= \int_{-\infty}^{\infty} V_{[x-b,x-a]}\left[t\mapsto e^{-ist}g(t)\right]\,dx\\ &\le 2|s|(b-a)\|g\|_1+2(b-a)Vg. \qquad\square\end{aligned}$$

PROPOSITION 15. *If f and g are in $\mathcal{HK}_{\rm loc}$ such that $\widehat{f}$ exists almost everywhere, $\widehat{g}\in L^1$, $s\mapsto s\,\widehat{g}(s)$ is in L^1 and $\widehat{g}\check{\ } = g$ almost everywhere then $f*g=(\widehat{f}\,\widehat{g})\check{\ }$ wherever $f*g$ exists.*

Proof. Let $x\in\mathbb{R}$. Then $\widehat{g}(x-t)$ exists for almost all $t\in\mathbb{R}$. And,

$$\begin{aligned}f*g(x) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(t)\int_{-\infty}^{\infty} e^{is(x-t)}\widehat{g}(s)\,ds\,dt\\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{isx}\,\widehat{g}(s)\int_{-\infty}^{\infty} e^{-ist}f(t)\,dt\,ds\\ &= (\widehat{f}\,\widehat{g})\check{\ }(x).\end{aligned}$$

Suppose $\widehat{f}$ exists at s_0. Then $V_I[t\mapsto e^{is(x-t)}e^{is_0t}\,\widehat{g}(s)]\le 2|\widehat{g}(s)||s-s_0||I|$ and the reversal of s and t integration order is by Lemma 25(a). $\square$

4. Inversion

A well-known inversion theorem states that if $\widehat{f}$ and $\widehat{f}^{\vee}$ are in L^1 then $f = \widehat{f}^{\vee}$ almost everywhere. These are rather restrictive conditions as both f and $\widehat{f}$ must be continuous (almost everywhere) and vanish at infinity. In Example 3(a) and (b), $\widehat{f}$ is a multiple of f and $\widehat{g}$ is a multiple of g so we certainly have $f = \widehat{f}^{\vee}$ and $g = \widehat{g}^{\vee}$ almost everywhere and yet none of these integrals exists in L^1. However, they do exist in $\mathcal{HK}$. And, we have a similar inversion theorem in $\mathcal{HK}$. First we need the following Parseval relation.

PROPOSITION 16. *Let ψ and ϕ be real-valued functions on $\mathbb{R}$. Suppose $\widehat{\psi}$ exists at some $s_0 \in \mathbb{R}$. Suppose $\phi \in L^1$ and the function $s \mapsto s\phi(s)$ is also in L^1. If $\int_{-\infty}^{\infty} \psi\,\widehat{\phi}$ exists, then $\widehat{\psi}$ exists almost everywhere and $\int_{-\infty}^{\infty} \psi\,\widehat{\phi} = \int_{-\infty}^{\infty} \widehat{\psi}\,\phi$.*

Proof. Let $f(x) = \psi(x)e^{-is_0x}$ and $g(x,y) = e^{i(s_0-y)x}\phi(y)$. A simple computation shows $V_{[a,b]}g(\cdot,y) = O((b-a)y\phi(y))$ as $|y| \to \infty$. The conditions of Lemma 25(a) are satisfied. $\square$

Now we have the inversion theorem. The proof uses the method of summability kernels. Using Proposition 16, one inserts a summability kernel in the inversion integral. There is a parameter $z = x + iy$ that is sent to x_0, yielding inversion at x_0. We can actually let $z \to x_0$ in the upper complex plane, provided the approach is non-tangential. This is analogous to the Fatou theorem for boundary values of harmonic functions. Define the upper half plane by $\Pi_+ = \{z = x + iy : x \in \mathbb{R},\ y > 0\}$. We identify $\partial\Pi_+$ with $\mathbb{R}$. For $x_0 \in \partial\Pi_+$, we say $z \to x_0$ non-tangentially in Π_+ if $z \in \Pi_+$ and $z \to x_0$ such that $|x - x_0|/y \le C$ for some $C > 0$.

DEFINITION 17 (Summability kernel). A summability kernel is a function $\Theta : \mathbb{R} \to \mathbb{R}$ such that $\Theta \in L^1 \cap AC$, $\Theta(0) = 1$, $s \mapsto s\,\Theta(s)$ is in L^1, $\widehat{\Theta} \in L^1 \cap \mathcal{BV}$, $\int_{-\infty}^{\infty} \widehat{\Theta} = 2\pi$, $s \mapsto s\widehat{\Theta}\,'(s)$ is in L^1, and $x \mapsto V_{[x,\infty)}\widehat{\Theta}$ and $x \mapsto V_{(-\infty,-x]}\widehat{\Theta}$ are $O(1/x)$ as $x \to \infty$.

THEOREM 18 (Inversion). *Let $f : \mathbb{R} \to \mathbb{R}$ such that $\widehat{f}$ exists almost everywhere. Define $F(x) = \int_{x_0}^{x} f$ for $x_0 \in \mathbb{R}$. If $F'(x_0) = f(x_0)$ and $f = \widehat{f}^{\vee}$ exists at x_0 then $f(x_0) = \widehat{f}^{\vee}(x_0)$. If $\widehat{f}^{\vee}$ exists almost everywhere then $f = \widehat{f}^{\vee}$ almost everywhere.*

Proof. Let $z = x + iy$ for $x \in \mathbb{R}$ and $y > 0$. Define $\phi_z : \mathbb{R} \to \mathbb{R}$ by $\phi_z(s) = \Theta(ys)e^{isx}$, where Θ is a summability kernel. Then $\widehat{\phi_z}(t) = \widehat{\Theta}((t-x)/y)/y$.

And,

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\phi_z(s)\widehat{f}(s)\,ds = \frac{1}{2\pi}\int_{-\infty}^{\infty}\widehat{\phi_z}(t)f(t)\,dt \tag{12}$$

$$= \frac{1}{2\pi y}\int_{-\infty}^{\infty}\widehat{\Theta}((t-x)/y)f(t)\,dt. \tag{13}$$

The inversion theorem now follows, provided we can prove the following:

(I) *The conditions of Proposition 16 are satisfied so that* (12) *is valid.*
(II) *As* $z \to x_0$ *non-tangentially in* Π_+*, the left side of* (12) *becomes* $\widehat{f}^{\smile}(x_0)$.
(III) *As* $z \to x_0$ *non-tangentially in* Π_+*,* (13) *becomes* $f(x_0)$.

(I) In Proposition 16, let $\psi = f$ and $\phi = \phi_z$. We have existence of $\widehat{f}$ at some $s_0 \in \mathbb{R}$. And, ϕ_z and $s \mapsto s\phi_z(s)$ are in L^1 if and only if Θ and $s \mapsto s\,\Theta(s)$ are in L^1. Since $\Theta \in L^1$, $\widehat{\phi_z}$ is continuous with limit 0 at infinity. So, if $\widehat{\phi_z}$ is of bounded variation at infinity, the integral $\int_{-\infty}^{\infty} f\widehat{\phi_z}$ will exist. It suffices to have $\widehat{\Theta}$ of bounded variation at infinity. Proposition 16 now applies.

(II) Write the left side of (12) as $(2\pi)^{-1}\int_{-\infty}^{\infty}\left[\Theta(ys)e^{is(x-x_0)}\right]\left[e^{isx_0}\widehat{f}(s)\right]\,ds$. The function $s \mapsto e^{isx_0}\widehat{f}(s)$ is in $\mathcal{HK}$. And, we have $V[s \mapsto \Theta(ys)e^{is(x-x_0)}] \leq 2V\Theta + 2\|\Theta\|_1|x-x_0|/y$. So, for non-tangential approach, this function is of bounded variation, uniformly as $z \to x_0$. This allows us to take the limit inside the integral on the left side of (12), yielding $\widehat{f}^{\smile}(x_0)$.

(III) Let $\delta > 0$. Write

$$\begin{aligned} &\frac{1}{y}\int_{-\infty}^{\infty}\widehat{\Theta}\left(\frac{t-x}{y}\right)f(t)\,dt \\ &\quad = \frac{1}{y}\int_{|t-x|<\delta}\widehat{\Theta}\left(\frac{t-x}{y}\right)f(t)\,dt + \frac{1}{y}\int_{|t-x|>\delta}\widehat{\Theta}\left(\frac{t-x}{y}\right)f(t)\,dt. \end{aligned} \tag{14}$$

Consider the last integral in (14). There is $s_0 \in \mathbb{R}$ such that $t \mapsto e^{-is_0t}f(t)$ is in $\mathcal{HK}$. Now,

$$V_{[x+\delta,\infty)}\left[t \mapsto \frac{1}{y}\widehat{\Theta}\left(\frac{t-x}{y}\right)e^{is_0t}\right] \leq \frac{2}{y}V_{[\delta/y,\infty)}\widehat{\Theta} + 2|s_0|\|\widehat{\Theta}\|_1.$$

With our assumptions on $\widehat{\Theta}$, this last expression is bounded as $z \to x_0$. And, when $\Theta \in L^1$, $\Theta \in AC_{\mathrm{loc}}$ and $\Theta' \in L^1$ then $\widehat{\Theta}(t) = o(1/t)$ as $t \to \infty$ [2, page 20]. The same applies on the interval $(-\infty, x-\delta]$. Hence, taking the limit $z \to x_0$ inside the integral yields 0 for each fixed $\delta > 0$.

Treat the first integral on the right side of (14) as follows. Because

$$\frac{1}{2\pi y}\int_{x-\delta}^{x+\delta}\widehat{\Theta}((t-x)/y)\,dt = \frac{1}{2\pi}\int_{-\delta/y}^{\delta/y}\widehat{\Theta}(t)\,dt \to 1$$

as $y \to 0^+$, we can assume $f(x_0) = 0$ (otherwise replace $f(\cdot)$ with $f(\cdot) - f(x_0)$). Let $F(t) = \int_{x_0}^t f$. We have $F'(x_0) = f(x_0) = 0$. And, $\widehat{\Theta}(s) = o(1/s)$ as $|s| \to \infty$. Given $\epsilon > 0$, we can take $0 < \delta < 1$ small enough such that $|F(x_0 + t)| \le \epsilon |t|$ and $|\widehat{\Theta}(1/t)| \le \epsilon |t|$ for all $0 < |t| \le 2\delta$. Without loss of generality, assume $x \ge x_0$. Take $|z - x_0| \le \delta$ with $|x - x_0|/y \le C$ for some constant $C > 0$. Integrate by parts,

$$\begin{aligned}(15)\qquad & \frac{1}{y}\int_{x-\delta}^{x+\delta} \widehat{\Theta}\left(\frac{t-x}{y}\right) f(t)\,dt \\ &\qquad = \frac{1}{y}\left[\widehat{\Theta}\left(\frac{\delta}{y}\right) F(x+\delta) - \widehat{\Theta}\left(-\frac{\delta}{y}\right) F(x-\delta)\right] - J_1 - J_2 - J_3,\end{aligned}$$

where $J_1 = y^{-2}\int_{x-\delta}^{x_0} \widehat{\Theta}'((t-x)/y)F(t)\,dt$, $J_2 = y^{-2}\int_{x_0}^{x} \widehat{\Theta}'((t-x)/y)F(t)\,dt$, $J_3 = y^{-2}\int_{x}^{x+\delta} \widehat{\Theta}'((t-x)/y)F(t)\,dt$. Note that if $0 < y \le \delta^2$ then $y/\delta \le \delta$ and $|x \pm \delta - x_0| < 2\delta$ so $|\widehat{\Theta}(\pm\delta/y)F(x \pm \delta)|/y \le 2\epsilon^2$.

Estimate J_1 by writing

$$\begin{aligned}(16)\qquad |J_1| &\le \frac{1}{y^2}\int_{x-\delta}^{x_0} (x_0 - t)\left|\widehat{\Theta}'\left(\frac{t-x}{y}\right)\right| \left|\frac{F(t)}{x_0 - t}\right| dt \\ &\le \frac{\epsilon}{y}\int_{-\delta/y}^{(x_0-x)/y} (x_0 - x - yt)\left|\widehat{\Theta}'(t)\right| dt \\ &\le \epsilon C\, V\widehat{\Theta} + \epsilon\int_{-\infty}^{0} |t|\left|\widehat{\Theta}'(t)\right| dt.\end{aligned}$$

Similarly,

$$(17)\qquad |J_3| \le \epsilon C\, V\widehat{\Theta} + \epsilon\int_0^\infty t\left|\widehat{\Theta}'(t)\right| dt.$$

For J_2 we have

$$\begin{aligned}(18)\qquad |J_2| &\le \frac{1}{y}\sup_{x_0 \le t \le x} |F(t)| \int_{(x_0-x)/y}^{0} \left|\widehat{\Theta}'(t)\right| dt \\ &\le \epsilon C\, V\widehat{\Theta}.\end{aligned}$$

Putting (16), (18) and (17) into (15) now shows that the first integral on the right side of (14) goes to 0 as $z \to x_0$ non-tangentially. This completes the proof of part (III). Since $F' = f$ almost everywhere, the proof of the theorem is now complete. □

REMARK 19. In place of the condition $t \mapsto t\,\widehat{\Theta}\,'(t)$ is in L^1 we can demand that $\widehat{\Theta}$ is increasing on $(-\infty, 0)$ and decreasing on $(0, \infty)$. The proof of (III) then follows with minor changes. The condition that $\Theta \in AC$ can also be weakened.

REMARK 20. The most commonly used summability kernels are:

$\Theta_1(x) = (1-\lvert x\rvert)\chi_{[-1,1]}(x)$	$\widehat{\Theta}_1(s) = \left[\frac{\sin(s/2)}{s/2}\right]^2$	Cesàro–Fejér
$\Theta_2(x) = e^{-\lvert x\rvert}$	$\widehat{\Theta}_2(s) = \frac{2}{1+s^2}$	Abel–Poisson
$\Theta_3(x) = e^{-x^2}$	$\widehat{\Theta}_3(s) = \sqrt{\pi}\, e^{-(s/2)^2}$	Gauss–Weierstrass.

The Abel and Gauss kernels are summability kernels according to Definition 17, while the Cesàro kernel does not satisfy this definition.

COROLLARY 21. *Let $f:\mathbb{R}\to\mathbb{R}$. Then $f=0$ almost everywhere if and only if $\widehat{f}=0$ almost everywhere.*

Proof. If $f=0$ almost everywhere then $\widehat{f}=0$ on $\mathbb{R}$. If $\widehat{f}=0$ almost everywhere then $\widehat{f}$ exists almost everywhere and $\widehat{f}^{\vee}$ exists almost everywhere. Therefore, by the Theorem, $\widehat{f}^{\vee}=f=0$, almost everywhere. $\square$

Note that the inversion theorem applies to Example 3(a)-(d). The condition that $\widehat{f}^{\vee}$ exists almost everywhere cannot be dropped. The following example shows that existence of $\widehat{f}$ on $\mathbb{R}$ does not guarantee existence of $\widehat{f}^{\vee}$ at any point in $\mathbb{R}$.

EXAMPLE 22. Let $f(x)=x^{\alpha}e^{ix^{\nu}}$ for $x\geq 0$ and $f(x)=0$ for $x<0$. Using the method of Lemma 23 we see that $\widehat{f}$ exists on $\mathbb{R}$ for $-1<\alpha<\nu-1$. And,

$$\begin{aligned}\widehat{f}(s) &= \int_0^{\infty} x^{\alpha}e^{i[x^{\nu}-sx]}\,dx \\ &= s^{\frac{\alpha+1}{\nu-1}}\int_0^{\infty} x^{\alpha}e^{ip[x^{\nu}-x]}\,dx \quad \left(p=s^{\nu/(\nu-1)}\right).\end{aligned} \tag{19}$$

Write $\phi(x)=x^{\nu}-x$. If $\nu>1$ then ϕ has a minimum at $x_0:=\nu^{-1/(\nu-1)}$. The method of stationary phase [7] shows that

$$\widehat{f}(s)\sim\sqrt{\tfrac{2\pi}{\nu(\nu-1)}}\,e^{i\pi/4}x_0^{\alpha-(\nu-2)/2}e^{i\phi(x_0)s^{\nu/(\nu-1)}}s^{\frac{2\alpha+2-\nu}{2(\nu-1)}}$$

as $s\to\infty$. Let $\nu>2$. It now follows from Lemma 23 that when $\nu/2\leq\alpha<\nu-1$, $\widehat{f}$ exists on $\mathbb{R}$ and $\widehat{f}^{\vee}$ diverges at each point of $\mathbb{R}$. Note that $f\in\mathcal{HK}$ but neither f nor $\widehat{f}$ is in any L^p space $(1\leq p\leq\infty)$.

5. Appendix

LEMMA 23. *If $\gamma>0$ and $\delta\in\mathbb{R}$ then:*

(a) *$\int_0^1 e^{ix^{-\gamma}}x^{\delta}dx$ exists in $\mathcal{HK}$ if and only if $\gamma+\delta+1>0$. The integral exists in L^1 if and only if $\delta>-1$.*

(b) *$\int_1^{\infty} e^{ix^{\gamma}}x^{\delta}dx$ exists in $\mathcal{HK}$ if and only if $\gamma>\delta+1$. The integral exists in L^1 if and only if $\delta<-1$.*

Proof. In (a), integrate by parts to get

$$\int_0^1 e^{ix^{-\gamma}} x^\delta dx = \frac{i}{\gamma}\left[e^i - \lim_{x\to 0^+} e^{ix^{-\gamma}} x^{\gamma+\delta+1}\right] - \frac{i(\gamma+\delta+1)}{\gamma}\int_0^1 e^{ix^{-\gamma}} x^{\gamma+\delta} dx.$$

The limit exists if and only if $\gamma+\delta+1>0$, the last integral then converging absolutely. Case (b) is similar. For L^1 convergence, we simply take the absolute value of each integrand. □

LEMMA 24. *Let $[a,b]\subset\overline{\mathbb{R}}$ and let $f\in\mathcal{HK}_{[a,b]}$ and $g\in\mathcal{BV}_{[a,b]}$. Then*

$$\left|\int_a^b fg\right| \le \left|\int_a^b f\right| \inf_{[a,b]}|g| + \|f\|_{[a,b]}V_{[a,b]}g.$$

Proof. Given $\epsilon>0$, take $c\in[a,b]$ such that $|g(c)|\le\epsilon+\inf_{[a,b]}|g|$. Integrate by parts:

$$\begin{aligned}\int_a^b fg &= \int_a^c fg + \int_c^b fg \\ &= g(c)\int_a^b f - \int_a^c\left(\int_a^x f\right)dg(x) + \int_c^b\left(\int_x^b f\right)dg(x).\end{aligned}$$

And,

$$\begin{aligned}\left|\int_a^b fg\right| &\le \left[\epsilon+\inf_{[a,b]}|g|\right]\left|\int_a^b f\right| + \sup_{a\le x\le c}\left|\int_a^x f\right| V_{[a,c]}g \\ &\quad + \sup_{c\le x\le b}\left|\int_x^b f\right| V_{[c,b]}g \\ &\le \left[\epsilon+\inf_{[a,b]}|g|\right]\left|\int_a^b f\right| + \|f\|_{[a,b]}V_{[a,b]}g.\end{aligned}$$ □

This lemma is an extension of inequalities proved in [8] and [4] (Theorem 45, page 36). Changing g on a set of measure 0, such as a singleton, does not affect the integral of fg but can make the infimum of $|g|$ equal to zero. However, this reduction in $\inf|g|$ is reflected by a corresponding increase in Vg. This redundancy can be eliminated by replacing g with its normalised version, i.e., for each $x\in[a,b)$ replace $g(x)$ with $\lim_{t\to x+} g(t)$ and redefine $g(b)=0$. Then the inequality becomes $\left|\int_a^b fg\right| \le \|f\|_{[a,b]}V_{[a,b]}g$.

The following lemma on interchange of iterated integrals is an extension of Theorem 57 on page 58 of [4].

LEMMA 25. *Let $f \in \mathcal{HK}$ and let $g:\mathbb{R}^2 \to \mathbb{R}$. Let $\mathcal{M}$ denote the measurable subsets of $\mathbb{R}$. For each $(A,B) \in \mathcal{BV} \times \mathcal{M}$, define the iterated integrals*

$$I_1(A,B) = \int_{x\in A}\int_{y\in B} f(x)g(x,y)\,dy\,dx,$$
$$I_2(A,B) = \int_{y\in B}\int_{x\in A} f(x)g(x,y)\,dx\,dy.$$

(a) *Assume that for each compact interval $I \subset \mathbb{R}$ there are constants $M_I > 0$ and $K_I > 0$ such that $\int_{\mathbb{R}} V_I g(\cdot,y)\,dy \le M_I$ and, for all $x \in I$, $\|g(x,\cdot)\|_1 \le K_I$. If I_1 exists on $\mathbb{R}\times\mathbb{R}$ then I_2 exists on $\mathcal{BV}\times\mathcal{M}$ and $I_1 = I_2$ on $\mathcal{BV}\times\mathcal{M}$.*

(b) *Assume there exist $M, G \in L^1$ such that, for almost all $y \in \mathbb{R}$, $Vg(\cdot,y) \le M(y)$ and, for all $x \in \mathbb{R}$, $|g(x,y)| \le G(y)$. Then $I_1 = I_2$ on $\mathcal{BV}\times\mathcal{M}$.*

Proof. (a) Let $\mathcal{I}$ be the open intervals in $\mathbb{R}$. First prove $I_1 = I_2$ on $\mathcal{I}\times\mathcal{I}$. Fix (a,b) and (α,β) in I. For $-\infty < a < t < \infty$, define

$$H_a(t) = I_2((a,t),(\alpha,\beta)) = \int_\alpha^\beta\int_a^t f(x)g(x,y)\,dx\,dy. \tag{20}$$

We will establish the equality of I_1 and I_2 by appealing to the necessary and sufficient conditions for interchanging repeated integrals [12, Corollary 6]. For this, we need to show that H_a is in ACG_* and that we can differentiate under the integral sign in (20). Let $F(x) = \int_{-\infty}^x f$. Integrate by parts:

$$H_a(t) = [F(t)-F(a)]\int_\alpha^\beta g(t,y)\,dy - \int_\alpha^\beta\int_a^t [F(x)-F(a)]\,d_1g(x,y)dy. \tag{21}$$

The integrator of the Riemann–Stieltjes integral over $x \in [a,t]$ is denoted by $d_1g(x,y)$. Now, by Lemma 24, $|H_a(t)| \le \|f\|[K_{[a,b]} + M_{[a,b]}]$, and $I_2(A,B)$ exists for all $A, B \in \mathcal{I}$ with A bounded.

We have $F \in ACG_*(\mathbb{R})$. So, there are $E_n \subset \mathbb{R}$ such that $\mathbb{R} = \cup E_n$ and F is AC_* on each E_n, i.e., for each $n \ge 1$, given $\epsilon > 0$, there is $\delta > 0$ such that if (s_i,t_i) are disjoint with $s_i, t_i \in E_n$ and $\sum |s_i - t_i| < \delta$ then $\sum \|f\|_{(s_i,t_i)} < \epsilon$. Fix $n \ge 1$ with E_n, ϵ and δ as above. Suppose (σ_i,τ_i) are disjoint with $\sigma_i, \tau_i \in E_n$ and $\sum |\sigma_i - \tau_i| < \delta$. With no loss of generality, we may assume E_n is a subset of a compact interval $[c,d]$. Then

$$\sup_{[p,q]\subset[\sigma_i,\tau_i]} |H_a(p) - H_a(q)| \le \|f\|_{[\sigma_i,\tau_i]}\left[K_{[c,d]} + M_{[c,d]}\right].$$

It follows that $H_a \in ACG_*(\mathbb{R})$.

We now show that we can differentiate under the integral sign to compute $H_a'(t)$. Let $0<|h|<1$ and $t\in\mathbb{R}$ such that $F'(t)=f(t)$. Then

$$\left|\frac{1}{h}\int_t^{t+h} f(x)g(x,y)\,dx\right| \le \sup_{0<|h|<1}\left|\frac{F(t+h)-F(t)}{h}\right| |g(t,y)| + \sup_{0<|h|<1}\left|\frac{1}{h}\right| \|f\|_{[t-|h|,t+|h|]} V_{[t-1,t+1]}g(\cdot,y).$$

It now follows from dominated convergence that $H_a'(t)=f(t)\int_\alpha^\beta g(t,y)\,dy$ for almost all $t\in\mathbb{R}$. And, by [12, Corollary 6], $I_1(A,B)=I_2(A,B)$ for all $A,B\in\mathcal{I}$ with A bounded.

By assumption, $I_1(\mathbb{R},\mathbb{R})$ exists. For $a\in\mathbb{R}$,

$$\begin{aligned}\int_a^\infty f(x)\int_\alpha^\beta g(x,y)\,dy\,dx &= \lim_{t\to\infty}\int_a^t f(x)\int_\alpha^\beta g(x,y)\,dy\,dx\\ &= \lim_{t\to\infty}\int_\alpha^\beta\int_a^t f(x)g(x,y)\,dx\,dy\\ &= \lim_{t\to\infty} H_a(t).\end{aligned}$$

Similarly, $\lim_{t\to-\infty} H_a(t)$ exists. Therefore, $H_{-\infty}$ is continuous on $\overline{\mathbb{R}}$ and hence in $ACG_*(\overline{\mathbb{R}})$. It follows from [12, Corollary 6] that $I_1(A,B)=I_2(A,B)$ for all $A,B\in\mathcal{I}$.

We have equality of I_1 and I_2 on $\mathcal{BV}\times\mathcal{M}$ upon replacing f with $f\chi_A$ and $g(x,\cdot)$ with $g(x,\cdot)\chi_B$ where $A\in\mathcal{BV}$ and $B\in\mathcal{M}$.

(b) This is similar to part (a), but now the conditions on g ensure the existence of I_2 on $\mathbb{R}\times\mathbb{R}$. As in (a), $H_a\in ACG_*(\mathbb{R})$. To show H_a is continuous on $\overline{\mathbb{R}}$, note that $|\int_\alpha^\beta\int_a^t f(x)g(x,y)\,dx\,dy| \le \|f\|(\|G\|_1+\|M\|_1)$ and $\lim_{t\to\infty}\int_a^t f(x)g(x,y)\,dx$ exists for almost all $y\in\mathbb{R}$. Whence, $\lim_{t\to\infty}H_a(t)$ exists and $H_{-\infty}$ is continuous on $\overline{\mathbb{R}}$. Using [12, Corollary 6], we now have equality of I_1 and I_2 on $\mathbb{R}\times\mathbb{R}$ and hence on $\mathcal{BV}\times\mathcal{M}$. □

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