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# ARGUMENT OF OUTER FUNCTIONS ON THE REAL LINE

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ABSTRACT. A complete description of the modulus of an outer function on the real line is well known. Indeed, this characterization is considered as one of the classical results of the theory of Hardy spaces. However, a satisfactory characterization of the argument of an outer function on the real line is not available yet. In this paper, we define some classes of real functions which can serve as the argument of an outer function. In particular, for any $0 < p \leq \infty$, an increasing bi-Lipschitz function is the argument of an outer function in $H^p(\mathbb{R})$.

## 1. Introduction

Let $h \geq 0$, $\log h \in L^1\left(\frac{dt}{1+t^2}\right)$, and let $\gamma$ be a real constant. Then, the function

$$O(z) = e^{i\,\gamma}\, \exp\left( \frac{i}{\pi} \int_{-\infty}^{\infty} \left( \frac{1}{z-t} + \frac{t}{1+t^2} \right) \log h(t)\; dt \right),$$

is an outer function in the upper half plane. If, moreover, $h \in L^p(dt)$, $0 < p \leq \infty$, then $O$ is also in the Hardy space $H^p(\mathbb{C}_+)$ [1, page 279]. Taking the limit of both sides of

$$\begin{aligned} O(z) = \exp&\left( \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{y}{(x-t)^2+y^2}\, \log h(t)\; dt \right) \\ &\times \exp\left( \frac{i}{\pi} \int_{-\infty}^{\infty} \left( \frac{x-t}{(x-t)^2+y^2} + \frac{t}{1+t^2} \right) \log h(t)\; dt + i\gamma \right) \end{aligned}$$

as $z$ non-tangentially tends to $x \in \mathbb{R}$, we get

$$O(x) = h(x)\, \exp\left( i\, \widetilde{\log h}\,(x) + i\gamma \right) \tag{1.1}$$

for almost all $x \in \mathbb{R}$, where

$$\widetilde{\log h}\,(x) = \frac{1}{\pi} ⨍_{\mathbb{R}} \left( \frac{1}{x-t} + \frac{t}{1+t^2} \right) \log h(t)\; dt$$

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is the Hilbert transform of $\log h$ [8, page 98], [2, page 192]. In particular, we have

$$|\,O(x)\,| = h(x)$$

for almost all $x \in \mathbb{R}$. Therefore, the conditions $h \geq 0$, $\log h \in L^1\big(\frac{dt}{1+t^2}\big)$ and $h \in L^p(dt)$ provide a *complete* description of the modulus of an outer function in $H^p(\mathbb{R})$. On the other hand, there is a natural question about the argument of an outer functions on $\mathbb{R}$:

Open Question. *For which measurable functions $\psi : \mathbb{R} \longrightarrow \mathbb{R}$ is there a nonnegative function $h \not\equiv 0$ such that $h\, e^{i\psi}$ represents an outer function on the real line?*

In other words, which real function $\psi$ can serve as the argument of an outer function on $\mathbb{R}$? The representation (1.1) immediately implies the following result.

Theorem 1.1. *Let $\psi$ be a real measurable function. Then, $\psi$ is the argument of an outer function on the real line if and only if there exists a real constant $\gamma$, a positive measurable function $h : \mathbb{R} \longrightarrow [0, \infty)$ with $\log h \in L^1\big(\frac{dt}{1+t^2}\big)$, and a measurable step function $S : \mathbb{R} \longrightarrow 2\pi\mathbb{Z}$ such that*

$$\psi(x) = \gamma + \widetilde{\log h}\,(x) + S(x) \tag{1.2}$$

*for almost all $x \in \mathbb{R}$. Moreover, $\psi$ is the argument of an outer function in $H^p(\mathbb{R})$, $0 < p \leq \infty$, if and only if $h$ satisfies the extra condition $h \in L^p(dt)$.*

Even though this theorem provides a necessary and sufficient condition for $\psi$ to be the argument of an outer function, it is not so useful in practice. To apply this theorem, we are supposed to find a positive function $h$ so that, among other things, the Hilbert transform of $\log h$ fulfills (1.2). Explicit evaluation of $\widetilde{\log h}$ is somewhat difficult [7], and thus there is normally no clue to ensure that $h$ exists.

A satisfactory answer to the open question is not available and the question is still considered *widely open*. However, in this paper, we define explicitly several classes of real functions which can serve as the argument of outer functions. We show that a *distorted sawtooth function*, a *mainly increasing Lipschitz function*, and an increasing *bi-Lipschitz* function are arguments of outer functions in *any* $H^p$ spaces. Using Levinson distribution, we give another family of arguments generated by zeros of functions in the Cartwright class.

## 2. A representation theorem

Suppose that $\{d_n\}_{n\in\mathbb{Z}}$ is a strictly increasing sequence of real numbers with

$$\lim_{n\to-\infty} d_n = -\infty \quad \text{and} \quad \lim_{n\to\infty} d_n = \infty.$$

Put $\ell_n = (d_n - d_{n-1})/2$ and $c_n = (d_n + d_{n-1})/2$. Suppose, furthermore, there are two positive constants $\ell$ and $L$ such that

$$0 < \ell \leq \ell_n \leq L < \infty \tag{2.1}$$

for all $n \in \mathbb{Z}$. In other words, the intervals $(d_n, d_{n+1})$ get neither too big nor too small. In more technical terms, the family $\{\,(d_n, d_{n+1})\,\}_{n\in\mathbb{Z}}$ is a *system of short intervals*, a notion occurring in some theorems of Fourier analysis.

Let $g$ be a real function such that, for each $d_n$,

$$\lim_{x\to d_n^+} g(x) = -\pi$$

and

$$\lim_{x\to d_n^-} g(x) = \pi.$$

Moreover, suppose that, for each $x_1$, $x_2 \in (d_n, d_{n+1})$, we have

$$|g(x_2) - g(x_1)| \leq \mathrm{Lip}_g \;\; |x_2 - x_1|, \tag{2.2}$$

where the constant $\mathrm{Lip}_g$ does not depend on $n$. Such a function is called a *distorted sawtooth function*. If $g$ is linear on each interval, then (2.2) is automatically fulfilled. In this section, we study the Hilbert transform of these functions.

This class of functions, and its generalizations, has been studied in [4]. Nevertheless, for the reader's convenience, we mention a special and reduced version of the representation theorem about the behavior of the Hilbert transform of distorted sawtooth functions (as defined here).

Let us start with the linear case. Let $v(x) = \pi\, x\, \chi_{[-1,1]}(x)$, where $\chi_{[-1,1]}$ is the characteristic function of $[-1, 1]$. Then, by a direct calculation, we have

$$\tilde{v}(x) = \frac{1}{\pi} \fint_{\mathbb{R}} \left( \frac{1}{x-t} + \frac{t}{1+t^2} \right) vu(t)dt = -\frac{\pi}{2} + x\, \log \left| \frac{x+1}{x-1} \right|. \tag{2.3}$$

One can easily verify that $\tilde{v}$ satisfies the following properties:

$$\begin{array}{ll} \tilde{v}(x) \geq -\pi/2 & \text{for } |x| < 1, \\ \tilde{v}(x) > 0 & \text{for } |x| > 1, \\ \tilde{v}(x) = O(\,1/x^2\,) & \text{as } |x| \to \infty. \end{array}$$

Let

$$u(x) = \sum_{n=-\infty}^{\infty} v\left( \frac{x - c_n}{\ell_n} \right).$$

For each $x \in \mathbb{R} \setminus \{d_n\}_{n\in\mathbb{Z}}$, at most one of the terms $v(\,(x - c_n)/\ell_n\,)$ is non-zero. The sum $u$ is thus a function with a graph shaped like a sawblade which grows linearly from $-\pi$ to $\pi$ on each $(d_{n-1}, d_n)$, and then jumps downward by $2\pi$

at each $d_n$. The slope of each line is at most $\pi/\ell$ and at least $\pi/L$. Since $u$ has a bounded primitive, for almost all $x \in \mathbb{R}$, we have

$$\tilde{u}(x) = \text{const} + \lim_{\varepsilon\to 0, N\to\infty} \frac{1}{\pi} \int_{I_{N\varepsilon}} \frac{u(t)}{x-t}\, dt,$$

where $I_{N\varepsilon} = (d_{-N}, x-\varepsilon) \cup (x+\varepsilon, d_N)$, and the passages to the limit can be taken in any order. The constant term is created by $\int_{\mathbb{R}} t/(1+t^2)\, u(t)\, dt$. Hence,

$$\begin{aligned} \tilde{u}(x) &= \text{const} + \lim_{\varepsilon\to 0, N\to\infty} \frac{1}{\pi} \int_{I_{N\varepsilon}} \frac{u(t)}{x-t}\, dt \\ &= \text{const} + \lim_{\varepsilon\to 0, N\to\infty} \frac{1}{\pi} \int_{I_{N\varepsilon}} \frac{\sum_{n=-N}^{N} v((t-c_n)/\ell_n)}{x-t}\, dt. \end{aligned}$$

If $d_{m-1} < x < d_m$, the integral on the right is equal, for large $N$ and $\varepsilon > 0$ small enough, to

$$\begin{aligned} \tilde{u}(x) = \text{const} + &\frac{1}{\pi} \sum_{|n|\le N,\, n\neq m} \int_{d_{n-1}}^{d_n} \frac{v((t-c_n)/\ell_n)}{x-t}\, dt \\ &+ \frac{1}{\pi} \int_{d_{m-1}<t<d_m,\ |t-x|>\varepsilon} \frac{v((t-c_m)/\ell_m)}{x-t}\, dt. \end{aligned}$$

For $n \neq m$, the substitution $\tau = (t-c_n)/\ell_n$ converts the corresponding term of the summation to

$$\frac{1}{\pi} \int_{-1}^{1} \frac{v(\tau)}{(\frac{x-c_n}{\ell_n}) - \tau}\, d\tau = \tilde{v}\left(\frac{x-c_n}{\ell_n}\right),$$

and the remaining integral is similarly seen to equal

$$\frac{1}{\pi} \int_{|\tau|<1,\ |\tau - \frac{x-c_m}{\ell_m}| > \frac{\varepsilon}{\ell_m}} \frac{v(\tau)}{(\frac{x-c_m}{\ell_m}) - \tau}\, d\tau.$$

Since $v(\tau) = 0$ for $|\tau| > 1$, this tends to $\tilde{v}(\,(x-c_m)/\ell_m\,)$ when $\varepsilon \to 0$. We therefore have

$$\lim_{\varepsilon\to 0} \int_{I_{N\varepsilon}} \frac{u(t)}{x-t}\, dt = \sum_{n=-N}^{N} \tilde{v}\left(\frac{x-c_n}{\ell_n}\right),$$

and finally

$$\tilde{u}(x) = \text{const} + \sum_{n=-\infty}^{\infty} \tilde{v}\left(\frac{x-c_n}{\ell_n}\right).$$

On $(c_m - \ell_m, c_m + \ell_m)$, $\tilde{v}(\,(x-c_m)/\ell_m\,) \geq -\pi/2$. All other terms in the summation are positive. Thus $\tilde{u}$ is also bounded below.

Let us look at the local behaviour of $\tilde{u}$. Clearly, on $\mathbb{R} \setminus \{d_n\}_{n\in\mathbb{Z}}$, $\tilde{u}$ is a $\mathcal{C}^\infty$ function. Now, fix $d_n = c_n + \ell_n = c_{n+1} - \ell_{n+1}$. Here, for $x \in (c_n, c_{n+1})$, a neighbourhood of $d_n$, we have, by (2.3),

$$\begin{aligned}\tilde{v}\left(\frac{x-c_n}{\ell_n}\right) &= -\frac{\pi}{2} + \left(\frac{x-d_n+\ell_n}{\ell_n}\right)\log\left|\frac{x-d_n+2\ell_n}{x-d_n}\right| \\ &= -\log|x-d_n| + v_l(x),\end{aligned}$$

and again

$$\begin{aligned}\tilde{v}\left(\frac{x-c_{n+1}}{\ell_{n+1}}\right) &= -\frac{\pi}{2} + \left(\frac{x-d_n-\ell_{n+1}}{\ell_{n+1}}\right)\log\left|\frac{x-d_n}{x-d_n-2\ell_{n+1}}\right| \\ &= -\log|x-d_n| + v_r(x),\end{aligned}$$

where $v_l$ and $v_r$ are continuous functions on $(c_n, c_{n+1})$. For $x \in (c_n, c_{n+1})$ and $m \geq n+2$ we also have

$$\begin{aligned}0 < \tilde{v}\left(\frac{x-c_m}{\ell_m}\right) &\leq \frac{C}{(\frac{x-c_m}{\ell_m})^2} \leq \frac{C}{(\frac{c_{n+1}-c_m}{\ell_m})^2} \\ &= \frac{C}{\left(\frac{\ell_{n+1}+2\ell_{n+2}+\cdots+2\ell_{m-1}+\ell_m}{\ell_m}\right)^2},\end{aligned}$$

and, for large values of $m$,

$$\frac{1}{\left(\frac{\ell_{n+1}+2\ell_{n+2}+\cdots+2\ell_{m-1}+\ell_m}{\ell_m}\right)^2} \leq \frac{1}{\left(\frac{2(m-n-1)\ell}{\ell_{max}}\right)^2 - 1} = O\left(\frac{1}{m^2}\right).$$

Thus

$$\sum_{m=n+2}^{\infty} \tilde{v}\left(\frac{x-c_m}{\ell_m}\right),$$

and similarly

$$\sum_{m=-\infty}^{n-1} \tilde{v}\left(\frac{x-c_m}{\ell_m}\right),$$

are continuous functions on $(c_n, c_{n+1})$. Therefore, for each $x \in (c_n, c_{n+1})$,

$$\tilde{u}(x) = -2\log|x-d_n| + w(x),$$

where $w$ is a continuous function. Hence, for any $m \geq 0$, the ratio

$$\frac{e^{-\tilde{u}(x)}}{\prod_{k=1}^{m}(x-d_{j_k})^2}, \tag{2.4}$$

formed using arbitrary distinct $d_{j_k}$ from among the $d_n$, is a bounded continuous function.

Now, we consider an arbitrary distorted sawtooth function. By (2.1) and (2.2), we have

$$|\,g(x)\,| \le |\,g(d_n^-)\,| + \mathrm{Lip}_g \;\; |x - d_n| \le \pi + 2\;\mathrm{Lip}_g\; L < \infty$$

for each $x \in (d_{n-1}, d_n)$. Therefore, $g$ is a bounded function. Let $u$ be the linear sawtooth function constructed with discontinuities at the $d_n$, and let

$$r = g - u, \tag{2.5}$$

with $r(d_n) = 0$, $n \in \mathbb{Z}$. Then $r$ is a bounded continuous function on $\mathbb{R}$. Moreover, for $x_1$, $x_2 \in (d_{n-1}, d_n)$, we have

$$\begin{aligned} \frac{|r(x_2) - r(x_1)|}{|x_2 - x_1|} &\le \frac{|g(x_2) - g(x_1)|}{|x_2 - x_1|} + \frac{|u(x_2) - u(x_1)|}{|x_2 - x_1|} \\ &\le \mathrm{Lip}_g + \frac{|u(d_n) - u(d_{n-1})|}{|d_n - d_{n-1}|} \\ &= \mathrm{Lip}_g + \frac{2\,\pi}{|d_n - d_{n-1}|} \le \mathrm{Lip}_g + \frac{\pi}{\ell}. \end{aligned}$$

Since $r$ is continuous, this inequality holds for all $x_1$, $x_2 \in [d_{n-1}, d_n]$. Hence it holds for all $x_1, x_2 \in \mathbb{R}$, i.e., for all $x_1$, $x_2 \in \mathbb{R}$,

$$|r(x_2) - r(x_1)| \le \mathrm{Lip}_r \;|x_2 - x_1|, \tag{2.6}$$

where

$$\mathrm{Lip}_r \le \mathrm{Lip}_g + \frac{\pi}{\ell}. \tag{2.7}$$

Since $r$ is of bounded variation on finite intervals, $r'(x)$ exists for almost all $x \in \mathbb{R}$, and

$$|r'(x)| \le \mathrm{Lip}_r\,. \tag{2.8}$$

LEMMA 2.1. *The Hilbert transform of $r$ grows logarithmically as $|x| \to \infty$, i.e.,*

$$|\tilde{r}(x)| \le \frac{1}{\pi}\left(4\,\|r\|_\infty + 6\,L\;\mathrm{Lip}_r\right)\log|x| + O(1) \quad \textit{for } |x| \to \infty.$$

*Proof.* Without loss of generality assume that $r(0) = 0$, since otherwise we can work with $r(x + d_0)$. Since $r$ is Lipschitz, and $r(0) = 0$, we can replace $t/(1 + t^2)$ by $1/t$ in the definition of Hilbert transform, i.e.,

$$\tilde{r}(x) = \mathrm{const} + \frac{1}{\pi}\, ⨍_{\mathbb{R}} \left(\frac{1}{x - t} + \frac{1}{t}\right) r(t)\,dt. \tag{2.9}$$

Let $x$ be a large positive number and suppose that $d_n < x < d_{n+1}$. We estimate the latter integral over three disjoint intervals $(-\infty, d_{n-1})$, $(d_{n-1}, d_{n+2})$

and $(d_{n+2}, \infty)$. By (2.1), we have

$$\left| \int_{d_{n+2}}^{\infty} \left( \frac{1}{x-t} + \frac{1}{t} \right) r(t)\, dt \right| \le \int_{d_{n+2}}^{\infty} \left( \frac{1}{t-x} - \frac{1}{t} \right) \|r\|_\infty \, dt$$
$$\le \|r\|_\infty \int_{x+2\ell}^{\infty} \left( \frac{1}{t-x} - \frac{1}{t} \right) dt$$
$$= \|r\|_\infty \log \left| \frac{x+2\ell}{2\ell} \right|.$$

Therefore,

$$\left| \int_{d_{n+2}}^{\infty} \left( \frac{1}{x-t} + \frac{1}{t} \right) r(t)\, dt \right| \le \|r\|_\infty \log|x| + O(1). \tag{2.10}$$

To estimate the integral in (2.9) over $(d_{n-1}, d_{n+2})$, we integrate by parts:

$$\left( \int_{d_{n-1}}^{x-\varepsilon} + \int_{x+\varepsilon}^{d_{n+2}} \right) \left( \frac{1}{x-t} + \frac{1}{t} \right) r(t)\, dt$$
$$= \log \left| \frac{t}{x-t} \right| r(t) \Big|_{d_{n-1}}^{x-\varepsilon} + \log \left| \frac{t}{x-t} \right| r(t) \Big|_{x+\varepsilon}^{d_{n+2}}$$
$$- \left( \int_{d_{n-1}}^{x-\varepsilon} + \int_{x+\varepsilon}^{d_{n+2}} \right) \log \left| \frac{t}{x-t} \right| r'(t)\ dt.$$

Since $r(d_{n-1}) = r(d_{n+2}) = 0$,

$$\left| \ \log \left| \frac{t}{x-t} \right| r(t) \Big|_{t=d_{n-1}}^{x-\varepsilon} + \log \left| \frac{t}{x-t} \right| r(t) \Big|_{t=x+\varepsilon}^{d_{n+2}} \right|$$
$$= \left| \ \log \left| \frac{x-\varepsilon}{\varepsilon} \right| r(x-\varepsilon) - \log \left| \frac{x+\varepsilon}{\varepsilon} \right| r(x+\varepsilon) \ \right|$$
$$\le \log \left( \frac{x-\varepsilon}{\varepsilon} \right) \ |r(x-\varepsilon) - r(x)|$$
$$+ \log \left( \frac{x+\varepsilon}{\varepsilon} \right) \ |r(x+\varepsilon) - r(x)|$$
$$+ \left| \ \log \left( \frac{x-\varepsilon}{\varepsilon} \right) - \log \left( \frac{x+\varepsilon}{\varepsilon} \right) \ \right| \ |r(x)|$$
$$\le \mathrm{Lip}_r\ \varepsilon \log \left( \frac{x-\varepsilon}{\varepsilon} \right) + \mathrm{Lip}_r\ \varepsilon \log \left( \frac{x+\varepsilon}{\varepsilon} \right)$$
$$+ \left| \ \log \left( \frac{x-\varepsilon}{x+\varepsilon} \right) \ \right| \|r\|_\infty \longrightarrow 0 \qquad \text{as} \qquad \varepsilon \to 0.$$

On the other hand, by (2.1) and (2.8), we have

$$\begin{aligned}
\Big| \left( \int_{d_{n-1}}^{x-\varepsilon} + \int_{x+\varepsilon}^{d_{n+2}} \right) \log \left| \frac{t}{x-t} \right| r'(t)\, dt \Big| &\leq \int_{d_{n-1}}^{d_{n+2}} \Big| \log \left| \frac{t}{x-t} \right| \Big| \, \|r'\|_\infty \, dt \\
&\leq \|r'\|_\infty \int_{d_{n-1}}^{d_{n+2}} \log t \; dt + 2\,\|r'\|_\infty \int_0^{d_{n+2}-d_{n-1}} \big| \log \tau \big| \, d\tau \\
&\leq \|r'\|_\infty \,(d_{n+2}-d_{n-1}) \, \log d_{n+2} + 2\,\|r'\|_\infty \int_0^{d_{n+2}-d_{n-1}} \big| \log \tau \big| \, d\tau \\
&\leq 6\,L\,\|r'\|_\infty \, \log\big(x+4\,L\big) + 2\,\|r'\|_\infty \int_0^{6\,L} \big| \log \tau \big| \, d\tau \\
&\leq 6\,L \;\mathrm{Lip}_r \, \log\big(x+4\,L\big) + 2 \;\mathrm{Lip}_r \int_0^{6\,L} \big| \log \tau \big| \, d\tau.
\end{aligned}$$

Therefore,

$$\left| \,⨍_{d_{n-1}}^{d_{n+2}} \left( \frac{1}{x-t} + \frac{1}{t} \right) r(t)\, dt \,\right| \leq 6\,L \;\mathrm{Lip}_r \, \log|x| + O(1). \tag{2.11}$$

To estimate the integral in (2.9) over $(-\infty, d_{n-1})$, due to the presence of the term $1/t$, we break this interval to three subintervals $(-\infty,-1)$, $(-1,1)$ and $(1, d_{n-1})$. We thus have

$$\begin{aligned}
\left| \int_{-\infty}^{-1} \left( \frac{1}{x-t} + \frac{1}{t} \right) r(t)\, dt \right| &\leq \int_{-\infty}^{-1} \left( \frac{1}{t-x} - \frac{1}{t} \right) \|r\|_\infty \; dt \\
&= \|r\|_\infty \, \log|x| + o(1),
\end{aligned} \tag{2.12}$$

$$\begin{aligned}
\left| \int_{-1}^{1} \left( \frac{1}{x-t} + \frac{1}{t} \right) r(t)\; dt \right| &\leq \int_{-1}^{1} \frac{|r(t)|}{x-t} \; dt + \int_{-1}^{1} \frac{|r(t)|}{|t|} \; dt \\
&\leq \|r\|_\infty \, \log \left| \frac{x+1}{x-1} \right| + 2 \;\mathrm{Lip}_r = O(1),
\end{aligned} \tag{2.13}$$

$$\begin{aligned}
\left| \int_{1}^{d_{n-1}} \left( \frac{1}{x-t} + \frac{1}{t} \right) r(t)\; dt \right| &\leq \int_{1}^{d_{n-1}} \left( \frac{1}{x-t} + \frac{1}{t} \right) \|r\|_\infty \; dt \\
&= \|r\|_\infty \, \log \left( \frac{d_{n-1}\,(x-1)}{x-d_{n-1}} \right) \leq \|r\|_\infty \, \log \left( \frac{x\,(x-1)}{2\,\ell} \right) \\
&= 2\,\|r\|_\infty \, \log|x| + O(1).
\end{aligned} \tag{2.14}$$

Finally, by (2.10), (2.11), (2.12), (2.13) and (2.14),

$$\begin{aligned}|\tilde{r}(x)| &= \left|\frac{1}{\pi} \, -\!\!\!\!\!\!\int_{\mathbb{R}} \left(\frac{1}{x-t} + \frac{1}{t}\right) r(t)\, dt\right| \\ &\le \frac{1}{\pi}\left(4\,\|r\|_\infty + 6\,L\ \mathrm{Lip}_r\right)\log|x| + O(1),\end{aligned}$$

for $|x|$ large enough. □

The following representation theorem, by itself, is an interesting result. Moreover, it plays a major role in characterizing some classes of the arguments of outer functions. Further generalizations with applications in model subspaces are available in [3], [4], and [6].

Theorem 2.2 (Representation Theorem). *Let $g$ be a distorted sawtooth function, and let $p > 0$. Then, there are a step function $S(x)$ with values all equal to integral multiples of $2\pi$ and also a measurable function $m \ge 0$ with $m \in L^\infty(dt) \cap L^p(dt)$ and $\log m \in L^1\left(\frac{dt}{1+t^2}\right)$, and a real constant $\gamma$, such that*

$$g = \gamma + \widetilde{\log m} + S.$$

*Proof.* Let

$$\alpha_r = \frac{1}{\pi}\left(4\,\|r\|_\infty + 6\,L\ \mathrm{Lip}_r\right),$$

and choose $n \in \mathbb{N}$ such that $n > \alpha_r/2 + 1/2p$, and choose any $n$ different points $d_{j_k}$ from among the $\{d_j\}_{j\in\mathbb{Z}}$. Put

$$m(x) = \frac{e^{-\tilde{g}(x)}}{\prod_{k=1}^{n}(x-d_{j_k})^2}.$$

Write $g = u + r$ as in (2.5). By (2.4), the function

$$e^{-\tilde{r}(x)} \cdot \frac{e^{-\tilde{u}(x)}}{\prod_{k=1}^{n}(x-d_{j_k})^2}$$

is bounded and continuous on any bounded interval. For large values of $|x|$, by (2.4) and Lemma 2.1,

$$e^{-\tilde{u}(x)} \cdot \frac{e^{-\tilde{r}(x)}}{|x|^{\alpha_r}}$$

is bounded. Hence, for large values of $|x|$,

$$m(x) \le \frac{C}{|x|^{2n-\alpha_r}}.$$

Therefore, $m \in L^p(\mathbb{R}) \cap L^\infty(\mathbb{R})$, and

$$\log m(x) = -\tilde{g}(x) - 2\sum_{k=1}^{n}\log|x-d_{j_k}| \in L^1\left(\frac{dt}{1+t^2}\right).$$

The Hilbert transform of $\log|t|$ equals to $-\pi/2\,\mathrm{sgn}(t)$. Hence

$$\begin{aligned}\widetilde{\log m}\,(x) &= -\widetilde{\widetilde{g}}(x) - 2\sum_{k=1}^{n}\Big(-\frac{\pi}{2}\,\mathrm{sgn}(x-d_{j_k})\Big)\\ &= -n\pi + \text{const} + g(x) + \pi\sum_{k=1}^{n}\Big(1+\mathrm{sgn}(x-d_{j_k})\Big).\end{aligned}$$

Thus

$$g = \gamma + \widetilde{\log m} + S,$$

where

$$S(x) = -\pi\sum_{k=1}^{n}\Big(1+\mathrm{sgn}(x-d_{j_k})\Big)$$

is a step function with values in $2\pi\mathbb{Z}$. □

Let $f$ be a real function defined on $\mathbb{R}$. Then, $f$ is called a *mainly increasing Lipschitz function* if $f$ is Lipschitz, i.e.,

$$|\,f(x_2) - f(x_1)\,| \le \mathrm{Lip}_f\ |x_2 - x_1|,$$

and there is an increasing sequence $\{d_n\}_{n\in\mathbb{Z}}$ such that, for each $n$,

$$f(d_n) = 2\,\pi\,n,$$

and

$$\sup_{n\in\mathbb{Z}}(d_{n+1}-d_n) < \infty.$$

In particular, a real function $\varphi : \mathbb{R} \longrightarrow \mathbb{R}$ is an increasing *bi-Lipschitz* function if there are $c, C > 0$ such that

$$c\,|x_2 - x_1| \le |\,\varphi(x_2) - \varphi(x_1)\,| \le C\,|x_2 - x_1|$$

for every $x_1,\, x_2 \in \mathbb{R}$, and

$$\lim_{x\to\pm\infty}\varphi(x) = \pm\infty.$$

Then, according to the intermediate value theorem, there is $d_n$, for each $n\in\mathbb{Z}$, such that $\varphi(d_n) = 2\,\pi\,n$. Since $\varphi$ is bi-Lipschitz, we have

$$\frac{2\,\pi}{C} \le d_{n+1} - d_n \le \frac{2\,\pi}{c}.$$

Therefore, $\varphi$ is a mainly increasing Lipschitz function.

Let $f$ be any mainly increasing Lipschitz function. Put $S_1(x) = 2n\pi$ for $x \in (d_n, d_{n+1})$. Then $g = f - S_1 - \pi$ is a distorted sawtooth function. According to Theorem 2.2, there are $m$, $\gamma_1$ and $S_2$ satisfying the required properties and, moreover, $g = \gamma_1 + \widetilde{\log m} + S_2$. Hence

$$f = \pi + \gamma_1 + \widetilde{\log m} + (S_1 + S_2). \tag{2.15}$$

## 3. A class of arguments of an outer functions

We are now ready to apply our representation theorem to show that a distorted sawtooth function, a mainly increasing Lipschitz function and, in particular, an increasing bi-Lipschitz function can serve as the argument of an outer function.

THEOREM 3.1. *Let $\varphi$ be a distorted sawtooth function on $\mathbb{R}$, and let $p>0$. Then there exists a function $h\geq 0$, $h\not\equiv 0$, such that $h\,e^{i\,\varphi}$ is an outer function in $H^p(\mathbb{R})\cap H^\infty(\mathbb{R})$.*

*Proof.* According to Theorem 2.2, there are functions $m$ and $S$ and a real constant $\gamma$, where $S$ is a step function with values all equal to integral multiples of $2\,\pi$ and $m\in L^p(\mathbb{R})\cap L^\infty(\mathbb{R})$ with $\log m\in L^1\big(\frac{dt}{1+t^2}\big)$, such that

$$\varphi=\gamma+\widetilde{\log m}+S. \tag{3.1}$$

Hence, by Theorem 1.1, $\varphi$ is the argument of an outer function in $H^p(\mathbb{R})\cap H^\infty(\mathbb{R})$. □

Similarly, applying Theorem 1.1 and (2.15), gives the following result:

COROLLARY 3.2. *Let $\varphi$ be a mainly increasing Lipschitz function on $\mathbb{R}$, and let $p>0$. Then, there exists a function $h\geq 0$, $h\not\equiv 0$, such that $h\,e^{i\,\varphi}$ is an outer function in $H^p(\mathbb{R})\cap H^\infty(\mathbb{R})$.*

COROLLARY 3.3. *Let $\varphi$ be an increasing bi-Lipschitz real function on $\mathbb{R}$, and let $p>0$. Then, there exists a function $h\geq 0$, $h\not\equiv 0$, such that $h\,e^{i\,\varphi}$ is an outer function in $H^p(\mathbb{R})\cap H^\infty(\mathbb{R})$.*

As a very particular example, let $\psi(x)=x$. On the one hand, $e^{ix}$ is an inner function, and on the other hand, Corollary 3.3 provides us an $h\geq 0$, $h\not\equiv 0$, such that $h(x)\,e^{i\,x}$ is an outer function. Hence we have a pair of inner and outer functions having the same argument on the real line.

## 4. Arguments constructed by Levinson's distributions

Let $\{x_n\}_{n\in\mathbb{Z}}$ be an increasing sequence of real numbers satisfying $\lim_{|n|\to\infty}|x_n|=\infty$ (repetition is allowed). The counting function $\nu_{\{x_n\}}(x)$ is defined to be constant between $x_{n-1}$ and $x_n$, and at each point $x_n$ jumps up by $k$ units, where $k$ is the number of times that $x_n$ repeats. The value of $\nu_{\{x_n\}}(x)$ at $x_n$ is defined such that $\nu_{\{x_n\}}$ is continuous from the right (for our application, this restriction is not necessary).

For a complex number $z=x+iy$, $y\neq 0$, let

$$\begin{aligned}\varphi_z(s)&=\int_0^s\frac{|y|}{|t-z|^2}\;dt\\&=\arctan(\,(s-x)/y\,)+\arctan(\,x/y\,).\end{aligned}$$

Let $\{z_k\}_{k\geq 1}$, $z_k \in \mathbb{C}$, be a Levinson distribution with density $\sigma$ [5]. Let $\{x_n\}_{n\in\mathbb{Z}}$ be the subsequence of real numbers in $\{z_k\}_{k\geq 1}$. We assume that they are indexed so that $\lim_{|n|\to\infty} |x_n| = \infty$, and that $x_n \leq x_m$ if $n \leq m$. According to Levinson's theorem

$$f(z) = \prod_k \left(1 - \frac{z}{z_k}\right) e^{z/z_k}$$

is an entire function of exponential type so that $\log|f(t)| \in L^1\left(\frac{dt}{1+t^2}\right)$ and

$$\limsup_{y\to+\infty} \frac{\log|f(iy)|}{y} + \limsup_{y\to+\infty} \frac{\log|f(-iy)|}{y} = 2\pi\sigma.$$

Therefore, according to the Main Theorem in [7],

$$\widetilde{\log|f|}(x) = \pi\sigma x - \pi\nu_{\{x_n\}}(x) - \sum_{\Im z_k\neq 0} \varphi_{z_k}(x).$$

Finally, in the light of Theorem 1.1, we see that, for any measurable step function $S : \mathbb{R} \longrightarrow 2\pi\mathbb{Z}$,

$$\psi(x) = S(x) + \pi\sigma x - \pi\nu_{\{x_n\}}(x) - \sum_{\Im z_k\neq 0} \varphi_{z_k}(x) \tag{4.1}$$

is the argument of an outer function.

If we know more about $f$ on the real line, we get more information about our outer function. For example, if $f$ is in the Paley-Wiener space, then we know that the outer function given in (4.1) is in $H^2(\mathbb{R})$.

If all the points of our sequence are on the real line, then we have somewhat simpler conditions.

THEOREM 4.1. *Let $\{x_n\}_{n\in\mathbb{Z}}$ be a non-decreasing sequence of real numbers such that $\lim_{|n|\to\infty} |x_n| = \infty$. Suppose that*

$$\lim_{|x|\to\infty} \frac{\nu_{\{x_n\}}(x)}{x}$$

*exists, say equal to $\sigma$, and*

$$\sum_{|x_n|\leq r,\ x_n\neq 0} \frac{1}{x_n}$$

*tends to a finite limit as $r \to \infty$. Then*

$$\psi(x) = \pi\sigma x - \pi\nu_{\{x_n\}}(x)$$

*is the argument of an outer function.*

Considering $\mathbb{Z}$ as our sequence, i.e., $x_n = n$, we see that the sawtooth function $\psi(x) = \pi(x - [x])$ is the argument of an outer function.

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