Illinois Journal of Mathematics
Volume 51, Number 2, Summer 2007, Pages 513–536
S 0019-2082

THE DIMENSIONS OF LIMITS OF VERTEX REPLACEMENT RULES

MICHELLE PREVITE

Abstract. Given an initial graph G, one may apply a rule $\mathcal{R}$ to G which replaces certain vertices of G with other graphs called replacement graphs to obtain a new graph $\mathcal{R}(G)$. By iterating this procedure on each resulting graph, a sequence of graphs $\{\mathcal{R}^n(G)\}$ is obtained. When the graphs in this sequence are normalized to have diameter one, questions of convergence can be investigated. Sufficient conditions for convergence in the Gromov-Hausdorff metric were given by J. Previte, M. Previte, and M. Vanderschoot for such normalized sequences of graphs when the replacement rule $\mathcal{R}$ has more than one replacement graph. M. Previte and H.S. Yang showed that under these conditions, the limits of such sequences have topological dimension one. In this paper, we compute the box and Hausdorff dimensions of limit spaces of normalized sequences of iterated vertex replacements when there is more than one replacement graph. Since the limit spaces have topological dimension one and typically have Hausdorff (and box) dimension greater than one, they are fractals. Finally, we give examples of vertex replacement rules that yield fractals.

1. Introduction

The notion of a vertex replacement rule was motivated by studying geodesic flows on two-dimensional singular spaces of nonpositive curvature (see [1]). The work in this paper is also related to a class of iterative systems, introduced by Aristid Lindenmayer (see [12] and [13]), which is used to model the growth of plants and simple multicellular organisms. Lindenmayer systems were later used in the areas of data and image compression. Since vertex replacement rules are more natural and geometric, they promise applications in the same fields that Lindenmayer impacted.

A vertex replacement rule $\mathcal{R}$ is a rule for substituting copies of finite graphs (called replacement graphs) for certain vertices in a given graph G. The result is a new graph $\mathcal{R}(G)$. Iterating $\mathcal{R}$ produces a sequence of graphs $\{\mathcal{R}^n(G)\}$.

Received May 9, 2005; received in final form June 29, 2005.
2000 *Mathematics Subject Classification.* 28A99, 05C55, 00A73, 28A80.

By letting $(\mathcal{R}^n(G), 1)$ be the metric space $\mathcal{R}^n(G)$ normalized to have diameter 1, the sequence of the normalized graphs can be studied.

Vertex replacement rules with one replacement graph were examined in [8] and [10]. Necessary and sufficient conditions were found for the sequence $\{(\mathcal{R}^n(G), 1)\}$ to converge in the Gromov-Hausdorff metric. Since the limit spaces have topological dimension one and, except for special cases, Hausdorff (and box) dimension greater than one, they are fractals.

Vertex replacement rules with at least two replacement graphs were studied in [9] and [11]. Sufficient conditions were found for $\{(\mathcal{R}^n(G), 1)\}$ to converge in the Gromov-Hausdorff metric. As in the case when the replacement rule has only one replacement graph, these limit spaces also have topological dimension one. We examine the box and Hausdorff dimensions of limits of such vertex replacement rules. The key results (Theorems 4.3 and 5.4) give formulas for each dimension. The final section provides examples of fractals which are the limits of replacement rules with more than one replacement graph.

2. Vertex replacement rules

In this section we define and provide some basic examples of vertex replacements. Throughout this paper we will assume that all graphs are connected, finite, unit metric graphs, i.e., each graph is a metric space and every edge has length one. In particular, the distance between two points in a graph will be measured by the shortest path in the graph between the two points.

Definition 2.1. A graph H with a designated set of vertices $\{v_1, \ldots, v_k\}$ is called *symmetric about* $\{v_1, \ldots, v_k\}$ if every permutation of $\{v_1, \ldots, v_k\}$ can be realized by an isometry of H. The vertices in such a designated set are called *boundary vertices* of H and are denoted by ∂H.

Definition 2.2. A *vertex replacement rule* $\mathcal{R}$ consists of a finite list of finite graphs (called *replacement graphs*) $\{H_1, \ldots, H_p\}$, each with a (symmetric) set ∂H_i of boundary vertices, so that $|\partial H_i| \neq |\partial H_j|$ for $i \neq j$, where $|\cdot|$ denotes the cardinality of a set.

Let G be a graph and let $\mathcal{R}$ be a vertex replacement rule given by the replacement graphs $H_1, \ldots, H_p$. Recall that the degree of a vertex v in G, denoted $\deg(v)$, is the number of edges in G adjacent to v.

Definition 2.3. A vertex v in G is called *replaceable* if $\deg(v) = |\partial H_i|$ for some replacement graph H_i in the replacement rule.

The replacement rule $\mathcal{R}$ acts on G by substituting each replaceable vertex in G with its corresponding replacement graph so that the $\deg(v)$ edges previously attached to v in G are attached to the $|\partial H_i|$ vertices of H_i. Since $|\partial H_i| \neq |\partial H_j|$ for $i \neq j$, each replaceable vertex has a unique corresponding replacement graph. Also, since each replacement graph H_i is symmetric about ∂H_i, it is irrelevant how the edges previously adjacent to v are attached to H_i. Thus, vertex replacement is a well defined procedure.

For example, we may define a vertex replacement rule $\mathcal{R}$ by the replacement graphs H_1 and H_2 depicted in Figure 1. The boundary vertices of the

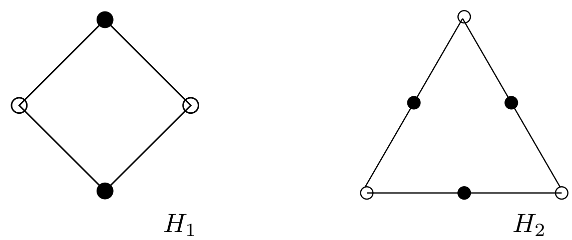

FIGURE 1. A replacement rule $\mathcal{R}$.

replacement graphs are shown with circles. Note that each replacement graph is symmetric about its set of boundary vertices. Let G be as depicted in Figure 2. Vertices w_1, w_2, and w_3 are replaceable by H_1, and vertices v_1, v_2,

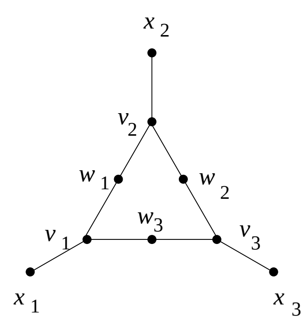

FIGURE 2. A graph G.

and v_3 are replaceable by H_2, but vertices x_1, x_2, and x_3 are not replaceable. Figure 3 shows $\mathcal{R}(G)$.

We extend the idea of a replaceable vertex to include the vertices of the replacement graphs themselves, but only after the replacement graphs have replaced some vertices. That is, one should not treat a replacement graph H_i as an initial graph G, but always view it as having already replaced some vertex. Hence we view each boundary vertex as having another edge attached.

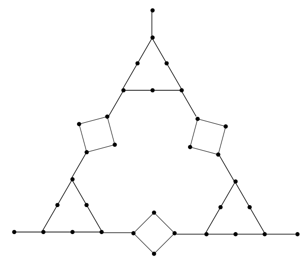

FIGURE 3. The graph $\mathcal{R}(G)$.

DEFINITION 2.4. A boundary vertex v is called *replaceable* if $\deg(v) = |\partial H_i| - 1$ for some replacement graph H_i in the replacement rule.

Notice that for the replacement rule in Figure 1, the boundary vertices of H_1 are replaceable by H_2 (each such vertex will have 3 edges adjacent after being inserted into a graph G) while the remaining vertices of H_1 are replaceable by H_1. Likewise, the boundary vertices of H_2 are replaceable by H_2 while the remaining vertices of H_2 are replaceable by H_1. Thus the replacement rule $\mathcal{R}$ may be iterated to create a sequence of graphs $\{\mathcal{R}^n(G)\}$. When each graph in this sequence is scaled to have diameter one, we obtain the sequence $\{(\mathcal{R}^n(G), 1)\}$ which, according to Theorem 3.3, converges in the Gromov-Hausdorff metric. Figure 4 shows the next two graphs in the sequence and the limit space of this sequence.

We now select some notation. There exists a pointwise map $\pi : \mathcal{R}(G) \to G$ which undoes replacement by crushing the inserted replacement graphs to the vertices they replaced. In general, for any set F in G, let $\mathcal{R}(F)$ be $\pi^{-1}(F)$. If $F \subset G$ contains no replaceable vertices, then $\mathcal{R}^n(F)$ can be identified with F and we label $\mathcal{R}^n(F)$ as $F \subset \mathcal{R}^n(G)$. Similarly, if $z \in G$ is not replaceable, label $\mathcal{R}^n(z)$ as $z \in \mathcal{R}^n(G)$. Observe that the inverse map π is only well defined when one is also given a specific replacement rule $\mathcal{R} : G \to \mathcal{R}(G)$. Otherwise, given a graph F, there might be two different graphs G_1 and G_2 such that $\mathcal{R}(G_1) = F = \mathcal{R}(G_2)$, and thus, two different inverse maps $\pi_1 : F \to G_1$ and $\pi_2 : F \to G_2$. For a finite graph F, let $N_i(F)$ denote the number of vertices in F which are replaceable by H_i, and let $N(F)$ be the total number of replaceable vertices in F. That is, for a replacement rule with p replacement graphs, $N(F) = \sum_{i=1}^{p} N_i(F)$. For a replacement graph H_i, we define $N_j(H_i)$ to be the number of vertices in H_i replaceable by H_j when one regards H_i as a subset of $\mathcal{R}(G)$. That is, $N_j(H_i)$ is the number of vertices v in H_i such that $\deg(v) = |\partial H_j| - 1$ if v is a boundary vertex or $\deg(v) = |\partial H_j|$ if v is not a boundary vertex.

Let H_i be a replacement graph in a replacement rule $\mathcal{R}$ and let v_i be a vertex in a graph G which is replaceable by H_i. Define the set $\partial\mathcal{R}^n(v_i)$ to be all vertices $w \in \mathcal{R}^n(v_i)$ that are adjacent to one of the $\deg(v_i)$ edges

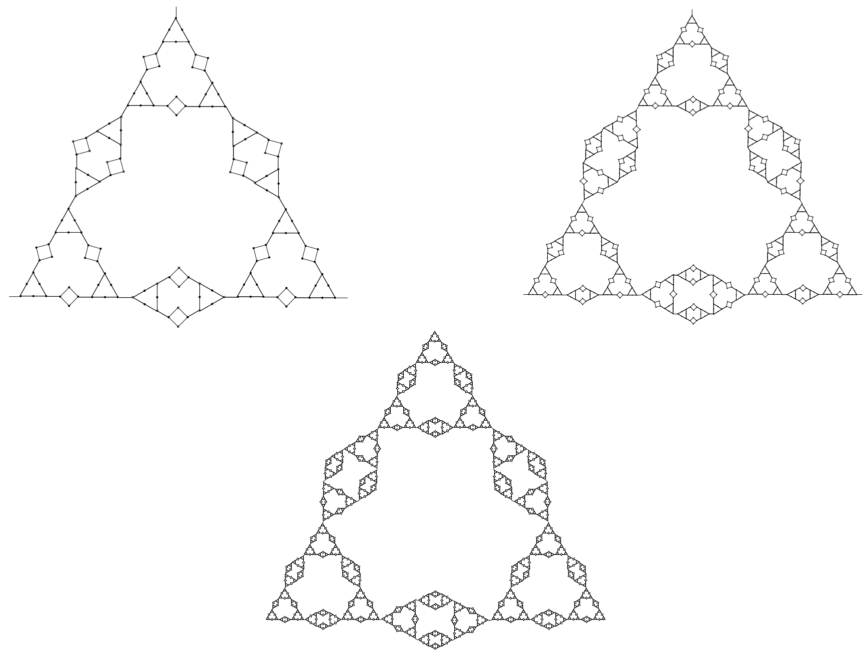

FIGURE 4. $(\mathcal{R}^2(G), 1)$, $(\mathcal{R}^3(G), 1)$, and the limit of $\{(\mathcal{R}^n(G), 1)\}$.

outside of $\mathcal{R}^n(v_i)$ that were adjacent to $v_i \in G$. So $\partial\mathcal{R}^n(v_i)$ is the set of possible vertices through which a path in $\mathcal{R}^n(G)$ must pass when entering or exiting $\mathcal{R}^n(v_i) \subset \mathcal{R}^n(G)$. Note that $|\partial\mathcal{R}^n(v_i)| = |\partial H_i|$. For example, if $\mathcal{R}$ is the replacement rule given in Figure 1 and w_3 is as in Figure 2, then Figure 5 depicts the two vertices in $\partial\mathcal{R}^3(w_3)$ with circles. To determine the growth of $\text{diam}(\mathcal{R}^n(G))$ (and the growth in complexity of $(\mathcal{R}^n(G), 1)$), we need to measure the distance between points in $\partial\mathcal{R}^n(v_i)$. Hence we define the function

$$a_i(n) = \text{dist}_{\mathcal{R}^n(v_i)}(u, u'),$$

where $u, u' \in \partial\mathcal{R}^n(v_i)$ for $u \neq u'$. We also define the function

$$b_i(n) = \sup_{z \in \mathcal{R}^n(v_i)} \{\text{dist}_{\mathcal{R}^n(v_i)}(u, z) \mid u \in \partial\mathcal{R}^n(v_i)\}.$$

By the symmetry of each H_i about ∂H_i, the above definitions are independent of the choices of u and u' in $\partial\mathcal{R}^n(v_i)$. Clearly $a_i(n) \leq b_i(n)$. Let

$$a_{\max}(n) = \max_{i=1,\ldots,p} a_i(n), \quad a_{\min}(n) = \min_{i=1,\ldots,p} a_i(n),$$
$$b_{\max}(n) = \max_{i=1,\ldots,p} b_i(n), \quad b_{\min}(n) = \min_{i=1,\ldots,p} b_i(n).$$

.

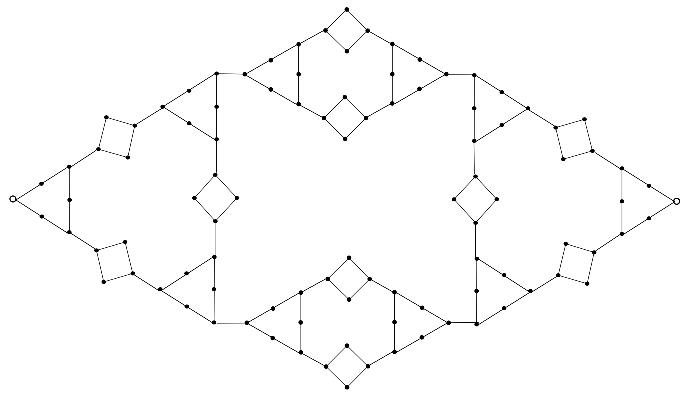

FIGURE 5. The graph $\mathcal{R}^3(w_3)$.

DEFINITION 2.5. A path σ in a replacement graph is called a *simple boundary connecting path* if σ is a simple path with boundary vertices for endpoints and no boundary vertices on its interior.

For each n, there is a path in $\mathcal{R}^n(v_i)$ that realizes $a_i(n)$ and projects via π^{n-1} to a simple boundary connecting path $\sigma_i(n)$ in H_i. It is extremely difficult to combinatorially determine the $\sigma_i(n)$ given an arbitrary replacement rule $\mathcal{R}$ since, in general, $\sigma_i(n) \neq \sigma_i(m)$ for $n \neq m$. Hence, we restrict to simple replacement rules.

For a simple path γ, let $L(\gamma)$ denote the length of γ.

DEFINITION 2.6. A replacement rule $\mathcal{R}$ given by the graphs $H_1, \ldots, H_p$ is *simple* if for all $i = 1, \ldots, p$ and any pair of simple boundary connecting paths σ_1 and σ_2 in H_i, we have $L(\sigma_1) = L(\sigma_2)$ and $N_j(\sigma_1) = N_j(\sigma_2)$ for all $j = 1, \ldots, p$.

DEFINITION 2.7. Let $\mathcal{R}$ be a replacement rule given by the graphs H_1, $\ldots, H_p$ and let $\{\sigma_1, \ldots, \sigma_p\}$ be a set of simple boundary connecting paths, where $\sigma_i \subset H_i$ for all $i = 1, \ldots, p$. We call the matrix

$$A = \begin{bmatrix} N_1(\sigma_1) & \cdots & N_p(\sigma_1) \\ \vdots & \ddots & \vdots \\ N_1(\sigma_p) & \cdots & N_p(\sigma_p) \end{bmatrix}$$

a *path matrix* of $\mathcal{R}$. If every path matrix A of $\mathcal{R}$ is primitive, i.e., A^k has only positive entries for some power k, then $\mathcal{R}$ is called *primitive*.

In the case where a replacement graph H_i has only one boundary vertex (and hence it has no path between distinct boundary vertices), we have

$L(\sigma_i) = 0$ and the row $N_1(\sigma_i), \ldots, N_p(\sigma_i)$ in the path matrix A above is either a row of zeros (when the boundary vertex of H_i is nonreplaceable) or else a row in which all but the jth entry is a zero (when the boundary vertex of H_i is replaceable by H_j).

The replacement rule $\mathcal{R}$ in Figure 1 is simple and primitive. So if σ_1 and σ_2 are simple boundary connecting paths in H_1 and H_2, respectively, then the path matrix of $\mathcal{R}$ is

$$A = \begin{bmatrix} N_1(\sigma_1) & N_2(\sigma_1) \\ N_1(\sigma_2) & N_2(\sigma_2) \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}.$$

Of course not every replacement rule is simple and primitive. Figure 6 shows an example of a replacement rule which is simple but not primitive since its path matrix is $\left[\begin{smallmatrix} 0 & 2 \\ 2 & 0 \end{smallmatrix}\right]$. The replacement rule in Figure 7 is not simple because there are two simple boundary connecting paths in H_1 which give rise to two different path matrices: $\left[\begin{smallmatrix} 1 & 2 \\ 1 & 2 \end{smallmatrix}\right]$ comes from taking the upper route between the boundary vertices of H_1 and $\left[\begin{smallmatrix} 0 & 3 \\ 1 & 2 \end{smallmatrix}\right]$ comes from taking the lower route between the boundary vertices of H_1.

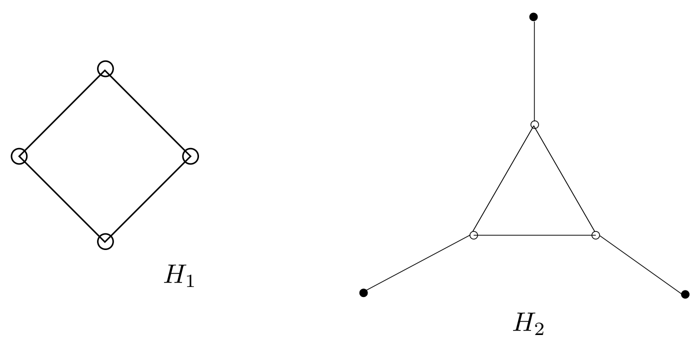

FIGURE 6. A replacement rule which is simple but not primitive.

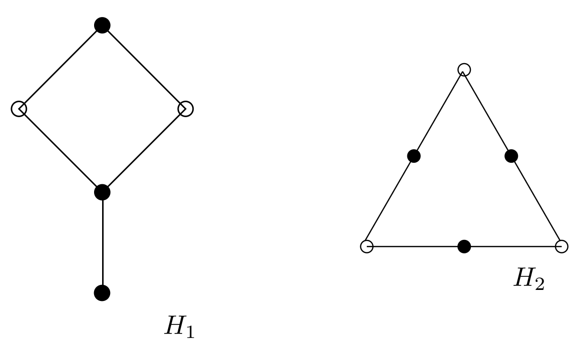

FIGURE 7. A replacement rule which is not simple.

Another important matrix associated with a replacement rule $\mathcal{R}$ is called the replaceable vertex matrix of $\mathcal{R}$. This matrix allows us to compute the total number of replaceable vertices in $\mathcal{R}^n(G)$ for any initial graph G.

Definition 2.8. Let $\mathcal{R}$ be a replacement rule given by the replacement graphs $H_1, \ldots, H_p$. The *replaceable vertex matrix* of $\mathcal{R}$ is defined as

$$M = \begin{bmatrix} N_1(H_1) & \cdots & N_p(H_1) \\ \vdots & \ddots & \vdots \\ N_1(H_p) & \cdots & N_p(H_p) \end{bmatrix}.$$

3. Convergence results

Before stating the convergence results, let us recall some facts about the Gromov-Hausdorff metric. For any metric space X, dist_X will denote the metric on X. Let Z be a metric space. For $C \subset Z$ and $\epsilon > 0$, let $C_\epsilon = \{z \in Z : \mathrm{dist}_Z(z, C) < \epsilon\}$.

Definition 3.1. The *Hausdorff distance* between two nonempty compact subsets A and B of a metric space Z is defined by

$$\mathrm{dist}_Z^{\mathrm{Haus}}(A, B) = \inf\{\epsilon > 0 : A \subseteq B_\epsilon \text{ and } B \subseteq A_\epsilon\}.$$

The Hausdorff distance defines a metric on the set of all compact subsets of Z.

We are now able to define the Gromov-Hausdorff distance. Informally, when measuring the Gromov-Hausdorff distance between spaces X and X', we place X and X' into some space in such a way that they are as close together as possible and then measure the resulting Hausdorff distance. Let $\mathcal{S}$ denote the collection of all isometry classes of compact metric spaces.

Definition 3.2. The *Gromov-Hausdorff distance* between two compact metric spaces X and X' is defined by

$$\mathrm{dist}_{\mathcal{S}}^{GH}(X, X') = \inf_{\substack{Z \in \mathcal{S} \\ I, J}} \{\epsilon > 0 : \mathrm{dist}_Z^{\mathrm{Haus}}(I(X), J(X')) < \epsilon\},$$

where I and J are isometric embeddings of X and X' into Z, respectively.

The space $(\mathcal{S}, \mathrm{dist}_{\mathcal{S}}^{GH})$ is a complete metric space. Moreover, $\mathrm{dist}_{\mathcal{S}}^{GH}(X, X') = 0$ if and only if X is isometric to X'. (See [5].)

For a finite graph G, let $(\mathcal{R}^n(G), 1)$ be the metric space $\mathcal{R}^n(G)$ normalized to have diameter 1, i.e., every edge in $(\mathcal{R}^n(G), 1)$ has length $1/\operatorname{diam}(\mathcal{R}^n(G))$.

Theorem 3.3 ([9]). *Let $H_1, \ldots, H_p$ define a simple, primitive vertex replacement rule $\mathcal{R}$ with $p \geq 2$ and let G be a finite graph with at least one replaceable vertex. Then the normalized sequence $\{(\mathcal{R}^n(G), 1)\}$ converges in the Gromov-Hausdorff metric.*

It should be noted that Theorem 3.3 extends to primitive replacement rules which are eventually simple (see [9]), but to streamline the arguments, this paper will discuss only simple replacement rules.

We now define convenient sets of points in a graph G which we will use to construct nets for $(\mathcal{R}^n(G), 1)$. The nets will be used in the proofs of the formulas for both the box and Hausdorff dimensions. Recall the definition of an ϵ-net.

Definition 3.4. Let $\epsilon > 0$. A finite subset S of a metric space X is an ϵ*−net* of X if $X = S_\epsilon := \{x \in X : \operatorname{dist}_X(x, S) < \epsilon\}$.

For any finite graph G, define $\Delta_0(G)$ to be the set of all midpoints of edges that are adjacent to replaceable vertices in G. Clearly $x \in \Delta_0(G)$ is not a vertex, and hence not replaceable. So for $n \geq 0$, we can identify $x \in \Delta_0(G) \subset G$ with $\mathcal{R}^n(x) \in \mathcal{R}^n(G)$. For notational purposes, we write $\mathcal{R}^n(x)$ as $x(n)$ and $\mathcal{R}^n(\Delta_0(G))$ as $\Delta_n(G)$. Note that $|\Delta_n(G)| = |\Delta_0(G)|$ for all n. For example, if the replacement rule $\mathcal{R}$ is as in Figure 1 and initial graph G is as in Figure 2, then Figure 8 points out the elements of $\Delta_0(G) \subset (G, 1), \Delta_1(G) \subset (\mathcal{R}(G), 1)$ and $\Delta_2(G) \subset (\mathcal{R}^2(G), 1)$ with arrows.

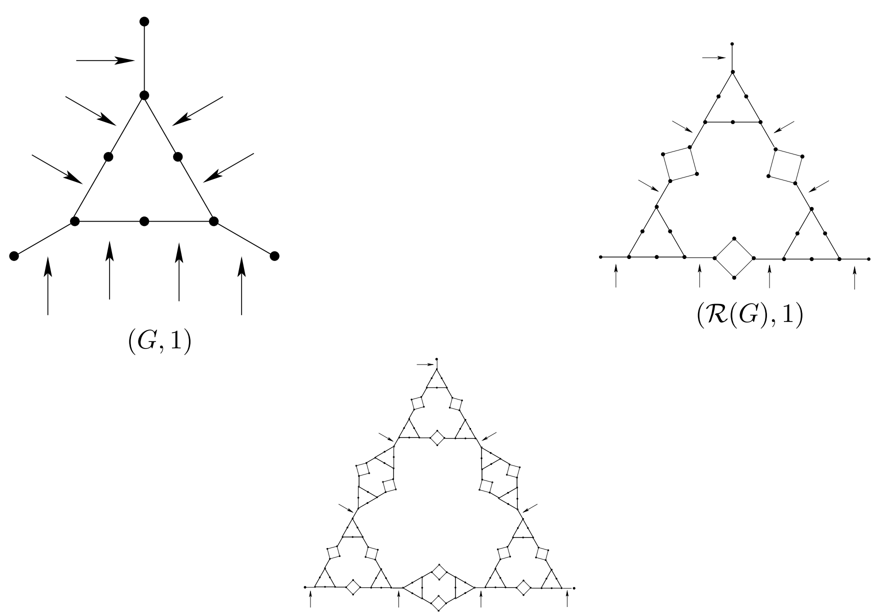

Figure 8. The elements of $\Delta_0(G)$, $\Delta_1(G)$, and $\Delta_2(G)$ pointed out with arrows.

Figure 9 points out the elements of $\Delta_0(\mathcal{R}(G)) \subset (\mathcal{R}(G), 1)$. Notice that $\Delta_0(\mathcal{R}(G))$ is a finer net of $(\mathcal{R}(G), 1)$ than $\Delta_1(G)$ is.

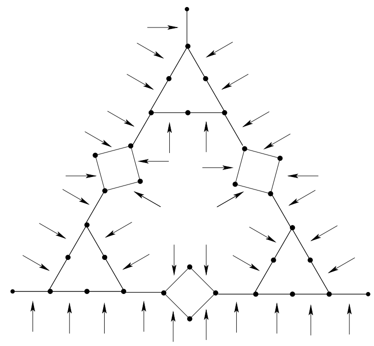

FIGURE 9. The elements of $\Delta_0(\mathcal{R}(G))$ are pointed out in $(\mathcal{R}(G), 1)$.

We conclude this section with two lemmas required to prove the formulas for the dimensions of limits of vertex replacements. The first lemma follows from Perron-Frobenius theory. See [2] for details.

LEMMA 3.5. *Let A be a nonnegative primitive matrix and let r be the spectral radius of A. Then $\lim_{n\to\infty}(A/r)^n$ is a positive matrix.*

The next lemma is a result from [9]. It says that the a_i's and b_i's grow at the same rate.

LEMMA 3.6. *If the replacement rule $\mathcal{R}$ is simple and primitive and ρ is the spectral radius of the path matrix of $\mathcal{R}$, then there exist positive constants $\widetilde{K}$, κ_1, and κ_2 such that*

$$\kappa_1 \leq \frac{a_i(n)}{b_j(n)} \leq \kappa_2, \tag{1}$$

$$\frac{\kappa_1}{\rho^m} \leq \frac{b_i(n)}{b_j(n+m)} \leq \frac{\kappa_2}{\rho^m}, \tag{2}$$

$$\frac{\kappa_1}{\rho^m} \leq \frac{a_i(n)}{b_j(n+m)} \leq \frac{\kappa_2}{\rho^m} \tag{3}$$

for all $i, j = 1, \ldots, p$ and for all $n > \widetilde{K}$.

4. The box dimension of limits of vertex replacements

Before calculating the box dimension of a limit of vertex replacements, recall the definition of the box dimension of a compact metric space.

DEFINITION 4.1. For any compact metric space X, define $\mathrm{Cov}(X, \epsilon)$ to be the smallest number of closed balls of radius ϵ which cover X.

DEFINITION 4.2. The *box dimension* (or *box-counting dimension*) of a compact metric space X is defined to be

$$\dim_{\mathrm{Box}}(X) = \limsup_{\epsilon \to 0} \frac{\ln(\mathrm{Cov}(X, \epsilon))}{-\ln \epsilon}.$$

THEOREM 4.3. *Let $\mathcal{R}$ be a simple primitive replacement rule and suppose that the sequence $\{(\mathcal{R}^n(G), 1)\}$ converges in the Gromov-Hausdorff metric to the metric space X. Then*

$$\dim_{\mathrm{Box}}(X) = \frac{\ln r}{\ln \rho},$$

where r and ρ are the spectral radii of the replaceable vertex and path matrices of $\mathcal{R}$.

Proof. We first show that $\dim_{\mathrm{Box}}(X) \leq \ln r / \ln \rho$. Let $\epsilon > 0$ be given. Choose m so that

$$\kappa_2/\rho^m < \epsilon \leq \kappa_2/\rho^{m-1}, \tag{4}$$

where κ_2 is the positive constant from Lemma 3.6. The idea of the proof is to construct an ϵ-net of $(\mathcal{R}^{n+m}(G), 1)$ which for large n yields a corresponding 2ϵ-net D in X. Thus, any element of a cover of X by 4ϵ-balls must contain at least one element of D. Since the number of elements in D is given in terms of M^m, where M is the replaceable vertex matrix of $\mathcal{R}$, the result follows from a simple computation.

We now show that the set $\Delta_n(\mathcal{R}^m(G))$ forms an ϵ-net of $(\mathcal{R}^{n+m}(G), 1)$ for all sufficiently large n. Recall $b_{\max}(n) = \max_i b_i(n)$. For each $x(n) \in \Delta_n(\mathcal{R}^m(G))$, consider the ball $B(x(n), b_{\max}(n)+1)$ of radius $b_{\max}(n)+1$ centered at $x(n)$ in the (unscaled) graph $\mathcal{R}^{m+n}(G)$. Let V denote the set of all replaceable vertices in $\mathcal{R}^m(G)$. Since $2b_{\max}(n) \geq \mathrm{diam}(\mathcal{R}^n(v))$ for any replaceable vertex v, the union of these balls covers $\mathcal{R}^n(V)$ for all $n \geq 1$. Moreover, since $\mathcal{R}^{n+m}(G) \setminus \mathcal{R}^n(V)$ contains no replaceable vertices for all n and $\mathrm{diam}(\mathcal{R}^{n+m}(G)) \to \infty$ as $n \to \infty$, for large enough n the balls $B(x_m(n), b_{\max}(n) + 1)$ form a cover of all of $\mathcal{R}^{n+m}(G)$.

Now in order to prove that $\Delta_n(\mathcal{R}^m(G))$ forms an ϵ-net of the normalized graph $(\mathcal{R}^{n+m}(G), 1)$, it remains to show that the scaled balls $B(x(n), (b_{\max}(n)+1)/\mathrm{diam}(\mathcal{R}^{n+m}(G)))$ have positive radius less than ϵ. Since $\mathcal{R}$ is primitive, we may assume without loss of generality that the initial graph G contains at least two replaceable vertices of each type. Therefore, $2b_{\max}(n+m) \le \operatorname{diam}(\mathcal{R}^{n+m}(G))$. Hence,

$$\frac{b_{\max}(n)+1}{\operatorname{diam}(\mathcal{R}^{n+m}(G))} \le \frac{b_{\max}(n)+1}{2b_{\max}(n+m)}.$$

Therefore, by Lemma 3.6 and inequality (4), we have

$$0 < \frac{b_{\max}(n)+1}{2b_{\max}(n+m)} \le \frac{\kappa_2}{\rho^m} < \epsilon$$

for all sufficiently large n. Thus, the set $\Delta_n(\mathcal{R}^m(G))$ forms an ϵ-net of $(\mathcal{R}^{n+m}(G), 1)$ for all sufficiently large n.

For large enough n, $\operatorname{dist}^{\mathrm{GH}}_{\mathcal{S}}(X, (\mathcal{R}^{n+m}(G), 1)) \le \epsilon$. Thus, by definition of the Gromov-Hausdorff metric, the set of points $\Delta_n(\mathcal{R}^m(G)) \subset \mathcal{R}^{n+m}(G)$ produces a corresponding 2ϵ-net D in X. So any element of a cover of X by 4ϵ-balls must contain at least one element of D. Therefore, $\operatorname{Cov}(X, 4\epsilon) \le |D| = |\Delta_n(\mathcal{R}^m(G))|$.

Let M be the replaceable vertex matrix of $\mathcal{R}$. Let $\mathbf{v}_G = [N_1(G), \ldots, N_p(G)]$ and $\mathbf{v}_{\mathcal{R}} = [|\partial H_1|, \ldots, |\partial H_p|]$. The number of points in $\Delta_n(\mathcal{R}^m(G))$ is at most

$$|\Delta_n(\mathcal{R}^m(G))| \le \mathbf{v}_G \cdot M^m \mathbf{v}_{\mathcal{R}}^T.$$

Hence, $\operatorname{Cov}(X, 4\epsilon) \le \mathbf{v}_G \cdot M^m \mathbf{v}_{\mathcal{R}}^T$. So

$$\frac{\ln(\operatorname{Cov}(X, 4\epsilon))}{-\ln(4\epsilon)} \le \frac{\ln(\mathbf{v}_G \cdot M^m \mathbf{v}_{\mathcal{R}}^T)}{\ln\left(\frac{\rho^{m-1}}{4\kappa_2}\right)}.$$

By inequality (4), when $\epsilon \to 0$, then $m \to \infty$. From Lemma 3.5, we have that $\lim_{m\to\infty}(M/r)^m$ is a positive matrix. So

$$\begin{aligned}\dim_{\mathrm{Box}}(X) &= \limsup_{\epsilon\to 0} \frac{\ln(\operatorname{Cov}(X, 4\epsilon))}{-\ln(4\epsilon)} \\ &\le \limsup_{m\to\infty} \frac{\ln\left(\mathbf{v}_G \cdot (M/r)^m \mathbf{v}_{\mathcal{R}}^T\right) + \ln(r^m)}{\ln(\rho^m) - \ln(4\kappa_2\rho)} = \frac{\ln r}{\ln \rho}.\end{aligned}$$

Next, we prove the reverse inequality $\dim_{\mathrm{Box}}(X) \ge \ln r / \ln \rho$. Let $\epsilon > 0$ be given. Fix m so that

$$\frac{\kappa_1}{N(G)\rho^m} \ge \epsilon > \frac{\kappa_1}{N(G)\rho^{m+1}}, \tag{5}$$

where κ_1 is as in Lemma 3.6. The idea of the proof is to create an $\epsilon/2$-separated set S_n in $(\mathcal{R}^{n+m}(G), 1)$ which for large n yields a corresponding $\epsilon/4$-separated set S in X. Thus, any element of a cover of X by $\epsilon/8$-balls can contain at most one element of S. Since the number of elements in S is given in terms of M^m, where M is the replaceable vertex matrix of $\mathcal{R}$, the result follows from a simple computation.

Let V be the set of all replaceable vertices in $\mathcal{R}^m(G)$. For each vertex $w \in V$, let w_n be the midpoint of a path in $\mathcal{R}^n(w) \subset \mathcal{R}^{n+m}(G)$ that realizes

$a_i(n)$, where i is such that w is replaceable by H_i. Let $S_n \subset \mathcal{R}^{n+m}(G)$ be the set of all such points.

We now show that the elements of S_n are $\epsilon/2$-separated in $(\mathcal{R}^{n+m}(G), 1)$. Let $B(w_n, a_i(n)/2)$ be the closed ball of radius $a_i(n)/2$ centered at w_n. By construction, the balls $B(w_n, a_i(n)/2)$ are disjoint. Recall from Lemma 3.6 that for large enough n, there are positive constants κ_1 and κ_2 so that

$$\frac{\kappa_2}{\rho^m} \geq \frac{a_{\min}(n)}{b_{\max}(n+m)} \geq \frac{\kappa_1}{\rho^m}. \tag{6}$$

Since $\operatorname{diam}(\mathcal{R}^{n+m}(G)) \leq N(G)b_{\max}(n+m) + \operatorname{diam}(G)$, for large enough n inequalities (5) and (6) imply that

$$\frac{a_{\min}(n)}{\operatorname{diam}(\mathcal{R}^{n+m}(G))} \geq \frac{a_{\min}(n)}{N(G)b_{\max}(n+m) + \operatorname{diam}(G)} \geq \frac{\kappa_1}{2N(G)\rho^m} \geq \epsilon/2.$$

Hence, the scaled balls $B(w_n, a_i(n)/2)$ in $(\mathcal{R}^{n+m}(G), 1)$ have radius at least $\epsilon/4$. So for large n, the points in S_n are more than $\epsilon/2$-separated in $(\mathcal{R}^{n+m}(G), 1)$.

Choose n large enough so that $\operatorname{dist}_{\mathcal{S}}^{\mathrm{GH}}(X, (\mathcal{R}^{n+m}(G), 1)) < \epsilon/8$. Then for each w_n in S_n, one can choose an associated point $x(w_n)$ in X and create an $\epsilon/4$-separated set S in X. Thus, any element of a cover of X by $\epsilon/8$-balls can contain at most one element of S. So the number of balls in such a cover is at least $|S| = |V| = \mathbf{v}_G \cdot M^m \mathbf{1}$, where M is the replaceable vertex matrix of $\mathcal{R}$, $\mathbf{1}$ is the vector in $\mathbb{R}^p$ consisting solely of ones, and $\mathbf{v}_G = [N_1(G), \ldots, N_p(G)]$.

Hence,

$$\operatorname{Cov}(X, \epsilon/8) \geq \mathbf{v}_G \cdot M^m \mathbf{1}$$

and

$$\frac{\ln(\operatorname{Cov}(X, \epsilon/8))}{\ln(8/\epsilon)} \geq \frac{\ln(\mathbf{v}_G \cdot M^m \mathbf{1})}{\ln(8N(G)\rho^{m+1}/\kappa_1)}.$$

By inequality (5), as $\epsilon \to 0$, we have $m \to \infty$. So by Lemma 3.5,

$$\begin{aligned}\dim_{\mathrm{Box}}(X) &= \limsup_{\epsilon \to 0} \frac{\ln(\operatorname{Cov}(X, \epsilon))}{-\ln(\epsilon)} \\ &\geq \limsup_{m \to \infty} \frac{\ln(\mathbf{v}_G \cdot (M/r)^m \mathbf{1}) + \ln(r^m)}{\ln(8N(G)\rho/\kappa_1) + \ln(\rho^m)} = \frac{\ln r}{\ln \rho}.\end{aligned}$$

Thus, $\dim_{\mathrm{Box}}(X) = \ln r / \ln \rho$, where r and ρ are the spectral radii of the replaceable vertex and path matrices of $\mathcal{R}$. □

5. The Hausdorff dimension of limits of vertex replacements

Before calculating the Hausdorff dimension of the limits of vertex replacements, let us recall the definition of the Hausdorff dimension of a metric space.

We say a cover $\mathcal{U}$ of a metric space X is a δ-cover if $0 < \operatorname{diam}(U) \leq \delta$ for all $U \in \mathcal{U}$. Let s be a non-negative number. For $\delta > 0$, define

$$\mathcal{H}^s_\delta(X) = \inf \sum_{i=1}^{\infty} (\operatorname{diam} U_i)^s,$$

where the infimum is over all countable δ-covers of X. Let

$$\mathcal{H}^s(X) = \lim_{\delta \to 0} \mathcal{H}^s_\delta(X) = \sup_{\delta > 0} \mathcal{H}^s_\delta(X).$$

Definition 5.1. The *Hausdorff dimension* of a metric space X, denoted by $\dim_H(X)$, is the unique s' such that $\mathcal{H}^s(X) = \infty$ for $0 \leq s < s'$ and $\mathcal{H}^s(X) = 0$ for $s > s'$.

It follows directly from the definitions (see [4]) that for any X,

$$\dim_H(X) \leq \dim_{\text{Box}}(X).$$

The above inequality gives:

Corollary 5.2. *If $\mathcal{R}$ is a simple primitive replacement rule and X is the limit space of a sequence $(\mathcal{R}^n(G), 1)$, then $\dim_H(X) \leq \ln r / \ln \rho$.*

We calculate the Hausdorff dimension of the limit space X of a sequence $\{(\mathcal{R}^n(G), 1)\}$ in the case when every replacement graph in the simple primitive replacement rule $\mathcal{R}$ has replaceable boundary vertices. When the boundary vertices of the replacement graphs are replaceable, then Lemma 5.3 shows that the function Q defined below remains constant under replacement.

For any edge e and any replaceable vertex v in any finite graph G, let $Q(e, v)$ be the number of distinct edges adjacent to v which can be connected to e by a nonreplaceable path (containing no interior replaceable vertices). Let $Q(e) = \sum_{v \in V(G)} Q(e, v)$, where $V(G)$ is the set of replaceable vertices in G.

To illustrate how $Q(e)$ is computed, consider the replacement rule given by the graphs H_1 and H_2 depicted in Figure 10. Notice that the boundary vertices of the replacement graphs are not replaceable. Figure 11 shows an edge e in $(G, 1)$ and the corresponding edges in $(\mathcal{R}(G), 1)$, $(\mathcal{R}^2(G), 1)$, and $(\mathcal{R}^3(G), 1)$ (also denoted by e). For any replaceable vertex v in $\mathcal{R}^n(G)$, we have that $Q(e, v)$ is either 1 or 0, depending upon where v is located in $\mathcal{R}^n(G)$. So $Q(e) = 2^n$, where e is the edge depicted in $\mathcal{R}^n(G)$.

If the boundary vertices of every replacement graph H_i are replaceable (i.e., have degree $|\partial H_j| - 1$ for some j), then we have that $Q(\mathcal{R}(e)) = Q(e)$ for $\mathcal{R}(e) \subset \mathcal{R}(G)$. Let $Q(G) = \max_{e \subset E(G)} Q(e)$, where $E(G)$ is the set of edges in G. Thus we have the following:

FIGURE 10. A replacement rule $\mathcal{R} = \{H_1, H_2\}$.

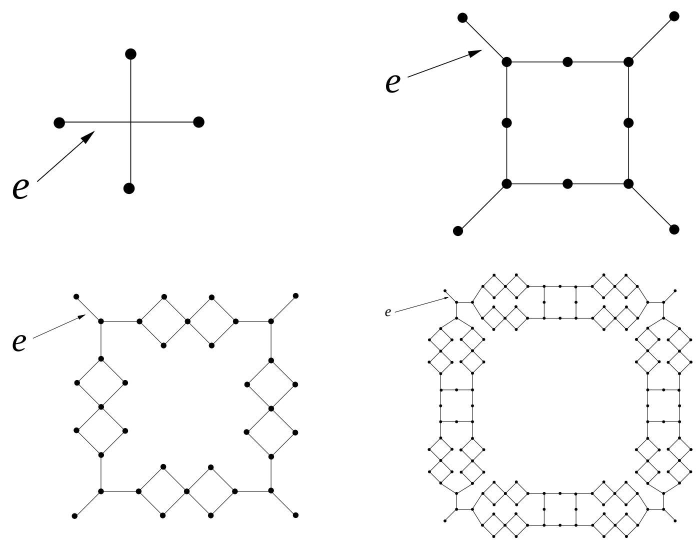

FIGURE 11. $(G, 1)$, $(\mathcal{R}(G), 1)$, $(\mathcal{R}^2(G), 1)$, and $(\mathcal{R}^3(G), 1)$.

LEMMA 5.3. *Let $\{H_1, \ldots, H_p\}$ define a simple primitive vertex replacement rule $\mathcal{R}$. Suppose that the boundary vertices of each replacement graph of $\mathcal{R}$ are replaceable. Then for all $n \geq 0$, we have $Q(\mathcal{R}^n(G)) = Q(\mathcal{R}(G))$.*

THEOREM 5.4. *Let $\{H_1, \ldots, H_p\}$ define a simple primitive vertex replacement rule $\mathcal{R}$. Suppose that the boundary vertices of each replacement graph of $\mathcal{R}$ are replaceable and that G is a graph containing replaceable vertices with $(\mathcal{R}^n(G), 1) \to X$ in the Gromov-Hausdorff metric as $n \to \infty$. Then $\dim_H(X) = \ln r / \ln \rho$.*

Proof. By Corollary 5.2, it suffices to show that $\dim_H(X) \geq \ln r / \ln \rho$. We will show that for $s = \ln r / \ln \rho$, there exists a positive constant K such that for all $\delta > 0$,

$$\mathcal{H}^s_\delta(X) = \inf \sum_{U \in \mathcal{U}} (\text{diam } U)^s \geq K > 0, \tag{7}$$

where the infimum is over all δ-covers $\mathcal{U}$ of X. By showing that $\mathcal{H}^s_\delta(X)$ is bounded away from 0 by a positive constant which is independent of δ, we obtain that $\mathcal{H}^s(X) = \sup_{\delta>0} \mathcal{H}^s_\delta(X)$ is bounded away from 0. Once we know that $\mathcal{H}^s(X) \neq 0$, then by definition of the Hausdorff dimension, $\dim_H(X) \geq s = \ln r/\ln \rho$, and we are done.

Consider a δ-cover $\mathcal{U}$ of X. Since $\mathcal{U}$ may not be open, create an open over $\widetilde{\mathcal{U}}$ of X as follows: for each $U \in \mathcal{U}$, define $\widetilde{U}$ to be all points in X which are less than $\text{diam}(U)$ away from a point in U. Then clearly $3 \text{ diam}(U) \geq \text{diam}(\widetilde{U})$. Thus

$$\sum_{U \in \mathcal{U}} (\text{diam } U)^s \geq \frac{1}{3^s} \sum_{U \in \widetilde{\mathcal{U}}} (\text{diam } \widetilde{U})^s. \tag{8}$$

Since X is compact, we may assume that $\widetilde{\mathcal{U}}$ is a finite cover.

For each $\widetilde{U}_\ell \in \widetilde{\mathcal{U}}$, choose m_ℓ so that

$$1/\rho^{m_\ell} > \text{diam}(\widetilde{U}_\ell) > 1/\rho^{m_\ell+2}. \tag{9}$$

By Lemma 3.6, for large enough n, there are positive constants κ_1 and κ_2 so that

$$\frac{\kappa_2}{\rho^m} > \frac{a_i(n-m)}{b_j(n)} > \frac{\kappa_1}{\rho^m} \tag{10}$$

for all $i, j = 1, \ldots, p$. Furthermore, since $(\mathcal{R}^n(G), 1) \to X$, we may also fix n so that the open cover $\widetilde{\mathcal{U}}$ has an associated open cover $\{V_\ell\}$ of $(\mathcal{R}^n(G), 1)$ satisfying

$$1/\rho^{m_\ell} \geq \text{diam}(V_\ell) \geq 1/\rho^{m_\ell+2}. \tag{11}$$

Hence

$$\begin{aligned}(\text{diam } V_\ell)^{\ln r/\ln \rho} &\geq \left(\frac{1}{\rho^{m_\ell+2}}\right)^{\ln r/\ln \rho} = e^{\ln\left[\rho^{-(m_\ell+2)\ln r/\ln \rho}\right]} \\ &= \frac{1}{r^{m_\ell+2}} = \frac{r^{n-m_\ell}}{r^{n+2}}.\end{aligned} \tag{12}$$

Thus

$$\sum_\ell (\text{diam } V_\ell)^{\ln r/\ln \rho} \geq \frac{1}{r^{n+2}} \sum_\ell r^{n-m_\ell}. \tag{13}$$

Since $\{V_\ell\}$ is a cover of $(\mathcal{R}^n(G), 1)$, the sum of the maximum number of replaceable vertices in the V_ℓ's must be bounded below by the number of replaceable vertices in $\mathcal{R}^n(G)$. We will find this sum and state it in terms of the right hand side of inequality (13) to get a lower bound for $\sum_\ell (\text{diam } V_\ell)^{\ln r/\ln \rho}$

which is independent of m_ℓ. This lower bound will then yield a lower bound for $\sum_{U\in\mathcal{U}}(\mathrm{diam}\ U)^{\ln r/\ln\rho}$ which is independent of the δ-cover $\mathcal{U}$.

Let w be a replaceable vertex in $\mathcal{R}^{m_\ell}(G)$. We will find the maximum number of replaceable vertices in a V_ℓ by counting the number of copies of $\mathcal{R}^{n-m_\ell}(w)$ contained in V_ℓ. Since $\mathrm{diam}(\mathcal{R}^n(G)) \le N(G)b_{\max}(n) + \mathrm{diam}(G)$, by inequality (10) one can fix a positive constant $C := \kappa_1/(2(N(G)+1))$ such that

$$\frac{a_{\min}(n-m_\ell)}{\mathrm{diam}(\mathcal{R}^n(G))} \ge C\frac{1}{\rho^{m_\ell}} \tag{14}$$

for all m_ℓ. Notice that C is independent of m_ℓ. Hence it is independent of the δ-cover $\mathcal{U}$. If V_ℓ contains an entire copy of $\mathcal{R}^{n-m_\ell}(w)$, then

$$\mathrm{diam}(V_\ell) \ge \mathrm{diam}((\mathcal{R}^{n-m_\ell}(w),1)) \ge \frac{a_{\min}(n-m_\ell)}{\mathrm{diam}(\mathcal{R}^n(G))}.$$

However, by inequalities (11) and (14),

$$\mathrm{diam}(V_\ell) \le \frac{1}{\rho^{m_\ell}} \le \frac{a_{\min}(n-m_\ell)}{C\,\mathrm{diam}(\mathcal{R}^n(G))}. \tag{15}$$

Therefore, any path in V_ℓ that realizes the diameter of V_ℓ can intersect at most $[1/C]+1$ copies of $\mathcal{R}^{n-m_\ell}(w)$, where $[\cdot]$ is the greatest integer function. Now because each replaceable vertex has degree at most $\max_i|\partial H_i|$ and each edge adjacent to a replaceable vertex in $\mathcal{R}^{m_\ell}(G)$ can be connected via nonreplaceable paths to at most $Q(\mathcal{R}^{m_\ell-1}(G))$ replaceable vertices, each replaceable vertex can be connected via nonreplaceable paths to at most $\max_i|\partial H_i|Q(\mathcal{R}^{m_\ell-1}(G))$ replaceable vertices. Therefore, the total number of copies of $\mathcal{R}^{n-m_\ell}(w)$ which V_ℓ can intersect is at most $C' := (\max_i|\partial H_i|Q(\mathcal{R}^{m_\ell-1}(G)))^{[1/C]+1}$.

For example, suppose $\max_i|\partial H_i| = 3$ and $Q(\mathcal{R}^{m_\ell-1}(G)) = 2$. Then a replaceable vertex $w \in \mathcal{R}^{m_\ell}(G)$ can be connected via nonreplaceable paths to no more than 6 replaceable vertices. If $C = 1/2$, then inequality (15) together with the fact that $a_{\min}(n-m_\ell) \le \mathrm{diam}(\mathcal{R}^{n-m_\ell}(w))$ imply that V_ℓ can intersect no more than $6^{[1/C]+1} = 6^3$ copies of $\mathcal{R}^{n-m_\ell}(w)$. Actually, this is a bit of overkill. Figure 12 illustrates that in this case, a better bound on the maximum number of copies of $\mathcal{R}^{n-m_\ell}(w)$ intersected by V_ℓ is $6\cdot 5+1$. Each vertex in the figure represents a copy of $\mathcal{R}^{n-m_\ell}(w)$ intersected by V_ℓ and each edge represents a nonreplaceable path between each of these copies. If, however, $C = 1/3$, then V_ℓ can intersect no more than $6^{[1/C]+1} = 6^4$ copies of $\mathcal{R}^{n-m_\ell}(w)$. Figure 13 illustrates that in this case, a better bound on the maximum number of copies of $\mathcal{R}^{n-m_\ell}(w)$ intersected by V_ℓ is $6\cdot 5^2+1$.

By Lemma 5.3, we have that $C' = (\max_i|\partial H_i|Q(\mathcal{R}(G)))^{[1/C]+1}$. Notice that C' is independent of m_ℓ and is therefore independent of the cover $\mathcal{U}$.

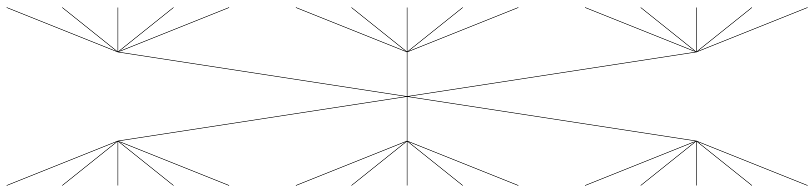

FIGURE 12. Counting copies of $\mathcal{R}^{n-m_\ell}(w)$ intersected by V_ℓ when $C = 1/2$.

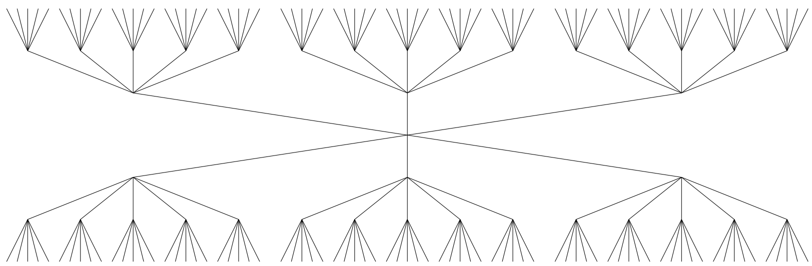

FIGURE 13. Counting copies of $\mathcal{R}^{n-m_\ell}(w)$ intersected by V_ℓ when $C = 1/3$.

Since for any replaceable vertex $w \in \mathcal{R}^{m_\ell}(G)$, we have that $\mathcal{R}^{n-m_\ell}(w)$ contains at most $\max_i N(H_i)^{n-m_\ell}$ replaceable vertices, V_ℓ can contain at most $C' \max_i N(H_i)^{n-m_\ell}$ replaceable vertices.

Now that we have determined the maximum number of replaceable vertices contained in each V_ℓ, we use the fact that $\{V_\ell\}$ is a cover of $(\mathcal{R}^n(G), 1)$ to get a lower bound for the sum of these maxima. This lower bound will then yield a lower bound for $\sum_\ell (\text{diam } V_\ell)^{\ln r/\ln \rho}$. Let M be the replaceable vertex matrix of $\mathcal{R}$, $\mathbf{v}_G = [N_1(G), \dots, N_p(G)]$, and let $\mathbf{1}$ be the vector in $\mathbb{R}^p$ containing all ones. As stated earlier, each V_ℓ has at most $C' \max_i N(H_i)^{n-m_\ell}$ replaceable vertices. However, $\mathcal{R}^n(G)$ contains exactly $\mathbf{v}_G \cdot M^n \mathbf{1}$ replaceable vertices. Since $\{V_\ell\}$ is a cover of $(\mathcal{R}^n(G), 1)$, the sum of the maximum number of replaceable vertices in the V_ℓ's must be bounded below by the number of replaceable vertices in $\mathcal{R}^n(G)$. That is,

$$\sum_\ell C' \max_i N(H_i)^{n-m_\ell} \geq \mathbf{v}_G \cdot M^n \mathbf{1}. \tag{16}$$

We now seek to restate inequalities (16) and (13) in terms of the replaceable vertex matrix M of $\mathcal{R}$. Fix $j \in \{1, \dots, p\}$ so that $N(H_j) = \max_i N(H_i)$. Then $\max_i N(H_i)^{n-m_\ell} = \mathbf{e}_j \cdot M^{n-m_\ell} \mathbf{1}$, where $\mathbf{e}_j$ is the j-th column of the $p \times p$

identity matrix. Hence,

$$\sum_{\ell} C' \max_i N(H_i)^{n-m_\ell} = \sum_{\ell} C' \mathbf{e}_j \cdot M^{n-m_\ell}\mathbf{1} \geq \mathbf{v}_G \cdot M^n \mathbf{1}. \tag{17}$$

Let $\mathbf{v}_M$ be the positive eigenvector of M corresponding to r whose j-th entry is 1. Then

$$r^{n-m_\ell} = r^{n-m_\ell}\mathbf{e}_j \cdot \mathbf{v}_M = \mathbf{e}_j \cdot M^{n-m_\ell}\mathbf{v}_M.$$

Let k be the minimum entry in $\mathbf{v}_M$. Then inequality (13) may be rewritten as

$$\begin{aligned} \sum_{\ell} (\text{diam } V_\ell)^{\ln r/\ln \rho} &\geq \frac{1}{r^{n+2}} \sum_{\ell} \mathbf{e}_j \cdot M^{n-m_\ell}\mathbf{v}_M \\ &\geq \frac{k}{r^{n+2}} \sum_{\ell} \mathbf{e}_j \cdot M^{n-m_\ell}\mathbf{1}. \end{aligned} \tag{18}$$

Inequalities (17) and (18) imply

$$\sum_{\ell} (\text{diam } V_\ell)^{\ln r/\ln \rho} \geq \frac{k\mathbf{v}_G \cdot M^n \mathbf{1}}{r^{n+2}C'}.$$

By Lemma 3.5, there is a positive matrix F so that

$$\sum_{\ell} (\text{diam } V_\ell)^{\ln r/\ln \rho} > \sum_{\ell} \frac{1}{\rho^{m_\ell+2}} \geq \frac{k\mathbf{v}_G \cdot F\mathbf{1}}{rC'}.$$

Hence by inequalities (8) and (9), for $K := (k\mathbf{v}_G \cdot F\mathbf{1})/(3^s C' r)$,

$$\sum_{U \in \mathcal{U}} (\text{diam } U)^{\ln r/\ln \rho} \geq K > 0$$

for all δ-covers $\mathcal{U}$ of X. That is,

$$\dim_H(X) \geq \frac{\ln r}{\ln \rho}.$$

Therefore, by Corollary 5.2, $\dim_H(X) = \ln r/\ln \rho$. □

6. Examples

Recall that a fractal is a metric space with Hausdorff dimension strictly greater than its topological dimension. It was shown in [11] that the limits of vertex replacements have topological dimension 1. The examples below illustrate a few replacement rules which yield metric spaces with Hausdorff dimension greater than one. As in earlier examples, the boundary vertices of the replacement graphs are depicted using circles.

EXAMPLE 1. In our first example, the replacement rule $\mathcal{R}$ (Figure 14) has path matrix $\left[\begin{smallmatrix}1 & 2\\ 1 & 2\end{smallmatrix}\right]$ and replaceable vertex matrix $\left[\begin{smallmatrix}1 & 4\\ 1 & 2\end{smallmatrix}\right]$. A few iterations of the replacement are shown in Figure 15. The sequence $\{(\mathcal{R}^n(G), 1)\}$ converges in the Gromov-Hausdorff metric to the Peter's Cross depicted in Figure 16 which has topological dimension 1 and Hausdorff dimension $(\ln(3+\sqrt{17}) - \ln 2)/(\ln 3)$.

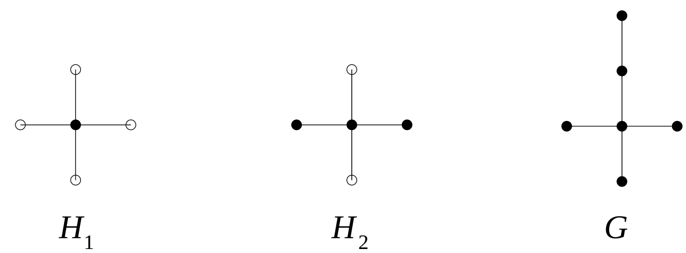

FIGURE 14. A replacement rule $\mathcal{R} = \{H_1, H_2\}$ and an initial graph G.

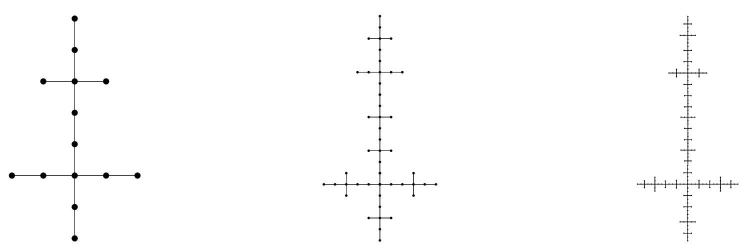

FIGURE 15. $(\mathcal{R}(G), 1)$, $(\mathcal{R}^2(G), 1)$, and $(\mathcal{R}^3(G), 1)$.

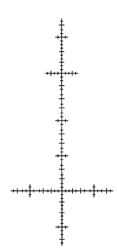

FIGURE 16. Peter's Cross.

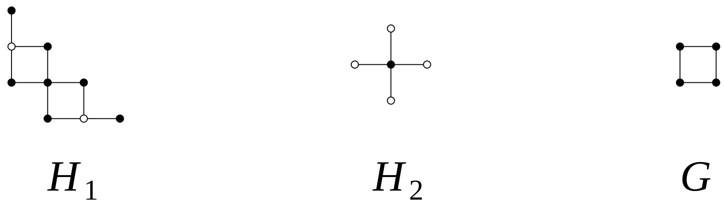

FIGURE 17. A replacement rule $\mathcal{R} = \{H_1, H_2\}$ and an initial graph G.

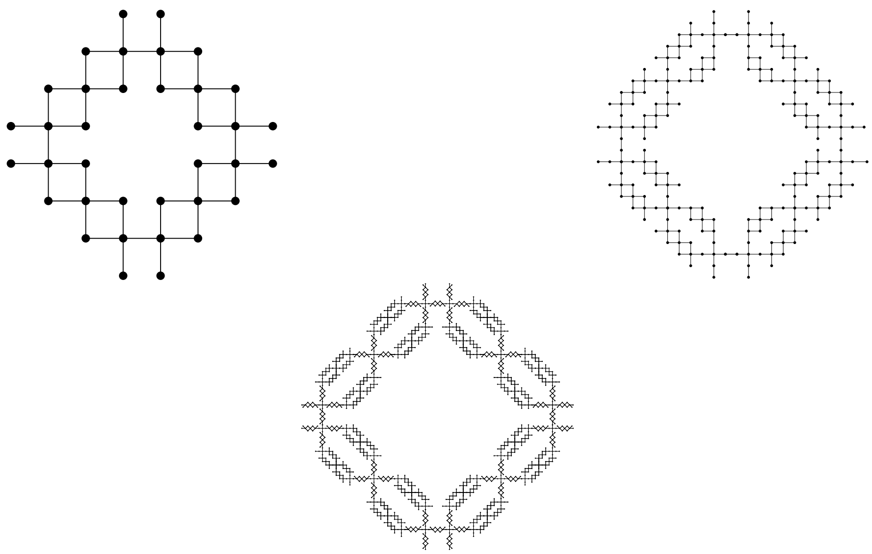

FIGURE 18. $(\mathcal{R}(G), 1)$, $(\mathcal{R}^2(G), 1)$, and $(\mathcal{R}^3(G), 1)$.

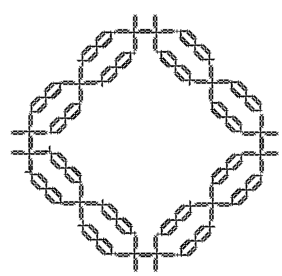

FIGURE 19. A doily.

EXAMPLE 2. For our second example, the replacement rule $\mathcal{R}$ is shown in Figure 17 and a few iterations of the replacement are shown in Figure 18.

Since $\mathcal{R}$ has path matrix $\left[\begin{smallmatrix} 2 & 3 \\ 2 & 1 \end{smallmatrix}\right]$ and replaceable vertex matrix $\left[\begin{smallmatrix} 4 & 3 \\ 4 & 1 \end{smallmatrix}\right]$, then the sequence $\{(\mathcal{R}^n(G), 1)\}$ converges to a metric space with topological dimension 1 and Hausdorff dimension $(\ln(5 + \sqrt{57}) - \ln 2)/(\ln 4)$. The limit is depicted in Figure 19.

Example 3. We now give an example which must be embedded in at least three dimensions. In this case, the replacement rule $\mathcal{R}$ (Figure 20) has path matrix $\left[\begin{smallmatrix} 2 & 1 \\ 1 & 2 \end{smallmatrix}\right]$ and replaceable vertex matrix $\left[\begin{smallmatrix} 2 & 1 \\ 6 & 4 \end{smallmatrix}\right]$. A few iterations of the replacement are shown in Figure 21, and the limit space of the sequence $\{(\mathcal{R}^n(G), 1)\}$ is depicted in Figure 22. Its Hausdorff dimension is $(\ln(3 + \sqrt{7}))/(\ln 3)$.

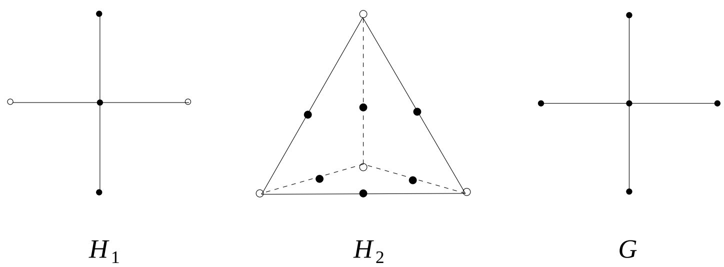

Figure 20. A replacement rule $\mathcal{R} = \{H_1, H_2\}$ and an initial graph G.

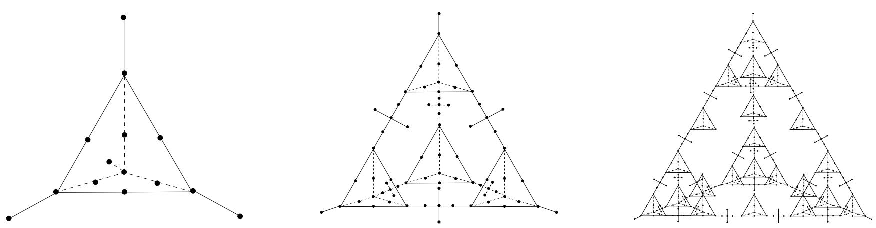

Figure 21. $(\mathcal{R}(G), 1)$, $(\mathcal{R}^2(G), 1)$, and $(\mathcal{R}^3(G), 1)$.

Example 4. In our final example, the replacement rule $\mathcal{R}$ (Figure 23) has path matrix $A = \left[\begin{smallmatrix} 0 & 2 \\ 1 & 2 \end{smallmatrix}\right]$ and replaceable vertex matrix $\left[\begin{smallmatrix} 0 & 2 \\ 6 & 4 \end{smallmatrix}\right]$. Note that although A is not positive, it is primitive since A^2 is positive. Consequently, $\mathcal{R}$ is a primitive replacement rule. Therefore, for any initial graph G with at least one replaceable vertex, the limit of the sequence $\{(\mathcal{R}^n(G), 1)\}$ has topological dimension 1 and Hausdorff dimension $(\ln 6)/(\ln(1 + \sqrt{3}))$. Figure 24 shows a few iterations of the replacement for the initial graph G given in Figure 23.

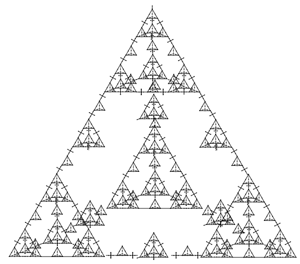

Figure 22. A modified Sierpinski tetrahedron.

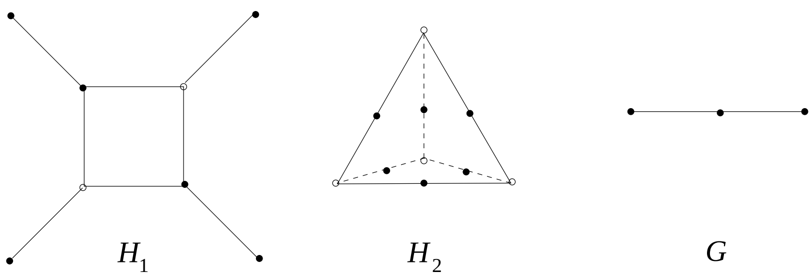

Figure 23. A replacement rule $\mathcal{R} = \{H_1, H_2\}$ and an initial graph G.

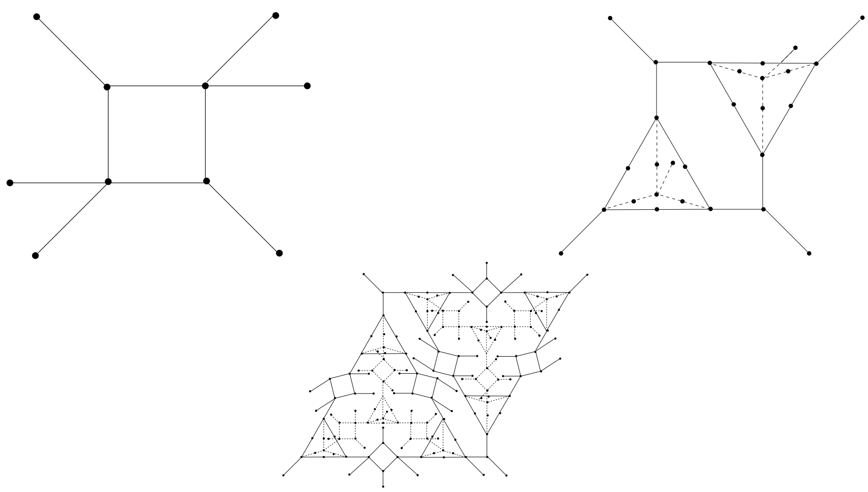

Figure 24. $(\mathcal{R}(G), 1)$, $(\mathcal{R}^2(G), 1)$, and $(\mathcal{R}^3(G), 1)$.

References

[1] W. Ballmann, *Lectures on spaces of nonpositive curvature*, DMV Seminar, vol. 25, Birkhäuser Verlag, Basel, 1995. MR 1377265 (97a:53053)

[2] A. Berman and R. J. Plemmons, *Nonnegative matrices in the mathematical sciences*, Classics in Applied Mathematics, vol. 9, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1994, Revised reprint of the 1979 original. MR 1298430 (95e:15013)

[3] K. J. Falconer, *The geometry of fractal sets*, Cambridge Tracts in Mathematics, vol. 85, Cambridge University Press, Cambridge, 1986. MR 867284 (88d:28001)

[4] ______, *Fractal geometry*, John Wiley & Sons Ltd., Chichester, 1990, Mathematical foundations and applications. MR 1102677 (92j:28008)

[5] M. Gromov, *Groups of polynomial growth and expanding maps*, Inst. Hautes Études Sci. Publ. Math. (1981), 53–73. MR 623534 (83b:53041)

[6] ______, *Hyperbolic groups*, Essays in group theory, Math. Sci. Res. Inst. Publ., vol. 8, Springer, New York, 1987, pp. 75–263. MR 919829 (89e:20070)

[7] ______, *Cell division and hyperbolic geometry*, preprint, IHES, June 1990.

[8] J. Previte, *Graph substitutions*, Ergodic Theory Dynam. Systems **18** (1998), 661–685. MR 1631736 (99h:58095)

[9] J. Previte, M. Previte, and M. Vanderschoot, *Limits of vertex peplacement rules*, Rocky Mountain J. Math. **37** (2007), 1001–1026.

[10] M. Previte and M. Vanderschoot, *The topological dimension of limits of graph substitutions*, Forum Math. **15** (2003), 477–487. MR 1977893 (2004h:54033)

[11] M. Previte and S. H. Yang, *The topological dimension of limits of vertex replacements*, Topology Appl. **153** (2006), 2013–2025. MR 2237593

[12] P. Prusinkiewicz and J. Hanan, *Lindenmayer systems, fractals, and plants*, Lecture Notes in Biomathematics, vol. 79, Springer-Verlag, New York, 1989, With contributions by A. Lindenmayer, F. D. Fracchia and K. Krithivasan. MR 1006213 (91c:68075)

[13] G. Rozenberg and A. Salomaa (Eds.), *Lindenmayer systems: Impacts on theoretical computer science, computer graphics, and developmental biology*, Springer-Verlag, Berlin, 1992. MR 1226682 (93m:68004)

Michelle Previte, School of Science, Penn State Erie, The Behrend College, 4205 College Drive, Erie, PA 16563, USA

E-mail address: MichellePrevite@psu.edu