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STRONGLY IRREDUCIBLE DECOMPOSITION AND SIMILARITY CLASSIFICATION OF OPERATORS

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ABSTRACT. Let $\mathcal{H}$ be a complex separable Hilbert space and let $\mathcal{L}(\mathcal{H})$ denote the collection of bounded linear operators on $\mathcal{H}$. In this paper, we show that if $T = A_1^{(n_1)} \oplus A_2^{(n_2)} \oplus \cdots \oplus A_k^{(n_k)}$, where $A_i \not\sim A_j$ for $1 \leq i \neq j \leq k$, and $\mathcal{A}'(A_i)/\operatorname{rad}\mathcal{A}'(A_i)$ is commutative, $K_0(\mathcal{A}'(A_i)) \cong Z$ for $i = 1, 2, \dots, k$, and for any positive integer n and minimal idempotent $P \in \mathcal{A}'(T^{(n)})$, $\mathcal{A}'(T^{(n)}|_{P\mathcal{H}^{(n)}})/\operatorname{rad}\mathcal{A}'(T^{(n)}|_{P\mathcal{H}^{(n)}})$ is commutative, then T is a stably finitely decomposable operator and has a stably unique (SI) decomposition up to similarity. Moreover, we give a similarity classification of the operators which satisfy the above conditions by using the K_0-group of the commutant algebra as an invariant.

1. Introduction

Let $\mathcal{H}$ be a complex separable Hilbert space, and let $\mathcal{L}(\mathcal{H})$ denote the collection of bounded linear operators on $\mathcal{H}$. A basic problem in operator theory is to determine when two operators A and B in $\mathcal{L}(\mathcal{H})$ are similar, that is, when there exists an invertible operator X on $\mathcal{H}$ satisfying $A = X^{-1}BX$. One of the most important problems in operator theory is to find invariants that can be used to determine when two operators are similar.

When $\mathcal{H}$ is a finite dimensional space, from the Jordan theorem we see that the characteristic roots and generalized characteristic subspaces of the operator are complete similarity invariants. When $\mathcal{H}$ is an infinite dimensional space, a general solution to this problem is not known; we can only find similarity invariants for some special classes of operators. For two star cyclic normal operators or star cyclic subnormal operators A and B, J.B. Conway showed that A and B are similar if and only if the scalar value spectral measures induced by A and B are equivalent (cf. [Co]). A.L. Shields (cf. [Sh]) proved that a complete similarity invariant for injective shift operators is the rate of the weighted sequence.

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As the basic element of non-commutative topology, K-theory opens up wide prospects for studying the structure theory of C^*-algebra. In K-theory one considers a pair of functors, K_0 and K_1. The group $K_0(\mathcal{A})$ is given an ordering that makes it an ordered Abelian group. For certain classes of C^*-algebras, the K-group is a complete invariant. In the early 1970s, G. Elliott (cf. [El1], [El2]) showed that AF-algebras (the so-called "approximately finite dimensional" C^*-algebras) are classified by their ordered K_0-groups. Today K-theory is an active research area, and a much used tool for the study of C^*-algebras.

In [EGL] and [Go], G. Elliott, G. Gong and L. Li successfully classified simple AH-algebras of finite dimensional local spectra using scaled order K-groups, spaces of tracial states and the relation between them as invariants. If one further assumes that the AH algebras are of real rank zero, then the scaled ordered K-group alone is an invariant (the other parts of the invariant are redundant). This result was previously obtained in [EG] and [DG].

In the spirit of the above work, we seek to obtain complete similarity invariants of operators in terms of the ordered K-groups of the commutant algebras of the operators.

For a unital Banach algebra $\mathcal{A}$, $\operatorname{rad}\mathcal{A}$ denotes the Jacobson radical of $\mathcal{A}$ and $\mathcal{A}'(T)$ denotes the commutant algebra of an operator T, i.e., $\mathcal{A}'(T) = \{S \in \mathcal{L}(\mathcal{H}) | ST = TS\}$ (cf. [Co], [Gi], [JW], [Jia]). Also, C denotes the complex plane, Z denotes the group of integers, and $N = \{0, 1, 2, \dots\}$.

The famous Jordan theorem in matrix algebra states that every $n \times n$ matrix can be written uniquely up to similarity as the direct sum of finite Jordan blocks, i.e., for all $A \in M_n(C)$, we have $A \sim \bigoplus_{i=1}^{l}(\lambda_i + J_{n_i})^{(m_i)}$, where $\lambda_i \in C$, J_{n_i} is an $n_i \times n_i$ nilpotent Jordan block, and $(\lambda_i, n_i) \neq (\lambda_j, n_j)$ for $i \neq j$. Note that J_{n_i} is not similar to J_{n_j} (denoted by $J_{n_i} \nsim J_{n_j}$) for $n_i \neq n_j$, and if $J_{n_i} \cdot S = S \cdot J_{n_j}$ and $J_{n_j} \cdot T = T \cdot J_{n_i}$, then $ST \in \operatorname{rad}\mathcal{A}'(J_{n_i})$. A simple computation shows that $\mathcal{A}'(A)/\operatorname{rad}\mathcal{A}'(A) \cong \bigoplus_{i=1}^{l} M_{m_i}(C)$. We can prove that two $n \times n$ matrices A and B are similar if and only if

$$(K_0(\mathcal{A}'(A \oplus B)), \bigvee(\mathcal{A}'(A \oplus B)), I) \overset{h}{\cong} (Z^{(l)}, N^{(l)}, 1),$$

and $h([I]) = \sum_{i=1}^{l} n_i e_i$, where I is the unit of $\mathcal{A}'(A \oplus B)$, and $\{e_i\}_{i=1}^{l}$ are the generators of $N^{(l)}$. The above theorem is equivalent to the Jordan theorem (cf. [CFJ]).

When $\mathcal{H}$ is an infinite dimensional Hilbert space, such results are only known for special classes of operators.

An operator A in $\mathcal{L}(\mathcal{H})$ is said to be strongly irreducible, and we write $A \in (SI)$, if $\mathcal{A}'(T)$ has no non-trivial idempotent. It is well-known that strongly irreducible operators are analogues of Jordan blocks in $\mathcal{L}(\mathcal{H})$ (cf. [Co], [Gi], [JW], [Ji]). Thus we consider the following generalization of the Jordan decomposition for an element $A \in \mathcal{L}(\mathcal{H})$: $A \sim \bigoplus_{i=1}^{l} A_i^{(m_i)}$, where

$A_i \in (SI)$, and $A_i \not\sim A_j$ when $i \neq j$. D.A. Herrero, C.L. Jiang and Z.Y. Wang (cf. [He1], [He2], [JW]) proved that the operator class $\mathcal{F} = \{T : T$ can be written as the direct sum of finite (SI) operators$\}$ is dense in $\mathcal{L}(\mathcal{H})$ under the norm topology. Therefore, it is of interest to find a complete similarity invariant of $\mathcal{F}$.

M.J. Cowen and R.G. Douglas (cf. [CD]) introduced a class of operators related to complex geometry, which are now referred to as Cowen-Douglas operators. The Cowen-Douglas operators play an important role in studying the structure of non-self-adjoint operators (cf. [He2], [JW]). Using techniques of complex geometry and K-theory, C.L. Jiang showed that two strongly irreducible Cowen-Douglas operators A and B are similar if and only if

$$(K_0(\mathcal{A}'(A \oplus B)), \bigvee(\mathcal{A}'(A \oplus B)), I) \cong (Z, N, 1),$$

where I is the unit of $\mathcal{A}'(A \oplus B)$ (cf. [Jia]). This shows that the scaled ordered K_0-group of the commutant algebra is a similarity invariant for strongly irreducible Cowen-Douglas operators.

The main result of this paper is as follows:

Main Theorem (Theorem 4.4). *Suppose $A, B \in \mathcal{L}(\mathcal{H})$, and*

$$A = A_1^{(n_1)} \oplus A_2^{(n_2)} \oplus \cdots \oplus A_k^{(n_k)},$$
$$B = B_1^{(m_1)} \oplus B_2^{(m_2)} \oplus \cdots \oplus B_l^{(m_l)},$$

where $A_i, B_j \in (SI)$ for $i = 1, 2, \ldots, k, j = 1, 2, \ldots, l$, A_i and B_j are not similarity equivalent to each other, and A, B and A_i, B_j ($i = 1, 2 \ldots, k$, $j = 1, 2, \ldots, l$) satisfy the following conditions:

(1) $K_0(\mathcal{A}'(A_i)) = Z$, $K_0(\mathcal{A}'(B_j)) = Z$ *for* $i = 1, 2, \ldots, k$, $j = 1, 2, \ldots, l$.
(2) *For any positive integer n and minimal idempotent $P \in \mathcal{A}'(T^{(n)})$, $\mathcal{A}'(T^{(n)}|_{P\mathcal{H}^{(n)}}) / \operatorname{rad} \mathcal{A}'(T^{(n)}|_{P\mathcal{H}^{(n)}})$ is commutative, where $T \in \{A, B\}$.*

Then $A \sim B$ if and only if:

(1) $(K_0(\mathcal{A}'(A \oplus B)), \bigvee(\mathcal{A}'(A \oplus B)), 1_{\mathcal{A}'(A \oplus B)}) \cong (Z^{(k)}, N^{(k)}, 1)$.
(2) *For all $\mathcal{J} \in m(A \oplus B)$, we have $\mathcal{A}'(A \oplus B)/\mathcal{J} \cong M_m(C)$, $m \in (2n_1, 2n_2, \ldots, 2n_k)$.*

The paper is organized as follows. In Section 2, we introduce some definitions and basic results. In Section 3, we consider the problem of stably finitely (SI) decomposition of operators. In Section 4, we prove our main theorem using the results of Section 3, and we complete the similarity classification of some classes of operators.

2. Preliminary results

To formulate our results, we need to introduce the following definitions, notations and theorems.

Definition 2.1. An operator $T \in L(\mathcal{H})$ is called strongly irreducible (SI) if T does not commute with any nontrivial idempotent operator, i.e., if there is no non-trivial idempotent operator in $\mathcal{A}'(T)$. Let $T \in L(\mathcal{H})$. A family of operators $\mathcal{P} = \{P_\lambda\}_{\lambda\in\Lambda} \subseteq \mathcal{A}'(T)$ is called a commutative idempotents set of T, if ${P_\lambda}^2 = P_\lambda$, $P_\lambda P_\eta = P_\eta P_\lambda$, $P_\lambda, P_\eta \in \mathcal{P}$. Naturally, each commutative idempotents set is contained in a maximal commutative idempotent set in $\mathcal{A}'(T)$. Let $P \in \mathcal{A}'(T)$ be a non-trivial idempotent. Then P is said to be a minimal idempotent if there is no nontrivial idempotent $Q \in \mathcal{A}'(T)$ such that $QP = PQ = Q$. Obviously, P is a minimal idempotent in $\mathcal{A}'(T)$ if and only if $T|_{P\mathcal{H}}$ is a strongly irreducible operator in $\mathcal{L}(P\mathcal{H})$.

Definition 2.2. $T \in \mathcal{L}(\mathcal{H})$ is called a finitely decomposable operator if the cardinality of an arbitrary maximal commutative idempotents set in $\mathcal{A}'(T)$ is finite, and T is called a stably finitely decomposable operator if $T^{(n)}$ is a finitely decomposable operator for all $n = 1, 2, 3, \ldots$.

Definition 2.3. Let $T \in \mathcal{L}(\mathcal{H})$, and let $\mathcal{P} = \{P_i\}_{i=1}^n$ $(n < \infty)$ be a family of idempotents in $\mathcal{A}'(T)$, satisfying:

(1) $0 \neq P_i \in \mathcal{A}'(T), 1 \leq i \leq n$.
(2) $P_iP_j = P_jP_i = 0, 1 \leq i \neq j \leq n$.
(3) $\sum_{i=1}^n P_i = I$.

Then $\mathcal{P} = \{P_i\}_{i=1}^n$ is called a unit finite (SI) decomposition of T.

If $T|_{P_i\mathcal{H}}$ is a strongly irreducible for $1 \leq i \leq n$, then we call $\mathcal{P}$ is a unit finite (SI) decomposition of T.

Definition 2.4. Let $T \in \mathcal{L}(\mathcal{H})$, and suppose that if $\mathcal{P} = \{P_i\}_{i=1}^n$ and $\mathcal{Q} = \{Q_i\}_{i=1}^m$ are both unit finite (SI) decompositions of T, then:

(1) $m = n$.
(2) There is an invertible operator $X \in \mathcal{A}'(T)$ and a permutation $\Pi \in S_n$ such that $XQ_{\Pi(i)}X^{-1} = P_i$ for $1 \leq i \leq n$.

Then we say that T has a unique finite (SI) decomposition up to similarity. We say that T has a unique stably finite (SI) decomposition up to similarity if $T^{(n)}$ has a unique finite (SI) decomposition up to similarity for all $n = 1, 2, 3, \ldots$

In [CD] M.J. Cowen and R.G. Douglas began a systematic study of a class of geometry operators, now called the Cowen-Douglas operators, which have an open set of eigenvalues.

DEFINITION 2.5. An operator $A \in \mathcal{L}(\mathcal{H})$ is said to be a Cowen-Douglas operator if there exists a connected open subset Ω of C and a positive integer n such that:

(1) $\Omega \subset \sigma(A) = \{z \in C | A - z \text{ is not invertible }\}$.
(2) $\operatorname{ran}(A - z) = \mathcal{H}, z \in \Omega$.
(3) $\bigvee_{z\in\Omega} \ker(A - z) = \mathcal{H}$.
(4) $\dim \ker(A - z) = n, z \in \Omega$.

C.L. Jiang proved that if T is a (SI) Cowen-Douglas operator, then $\mathcal{A}'(T)$ is an essentially commutative algebra, i.e., $\mathcal{A}'(T)/\operatorname{rad}\mathcal{A}'(T)$ is commutative, where $\operatorname{rad}\mathcal{A}'(T)$ is the Jacobson radical of $\mathcal{A}'(T)$ (cf. [Jia]). We note that for each Cowen-Douglas operator T, if $T = A_1 \oplus A_2 \cdots \oplus A_n$ is a unit (SI) decomposition of T, then A_i $(i = 1, 2, \ldots, n)$ are all Cowen-Douglas operators.

To proceed further, we recall briefly some notations of K-theory. Let $\mathcal{A}$ be a unital Banach algebra. Let e and f be idempotents in $\mathcal{A}$. Then e and f are said to be algebraically equivalent (denoted by $\sim_a$), if there exist $x, y \in \mathcal{A}$ such that $xy = e$, $yx = f$. Moreover, e and f are said to be similarity equivalent, if there exists an invertible element $z \in \mathcal{A}$ such that $zez^{-1} = f$.

Let $M_n(\mathcal{A}) = \{(a_{ij})_{n\times n} | a_{ij} \in \mathcal{A}\}$. Then $M_\infty(\mathcal{A})$ is the algebraic direct limit of $M_n(\mathcal{A})$, under the embedding $a \to \operatorname{diag}(a, 0) = (a \oplus 0)$.

The symbol $\operatorname{Proj}(M_n(\mathcal{A}))$ denotes the set of algebraic equivalence classes of idempotents in $M_\infty(\mathcal{A})$, and we let $\bigvee(\mathcal{A}) = \operatorname{Proj}(M_n(\mathcal{A}))$.

There is a binary operation (orthogonal addition) on $\bigvee(\mathcal{A})$ defined as follows: If $[e], [f] \in \bigvee(\mathcal{A})$, choose $e' \in [e], f' \in [f]$ with $e'f' = f'e' = 0$. Then $[e] + [f] = [e' + f']$. Since for all e we have $e \oplus 0 \sim_a e \sim_a 0 \oplus e$, we can choose $e' = e \oplus 0, f' = 0 \oplus f$. Thus $[e] + [f] = [e \oplus f]$. This operation is well defined and it makes $\bigvee(\mathcal{A})$ an Abelian semigroup with identity. From classic K-theory one obtains exactly the same semigroup starting with $\sim (\mathcal{A})$ instead of $\sim_a$, since the two notions coincide on $M_\infty(\mathcal{A})$.

Note that $\bigvee(\mathcal{A})$ depends on $\mathcal{A}$ only up to stable isomorphism. If $M_\infty(\mathcal{A}_1)$ is isomorphic ($\cong$) to $M_\infty(\mathcal{A}_2)$, then $\bigvee(\mathcal{A}_1) \cong \bigvee(\mathcal{A}_2)$, and $K_0(\mathcal{A})$ is the Grothendick group of $\bigvee(\mathcal{A})$.

From basic results of operator theory we deduce the following properties:

(2.1) If $T = T_1 \oplus T_2 \oplus \ldots T_n$, then $\mathcal{A}'(T) = \{(S_{ij})_{n\times n} \mid S_{ij} \in \ker\tau_{T_i,T_j}, 1 \leq i, j \leq n\}$ is a unital Banach algebra, where τ_{T_i,T_j} is the Rosenblum operator defined by $\tau_{T_i,T_j}(C) = T_iC - CT_j, C \in \mathcal{L}(\mathcal{H}_j, \mathcal{H}_i)$.
(2.2) $\ker\tau_{T_i,T_j}$ is a linear space, and $\ker\tau_{T_i,T_i} = \mathcal{A}'(T_i)$ is a unital Banach algebra.
(2.3) Let $e_{\mathcal{A}'(T)}$ denote the unit of $\mathcal{A}'(T)$. Then $e_{\mathcal{A}'(T)} = e_{\mathcal{A}'(T_1)} \oplus \cdots \oplus e_{\mathcal{A}'(T_n)}$.
(2.4) If $S_{ij} \in \ker\tau_{T_i,T_j}$, and $S_{jk} \in \ker\tau_{T_j,T_k}$, then $S_{ij}S_{jk} \in \ker\tau_{T_i,T_k}$. In particular, if $S_{ij} \in \ker\tau_{T_i,T_j}$, $S_{ji} \in \ker\tau_{T_j,T_i}$, then $S_{ij}S_{ji} \in \mathcal{A}'(T_i)$.

(2.5) If $S=(S_{ij})_{n\times n}\in\mathcal{A}'(T)$, then

$$S(i,j)\triangleq\begin{bmatrix}0&\dots&0&\dots&0\\ \vdots&&\vdots&&\vdots\\ 0&\dots&S_{ij}&\dots&0\\ \vdots&&\vdots&&\vdots\\ 0&\dots&0&\dots&0\end{bmatrix}\in\mathcal{A}'(T).$$

(2.6) By Property (2.5), we can define a canonical map Φ_{ij} from $\mathcal{A}'(T)$ to $\ker\tau_{T_i,T_j}$ by $\Phi_{ij}(S)=S_{ij}$, for $S=(S_{ij})_{n\times n}\in\mathcal{A}'(T)$. Then Φ_{ij} is a linear map and $\Phi_{ii}(S)\in\mathcal{A}'(T_i)$ for $S\in\mathcal{A}'(T)$.

(2.7) Throughout this paper an ideal $\mathcal{J}$ means a proper two-sided ideal. Let $\mathcal{J}$ be an ideal of $\mathcal{A}'(T)$, and define

$$\mathcal{J}_{ij}=\left\{S_{ij}\mid S_{ij}\in\ker\tau_{T_i,T_j},\quad\begin{bmatrix}0&\dots&0&\dots&0\\ \vdots&&\vdots&&\vdots\\ 0&\dots&S_{ij}&\dots&0\\ \vdots&&\vdots&&\vdots\\ 0&\dots&0&\dots&0\end{bmatrix}\in\mathcal{J}\right\}.$$

Then we have:

(2.7.1) $\mathcal{J}_{ii}$ is an ideal of $\mathcal{A}'(T_i)$ or $\mathcal{J}_{ii}=\mathcal{A}'(T_i)$.

(2.7.2) $\mathcal{J}_{ij}$ is a subspace of $\ker\tau_{T_i,T_j}$.

(2.7.3) $S=(S_{ij})_{n\times n}\in\mathcal{J}$ for $S(i,j)\in\mathcal{J}$.

(2.8) Let $\mathcal{J}$ be a closed ideal of $\mathcal{A}'(T)$. By Property (2.7), we can define a canonical map from $\ker\tau_{T_i,T_j}$ to $\ker\tau_{T_i,T_j}/\Phi_{ij}(\mathcal{J})$ by $S_{ij}\longrightarrow[S_{ij}]_{\mathcal{J}}$, where $\ker\tau_{T_i,T_j}/\Phi_{ij}(\mathcal{J})$ is the quotient space of $\ker\tau_{T_i,T_j}$ by the subspace $\Phi_{ij}(\mathcal{J})$. If $\mathcal{J}$ is closed, then $\mathcal{A}'(T)/\mathcal{J}=\{([S_{ij}]_{\mathcal{J}})_{n\times n}|S_{ij}\in\ker\tau_{T_i,T_j}\}$ is a unital Banach algebra. It is easy to see that the canonical map $\Phi_{\mathcal{J}}$ from $\mathcal{A}'(T)$ to $\mathcal{A}'(T)/\mathcal{J}$ is $\Phi_{\mathcal{J}}((S_{ij})_{n\times n})=([S_{ij}]_{\mathcal{J}})_{n\times n}$. Moreover, if $([S_{ij}]_{\mathcal{J}})_{n\times n}=\Phi_{\mathcal{J}}(S)\in\mathcal{A}'(T)/\mathcal{J}$, then

$$\begin{bmatrix}0&\dots&0&\dots&0\\ \vdots&&\vdots&&\vdots\\ 0&\dots&[S_{ij}]_{\mathcal{J}}&\dots&0\\ \vdots&&\vdots&&\vdots\\ 0&\dots&0&\dots&0\end{bmatrix}=\Phi_{\mathcal{J}}(S(i,j))\in\mathcal{A}'(T)/\mathcal{J}.$$

Definition 2.6. A finite irreducible algebra $\mathcal{A}$ is a Banach algebra such that for every continuous irreducible representation π on a Banach space X of $\mathcal{A}$, $\pi(\mathcal{A})$ is finite-dimensional, i.e., $\dim X<\infty$. A Banach algebra $\mathcal{A}$ is said to be n-homogeneous if all its continuous irreducible representations are isomorphism to $M_n(C)$.

By Gelfand theory, if $\mathcal{A}$ is a Banach algebra, and if $\mathcal{A}/\operatorname{rad}\mathcal{A}$ is commutative, then $\mathcal{A}$ is a 1-homogeneous algebra. Conversely, a 1-homogeneous algebra $\mathcal{A}$ must be essentially commutative, i.e., $\mathcal{A}/\operatorname{rad}\mathcal{A}$ is commutative.

Lemma 2.7 (cf. [Jia]). *Let $T = \bigoplus_{k=1}^{n} T_k$, and suppose $\mathcal{J}_1$ is an ideal of $\mathcal{A}'(T_1)$. Then there exists an ideal $\mathcal{J}$ of $\mathcal{A}'(T)$ satisfying $\Phi_{11}(\mathcal{J}) = \mathcal{J}_1$, and if there is another ideal $\mathcal{J}'$ of $\mathcal{A}'(T)$ such that $\Phi_{11}(\mathcal{J}') = \mathcal{J}_1$, then $\mathcal{J} \subseteq \mathcal{J}'$.*

Lemma 2.8 (cf. [Jia]). *Let $T = \bigoplus_{k=1}^{n} T_k$, and suppose that $\mathcal{J} \in m(\mathcal{A}'(T))$. Then $\Phi_{kk}(\mathcal{J}) = \mathcal{A}'(T_k)$ or $\Phi_{kk}(\mathcal{J}) \in m(\mathcal{A}'(T_k))$, $k = 1, 2, \ldots, n$.*

Lemma 2.9 (cf. [Jia]). *If T is a strongly irreducible Cowen-Douglas operator, then $K_0(\mathcal{A}'(T)) \cong Z$, $\bigvee(\mathcal{A}'(T)) \cong N$.*

3. The stably finite (SI) decomposition of operators

Lemma 3.1 (Theorem CFJ, cf. [CFJ]). *Let $T \in \mathcal{L}(\mathcal{H})$, and let $\mathcal{H}^{(n)}$ denote the direct sum of n copies of Hilbert space $\mathcal{H}$, and $T^{(n)}$ the operator $\bigoplus_1^n T$ on $\mathcal{H}^{(n)}$. Then the following are equivalent:*

1. *T is similar to $(\sim)$ $\bigoplus_{i=1}^{k} A_i^{(n_i)}$ with respect to the decomposition $\mathcal{H} = \bigoplus_{i=1}^{k} \mathcal{H}_i^{(n_i)}$, and for each natural number n, $T^{(n)}$ has a unique finite (SI) decomposition, where $k, n_1, \ldots, n_k$ are natural numbers, $A_1, \ldots, A_k$ are strongly irreducible operators, and $A_i \not\sim A_j$ for $1 \le i \ne j \le k$.*
2. *$K_0(\mathcal{A}'(T)) \cong Z^{(k)}$ and $V(\mathcal{A}'(T)) \cong N^{(k)}$. If h denotes the isomorphism from $V(\mathcal{A}'(T))$ to $N^{(k)}$, then h sends $[I]$ to $(n_1, n_2, \ldots, n_k)$, i.e., $h([I]) = n_1 e_1 + n_2 e_2 + \cdots + n_k e_k$, where k, $n_1, \ldots, n_k$ are natural numbers and $\{e_i\}_{i=1}^{k}$ are generators of $N^{(k)}$.*

Lemma 3.2. *Let $T = A_1^{(m_1)} \oplus A_2^{(m_2)} \oplus \cdots \oplus A_k^{(m_k)}$, $A_i \subset (SI)$, and $A_i \not\sim A_j$ for $1 \le i \ne j \le k$. Then $\bigvee(\mathcal{A}'(T)) \cong N^{(k)} \Leftrightarrow \bigvee(\mathcal{A}'(\bigoplus_{i=1}^{k} A_i^{(n_i)}) \cong N^{(k)}$, where $\{m_1, \ldots, m_k\}$ and $\{n_1, \ldots, n_k\}$ are positive integers.*

Proof. We need only to prove "$\Rightarrow$". By Theorem CFJ, $\bigvee(\mathcal{A}'(T)) \cong N^{(k)}$ implies that $\bigoplus_{i=1}^{k} A_i^{(mm_i)}$ is finitely decomposable, where $m = \sum_{i=1}^{k} nn_i$. Let $T_1 = \bigoplus_{i=1}^{k} A_i^{(n_i)}$. Then $T_1^{(n)} = \bigoplus_{i=1}^{k} A_i^{(nn_i)}$. Note that $mm_i \ge nn_i$ for $1 \le i \le k$ and $\bigoplus_{i=1}^{k} A_i^{(mm_i)} = T^{(n)} \oplus \bigoplus_{i=1}^{k} A_i^{(mm_i - nn_i)}$. So $T^{(n)}$ is a finitely decomposable operator and has a unique (SI) decomposition up to similarity. By Theorem CFJ again, we have $\bigvee(\mathcal{A}'(\bigoplus_{i=1}^{k} A_i^{(n_i)})) \cong N^{(k)}$. □

Lemma 3.3. *Suppose $\mathcal{A}$ is a unital finite irreducible Banach algebra, $\mathcal{J} \subseteq \mathcal{A}$ is a closed ideal, and $0 \to \mathcal{J} \to \mathcal{A} \to \mathcal{A}/\mathcal{J} \to 0$ is the short exact sequence. If $V(\mathcal{A}) \cong N$ and $[1_{\mathcal{A}}] = 1$, then $\pi_* : K_0(\mathcal{A}) \to K_0(\mathcal{A}/\mathcal{J})$ is injective.*

Proof. Let n be a positive integer and $p, q \in M_n(\mathcal{A})$ be two idempotents. Since $V(\mathcal{A}) \cong N$, we have $[p] = [e_r]$, $[q] = [e_s]$, where $e_k = \operatorname{diag}(1_{\mathcal{A}}, \ldots, 1_{\mathcal{A}}, 0, \ldots, 0)$ with k terms $1_{\mathcal{A}}$ on the diagonal. If $\pi_*([p]) = \pi_*([q])$, then $[\pi(e_r)] = [\pi(e_s)]$. Since $\mathcal{A}$ is a unital finite irreducible Banach algebra, $\mathcal{A}/\mathcal{J}$ is also a unital finite irreducible Banach algebra, $\mathcal{A}/\mathcal{J}$ is stably finite, and $r = s$. Therefore the map $\pi_* : K_0(\mathcal{A}) \to K_0(\mathcal{A}/\mathcal{J})$ is injective. $\square$

LEMMA 3.4. *Suppose $\mathcal{A}$ is a unital finite irreducible Banach algebra, and $\mathcal{J}_1 \neq \mathcal{J}_2$ are two maximal ideals of $\mathcal{A}$. Let $\mathcal{J} = \mathcal{J}_1 \cap \mathcal{J}_2$. Then $\mathcal{A}/\mathcal{J} \cong \mathcal{A}_1/\mathcal{J}_1 \oplus \mathcal{A}_2/\mathcal{J}_2$.*

Proof. Let ϕ_i $(i = 1, 2)$ be the quotient from $\mathcal{A}$ to $\mathcal{A}/\mathcal{J}_i$. Define $\phi : \mathcal{A} \longrightarrow \mathcal{A}_1/\mathcal{J}_1 \oplus \mathcal{A}_2/\mathcal{J}_2$ by $\phi(a) = \phi_1(a) \oplus \phi_2(a)$. Then ϕ is a homomorphism. By the Chinese Remainder Theorem, ϕ is onto. Note that $\ker \phi = \ker \phi_1 \cap \ker \phi_2 = \mathcal{J}_1 \cap \mathcal{J}_2 = \mathcal{J}$. So $\mathcal{A}/\mathcal{J} \cong \mathcal{A}_1/\mathcal{J}_1 \oplus \mathcal{A}_2/\mathcal{J}_2$. $\square$

LEMMA 3.5 (cf. [CFJ]). *Let $T \in \mathcal{L}(\mathcal{H})$, and suppose P_1, P_2 are idempotents of $\mathcal{A}'(T)$. If $P_1 \sim_a P_2$ in $(\mathcal{A}'(T))$, then $T|_{P_1\mathcal{H}} \sim T|_{P_2\mathcal{H}}$.*

LEMMA 3.6. *Let $A_i \in (SI)$ and $A_i \not\sim A_j$ for $1 \leq i \neq j \leq k$. Let $\{n_1, n_2, \ldots, n_k\}$ be positive integers. Let $T = \bigoplus_{i=1}^{k} A_i^{(n_i)} \oplus B$, $S_1 = A_1^{(n_1)} \oplus B$, $S_2 = \bigoplus_{i=2}^{k} A_i^{(n_i)}$, where B is an arbitrary bounded operator. Note that $T = S_1 \oplus S_2$. Let $\mathcal{A}_{12} = \ker \tau_{S_1,S_2}$, $\mathcal{A}_{21} = \ker \tau_{S_2,S_1}$. If $\bigvee(\mathcal{A}'(S_2)) \cong N^{(k-1)}$, then*

$$\hat{\mathcal{J}}_1 = \left\{ \sum_{i=1}^{n} x_i y_i, \quad x_i \in \mathcal{A}_{12}, \quad y_i \in \mathcal{A}_{21}, \quad 1 \leq i \leq n, \quad n = 1, 2, \ldots \right\}$$

is a proper ideal of $\mathcal{A}'(S_1)$.

Proof. By Property (2.7.1), $\hat{\mathcal{J}}_1 = \mathcal{A}'(S_1)$ or $\hat{\mathcal{J}}_1$ is a proper ideal of $\mathcal{A}'(S_1)$. If $\hat{\mathcal{J}}_1 = \mathcal{A}'(S_1)$, then there exist $x_1, x_2, \ldots, x_n \in \ker \tau_{S_1,S_2}$, and $y_1, y_2, \ldots, y_n \in \ker \tau_{S_2,S_1}$ such that $x_1 y_1 + \cdots + x_n y_n = 1_{\mathcal{A}'(S_1)}$. It is easy to see that there exists an idempotent $P \in M_n(\mathcal{A}'(S_2))$ such that $1_{\mathcal{A}'(S_1)} \oplus 0 \sim_a 0 \oplus P$ in $\mathcal{A}'(S_1 \oplus S_2^{(n)})$. Assume $S_1 \in \mathcal{B}(\mathcal{K}_1)$, $S_2^{(n)} \in \mathcal{B}(\mathcal{K}_2)$. By Lemma 3.5, we have

$$\begin{aligned} A_1^{(n)} \oplus B = S_1 = (S_1 \oplus S_2^{(n)})|_{(1_{\mathcal{A}'(A_1)} \oplus 0)(\mathcal{K}_1 \oplus \mathcal{K}_2)} &\sim (S_1 \oplus S_2^{(n)})|_{(0 \oplus P)(\mathcal{K}_1 \oplus \mathcal{K}_2)} \\ &= S_2^{(n)}|_{P\mathcal{K}_2}. \end{aligned}$$

Since $\bigvee(\mathcal{A}'(S_2)) \cong N^{(k-1)}$, $S_2^{(n)}$ has a unique (SI) decomposition up to similarity by Theorem CFJ. Since $A_1^{(n)} \oplus B \sim S_2^{(n)}|_{P\mathcal{K}_2}$, we have $A_1 \sim A_j$ for some $2 \leq j \leq k$. This contradicts our assumption that $A_i \not\sim A_j$, $1 \leq i \neq j \leq k$. So $\hat{\mathcal{J}}_1$ is an ideal of $\mathcal{A}'(S_1)$. $\square$

Lemma 3.7 (cf. [Jia]). *Let $T = \bigoplus_{k=1}^{n} T_k$, where $\mathcal{A}'(T_i)/\operatorname{rad}\mathcal{A}'(T_i)$ is commutative for $1 \le i \le n$. Then for each $\mathcal{J} \in m(\mathcal{A}'(T))$ there exists a positive integer $l_{\mathcal{J}} \le n$ such that $\mathcal{A}'(T)/\mathcal{J} \cong M_{l_{\mathcal{J}}}$.*

Lemma 3.8. *Let $A \in \mathcal{L}(\mathcal{H})$ be a strongly irreducible operator, such that $\mathcal{A}'(A)/\operatorname{rad}\,\mathcal{A}'(A)$ is commutative. Then A is a stably finitely decomposable operator. Furthermore, $A^{(n)}$ has a stably (SI) decomposition for $n = 1, 2, \dots,$ if and only if $K_0(\mathcal{A}'(A)) \cong Z$.*

Proof. Let $\{P_1, P_2, \dots, P_m\}$ be a unit decomposition of $A^{(n)}$. Then from the proof of Lemma 3.7 we see that there exists a continuous natural homomorphism $\Phi : \mathcal{A}'(A^{(n)}) \to M_n(C(m(A)))$, where $m(A)$ is the set of maximal ideals of $\mathcal{A}'(A)$. Let $P_k = (P_{ij}^k)_{n\times n}$, $1 \le k \le m$. Then $\Phi(P_k) = (P_{ij}^k(J))_{n\times n}$, where $(P_{ij}^k(J))$ is a continuous function on $m(A)$, $1 \le i, j \le k$. Hence $(\operatorname{tr} P_k)(J) = \sum_{i=1}^{n} (P_{ij}^k(J))$ is continuous on $m(A)$. By the Shilov idempotent theorem, $m(A)$ is connected. Therefore $(\operatorname{tr} P_k)(J) \equiv n_k \ge 1$. Note that for $J \in m(A)$, $\{(P_{ij}^k)\}_{n\times n}$ is a unit idempotent decomposition in $M_n(C)$. Therefore $\sum_{k=1}^{m} (\operatorname{tr} P_k)(J) = n$, that is, $\sum_{k=1}^{m} n_k = n$. So $m \le n$. Thus A is a stably finite decomposition operator.

For $P \in \mathcal{A}'(A_n)$, let $S = A^{(n)}|P\mathcal{H}$. Then we can prove that $\mathcal{A}'(S)$ is a homogeneous algebra. In fact, we see that $(\operatorname{tr} P)(J) \equiv k$ for all $J \in m(A)$ for $J_1 \in m(S)$, and there exists a unique $J \in m(A^{(n)})$ such that $J \cap \mathcal{A}'(S) = J_1$. Hence

$$\Phi_{J_1}(\mathcal{A}'(S)) = \phi_J(P)\phi_J(\mathcal{A}'(A^{(n)}))\phi_J(P) \cong M_k(C),$$

where ϕ_J is the canonical quotient homomorphism from $\mathcal{A}'(A^{(n)})$ to $\mathcal{A}'(A^{(n)})/J$. Therefore $\mathcal{A}'(S)$ is a k-homomorphism algebra.

The "only if" part follows from Lemma 3.1, so it remains to show the "if" part. We know that $(K_0(\mathcal{A}'(A)), \bigvee(\mathcal{A}'(A)))$ is an ordered group. We also have $K_0(\mathcal{A}'(A)) \cong Z$. Now, if $G = Z$ and (G, G_+) is an order group, then there exists an isomorphism ϕ from G to Z such that $\phi(G_+) \subseteq N$. Thus we may assume that $\bigvee(\mathcal{A}'(A)) \subseteq N$.

Let $p = \operatorname{diag}(I, 0, 0 \dots) \in M_\infty(\mathcal{A}'(A))$ and $r = [p] \in \bigvee(\mathcal{A}'(A))$, $r \in N$. Let $q \in M_n(\mathcal{A}'(A))$ be a non-zero idempotent. Then $0 \ne [q] = s \in \bigvee(\mathcal{A}'(A))$. Let $B = A^{(n)}|_{qH^{(n)}}$. From the above proof we see that $\mathcal{A}'(B)$ is a k-homogeneous algebra. Note that $rs = r[q] = s[p]$. So there exists $n' \ge n$ such that

$$Q = \operatorname{diag}(q, \dots, q_{(r)}, 0, \dots, 0) \sim_a \operatorname{diag}(p, \dots, p_{(s)}, 0, \dots, 0) = P,$$

where $Q, P \in \mathcal{P}(H^{(n')})$. By Lemma 3.4, $B^{(r)} = A^{(n')}|_{QH^{(n')}} \sim A^{(n')}|_{PH^{(n')}} = A^{(s)}$. Therefore $\mathcal{A}'(B^{(r)}) \cong \mathcal{A}'(A^{(s)})$, i.e., $M_r(\mathcal{A}'(B)) \cong M_s(\mathcal{A}'(A))$. Note that $M_r(\mathcal{A}'(B))$ is an (rk)-homogeneous algebra. But $M_s(\mathcal{A}'(A))$ is s-homogeneous, so $s = rk$ and $\bigvee(\mathcal{A}'(A)) = \{kr, k = 0, 1, 2, \dots\}$. Since $(K_0(\mathcal{A}'(A)), \bigvee(\mathcal{A}'(A)))$

is an ordered group, $r = 1$, and $\bigvee(\mathcal{A}'(A)) \cong N$. In view of Theorem CFJ, the "if" part is proved. $\square$

LEMMA 3.9. *Let $T = A_1^{(n_1)} \oplus A_2^{(n_2)} \oplus \cdots \oplus A_k^{(n_k)}$, where A_i is a strongly irreducible operator, $A_i \not\sim A_j$ for $1 \leq i \not\sim j \leq n$, and $\mathcal{A}'(A_i)/\operatorname{rad}\mathcal{A}'(A_i)$ is commutative. Suppose the following conditions are satisfied:*

(1) *$K_0(\mathcal{A}'(A_i)) = Z$ for $i = 1, 2, \ldots, n$.*
(2) *For any positive integer n and any minimal idempotent $P \in \mathcal{A}'(T^{(n)})$, $\mathcal{A}'(T^{(n)}|_{PH})/\operatorname{rad}\mathcal{A}'(T^{(n)}|_{PH})$ is commutative.*

Then T is a stably finitely decomposable operator and T has a stably unique (SI) decomposition up to similarity.

Proof. By Lemma 3.2, we may assume that $T = A_1 \oplus A_2 \oplus \cdots \oplus A_k$, and we only need to prove that, for all $n \in N$, $T^{(n)}$ has a unique (SI) decomposition up to similarity.

If $\mathcal{A}'(T)/\operatorname{rad}\mathcal{A}'(T)$ is commutative, i.e., $\mathcal{A}'(T)$ is a 1-homogeneous algebra, then

$$\begin{aligned}\mathcal{A}'(T)/\operatorname{rad}\mathcal{A}'(T) \cong (\mathcal{A}'(A_1)/\operatorname{rad}\mathcal{A}'(A_1)) \oplus (\mathcal{A}'(A_2)/\operatorname{rad}\mathcal{A}'(A_2))\\ \oplus \ldots \cdots \oplus (\mathcal{A}'(A_k)/\operatorname{rad}\mathcal{A}'(A_k))\end{aligned}$$

by Lemma 3.8. We know that $\bigvee \mathcal{A}'(A_i) \cong N$. Therefore

$$\begin{aligned}\bigvee(\mathcal{A}'(T)) &\cong \bigvee(\mathcal{A}'(T)/\operatorname{rad}\mathcal{A}'(T))\\ &\cong \bigvee(\mathcal{A}'(A_1)/\operatorname{rad}\mathcal{A}'(A_1)) \oplus \bigvee \mathcal{A}'(A_2)/\operatorname{rad}\mathcal{A}'(A_2))\\ &\quad\oplus \cdots \oplus \bigvee(\mathcal{A}'(A_k)/\operatorname{rad}\mathcal{A}'(A_k))\\ &\cong N^{(2)}.\end{aligned}$$

By Lemma 3.1, the result follows for this case.

Therefore, we can assume that there exists $\breve{\mathcal{J}} \in m(T)$ such that $\mathcal{A}'(T)/\breve{\mathcal{J}} \cong M_r(C)$ for $r \geq 2$, where $m(T)$ denotes the set of the maximal ideals of $\mathcal{A}'(T)$.

Let $T^{(n)} = A_1^{(n)} \oplus A_2^{(n)} \oplus \cdots \oplus A_k^{(n)}$. Suppose there exists another finite (SI) decomposition of $T^{(n)}$,

$$T^{(n)} \sim A_1^{(m_1)} \oplus A_2^{(m_2)} \oplus \cdots \oplus A_k^{(m_k)} \oplus B_1 \oplus \cdots \oplus B_m,$$

where $m_i \geq 0$, $i = 1, 2, \ldots, k, m \geq 0$, and $B_j \in (SI)$, and $B_j \not\sim A_i$ for all $1 \leq i \leq k$, $1 \leq j \leq m$.

CLAIM 1. *$m_i + m \leq n$ for $i = 1, 2, \ldots, k$. Therefore T is a stably finitely decomposable operator.*

If the claim is not true, then, without loss of generality, we can assume $m_1 + m > n$.

Let $B = B_1 \oplus B_2 \oplus \cdots \oplus B_m$ and $S = A_2 \oplus A_3 \oplus \cdots \oplus A_k, R = T^{(n)} \oplus S$. We proceed by induction on k. By Lemma 3.1 and Lemma 3.7, we know the result is true when $k = 1$. We now assume Lemma 3.9 to be true when $n \leq k-1$. Let $\hat{\mathcal{J}}$ be the subalgebra of $\mathcal{A}'(T^{(n)})$ generated by $\ker \tau_{T^{(n)},S}$ and $\ker \tau_{S,T^{(n)}}$. By Lemma 3.6, $\hat{\mathcal{J}}$ is a proper ideal of $\mathcal{A}'(T^{(n)})$. Let $\mathcal{J}_1$ be the closure of $\hat{\mathcal{J}}$ in $\mathcal{A}'(T^{(n)})$. Then $\mathcal{J}_1$ is a closed ideal of $\mathcal{A}'(T^{(n)})$.

Let

$$\mathcal{J} = \begin{bmatrix} \mathcal{J}_1 & \ker \tau_{T^{(n)},S} \\ \ker \tau_{S,T^{(n)}} & \mathcal{A}'(S) \end{bmatrix} \subseteq \mathcal{A}'(R).$$

Then $\mathcal{J}$ is a closed ideal of $\mathcal{A}'(R)$, and $\mathcal{A}'(R)/\mathcal{J} = \mathcal{A}'(T^{(n)})/\mathcal{J}_1 \oplus 0$. Let $\mathcal{A} = \mathcal{A}'(T^{(n)})/\mathcal{J}_1$.

When $T^{(n)} = A_1^{(n)} \oplus A_2^{(n)} \oplus \cdots \oplus A_k^{(n)}$, we have

$$\ker \tau_{T^{(n)},S} = \begin{bmatrix} \ker \tau_{A_1,S} \\ \vdots \\ \ker \tau_{A_1,S} \\ \mathcal{A}'(S) \\ \cdots \\ \mathcal{A}'(S) \end{bmatrix}, \tag{1}$$

$$\ker \tau_{S,T^{(n)}} = [\ker \tau_{S,A_1}, \ldots, \ker \tau_{S,A_1}, \mathcal{A}'(S), \ldots, \mathcal{A}'(S)],$$

$$\ker \tau_{T^{(n)},S} \cdot \ker \tau_{S,T^{(n)}} = \begin{bmatrix} [\ker \tau_{A_1,S} \cdot \ker \tau_{S,A_1}]_{n\times n} & * \\ * & [\mathcal{A}'(S)]_{n\times n} \end{bmatrix},$$

where $S = A_2^{(n)} \oplus \cdots \oplus A_k^{(n)}$.

Next, consider the case

$$\begin{aligned} T^{(n)} &\sim A_1^{(m_1)} \oplus A_2^{(m_2)} \oplus B_1 \oplus \cdots \oplus B_m \\ &= A_1^{(m_1)} \oplus A_2^{(m_2)} \oplus B \sim A_1^{(m_1)} \oplus B \oplus A_2^{(m_2)}. \end{aligned}$$

By a simple computation we get

$$\ker \tau_{T^{(n)},S} \cdot \ker \tau_{S,T^{(n)}} = \operatorname{diag}(A_1 \oplus B_1 \oplus \ldots B_m \oplus A_2 \oplus \ldots A_k),$$

where

$$\begin{aligned} A_1 &= [\ker \tau_{A_1,S} \cdot \ker \tau_{S,A_1}]_{m_1\times m_1}, \\ B_i &= \ker \tau_{B_i,A_2} \cdot \ker \tau_{A_2,B_i}, \quad i = 1, 2, \ldots, m, \\ A_i &= [\mathcal{A}'(A_i)]_{m_i\times m_i}. \end{aligned}$$

Note that $m_1 + m > n$. Therefore $\mathcal{J}$ is not a maximal ideal. In fact, by the proof of Lemma 3.6 and $\bigvee(\mathcal{A}'(S)) \cong N$, we have

$$\begin{aligned} \mathcal{J}_{B_j} = \{x_1y_1 + x_2y_2 + \cdots + x_ny_n,\ & x_i \in \ker \tau_{B_j,S}, \\ & y_i \in \ker \tau_{S,B_j},\ i = 1, 2, \ldots\} \end{aligned}$$

and the closure of $\mathcal{J}_{B_j}$ is not equal to $\mathcal{A}'(B_j)$ for $1 \leq j \leq m$. If $\mathcal{J}$ is a maximal ideal, then $\mathcal{A}'(T^{(n)})/\mathcal{J}_1 \cong M_n(C)$ and $\mathcal{A}'(A_1^{(m_1)} \oplus B)/\mathcal{J} \cong M_{m_1+m}(C)$. Since $m_1 + m > n$, we get a contradiction.

Note that $\mathcal{A}'(T^{(n)}) = \operatorname{diag}(\mathcal{A}'(A_1^{(n)}), \mathcal{A}'(S^{(n)}))$. Let $J_1 = \operatorname{diag}(J_{11}, J_{22})$. By (1), $\bigvee(\mathcal{A}'(S)) \cong N$, and Lemma 3.6, J_{11} is a closed ideal of $\mathcal{A}'(A_1^{(n)})$ and $J_{22} = \mathcal{A}'(S^{(n)}) = [\mathcal{A}'(S)]_{n\times n}$. So

$$\mathcal{A} = \mathcal{A}'(T^{(n)})/J_1 = \mathcal{A}'(A_1^{(n)})/J_{11} \oplus 0. \tag{3}$$

On the other hand,

$$\mathcal{A}'(T^{(n)}) = \operatorname{diag}(\mathcal{A}'(A_1^{(m_1)}), \mathcal{A}'(B_1), \ldots, \mathcal{A}'(B_m), \mathcal{A}'(A_2^{(m_2)}), \ldots, \mathcal{A}'(A_k^{(m_k)})).$$

Similarly, from (2), $\bigvee(\mathcal{A}'(A_2)) \cong N$, and Lemma 3.6 we obtain

$$\mathcal{J}_1 = \operatorname{diag}(\mathcal{J}_{11}, \mathcal{J}_{22}, \ldots, \mathcal{J}_{m+k,m+k}),$$

where $\mathcal{J}_{11}$ is a closed ideal of $\mathcal{A}'(A_1^{(m_1)})$, $\mathcal{J}_{ii}$ is a closed ideal of $\mathcal{A}'(B_{i-1})$, and $\mathcal{J}_{m+j,m+j} = \mathcal{A}'(A_j^{(m_j)})$ for $j = 2, 3, \ldots, k$. Therefore

$$\mathcal{A} = \mathcal{A}'(T^{(n)})/\mathcal{J}_1 = \mathcal{A}'(A_1^{(m_1)} \oplus B)/\mathcal{J}_1' \oplus 0, \tag{4}$$

where $\mathcal{J}_1' = \operatorname{diag}(\mathcal{J}_{11}, \mathcal{J}_{22}, \ldots, \mathcal{J}_{m+1,m+1})$.

Without loss of generality, we may assume that $m_1, m_2 > 0$; otherwise, we can consider

$$T^{(2n)} = T^{(n)} \oplus T^{(n)} \sim A_1^{(n+m_1)} \oplus A_2^{(n+m_2)} \oplus \cdots \oplus A_k^{(n+m_k)} \oplus B_1 \oplus \cdots \oplus B_m,$$

and

$$T^{(2n)} = A_1^{(2n)} \oplus A_2^{(2n)} \oplus \cdots \oplus A_k^{(2n)}.$$

By (4), there exists a homomorphism $\phi : \mathcal{A}'(A_1^{(m_1)} \oplus B) \longrightarrow \mathcal{A}$, which is onto. By (3) and since $\mathcal{A}'(A_1)/\operatorname{rad}\mathcal{A}'(A_1)$ is commutative, $\mathcal{A}'(A_1^{(m_1)})$ is n-homogeneous. So $\mathcal{A}$ is n-homogeneous. By the second decomposition of $T^{(n)}$ and (4), we have

$$\begin{aligned}\mathcal{A} &= \operatorname{diag}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_{1+m}) \\ &= (\mathcal{A}'(A_1^{(m_1)})/\mathcal{J}_{11}, \mathcal{A}'(B_1)/\mathcal{J}_{22}, \ldots, \mathcal{A}'(B_m)/\mathcal{J}_{m+1,m+1}).\end{aligned}$$

Since $\mathcal{J}_{ii}$ are all proper ideals, $\mathcal{A}_i \neq 0$ for $i = 1, 2, \ldots, 1+m$. Let $\mathcal{J}_{11}'$ be a maximal ideal of $\mathcal{A}_1$. By Kaplansky's theorem, there exists a unique maximal ideal $\mathcal{J}_2 \subseteq \mathcal{A}$ such that $\Phi_{11}(\mathcal{J}_2) = \mathcal{J}_{11}'$, where Φ_{11} is the projection from $\mathcal{A}$ to $\mathcal{A}_1$. Therefore

$$\mathcal{A}/\mathcal{J}_2 = \operatorname{diag}(\mathcal{A}_1/\mathcal{J}_{11}', \mathcal{A}_2/\Phi_{22}(\mathcal{J}_2), \ldots, \mathcal{A}_{1+m}/\Phi_{1+m,1+m}(\mathcal{J}_2)).$$

Since $\mathcal{A}$ is n-homogeneous, $\mathcal{A}/\mathcal{J}_2 \cong M_n(C)$. Since $m_1 + m > n$, there exists $1 \leq j \leq m$ such that $\mathcal{A}_j/\Phi_{jj}(\mathcal{J}_2) = 0$. We may assume $j = m+1$, or, equivalently, $\mathcal{A}_{1+m,1+m}/\Phi_{1+m,1+m}(\mathcal{J}_2) = 0$. Let $\mathcal{J}_{1+m,1+m}'$ be a maximal ideal of $\mathcal{A}_{1+m}$. By Kaplansky's theorem, there exists $\mathcal{J}_3$, a maximal ideal of $\mathcal{A}$, such that $\Phi_{1+m,1+m}(\mathcal{J}_3) = \mathcal{J}'_{1+m,1+m}$. Then $\mathcal{J}_2 \neq \mathcal{J}_3$. Since $\mathcal{A}$ is n-homogeneous, $\mathcal{A}/\mathcal{J}_3 \cong M_n(C)$. Let $\mathcal{J}_4 = \mathcal{J}_2 \cap \mathcal{J}_3$. By Lemma 3.4, there exists a homomorphism $\Phi_1 : \mathcal{A} \longrightarrow \mathcal{A}/\mathcal{J}_4 \cong M_n(C) \oplus M_n(C)$, such that

$$\Phi_1(1_{\mathcal{A}_1} \oplus 0 \oplus \cdots \oplus 0) = (1 \oplus 0 \oplus \cdots \oplus 0) \oplus P,$$
$$\Phi_1(0 \oplus \cdots \oplus 0 \oplus 1_{\mathcal{A}_{1+m}}) = 0 \oplus (0 \oplus \cdots \oplus 0 \oplus 1).$$

Let $\Phi = \Phi_1 \cdot \phi$. Then Φ is a homomorphism from $\mathcal{A}'(A_1^{(m_1)} \oplus B)$ onto $M_n(C) \oplus M_n(C)$, such that

$$\Phi(1_{\mathcal{A}'(A_1^{(m_1)})} \oplus 0 \oplus \cdots \oplus 0) = (1 \oplus 0 \oplus \cdots \oplus 0) \oplus P,$$
$$\Phi(0 \oplus \cdots \oplus 0 \oplus 1_{\mathcal{A}'(B_m)}) = 0 \oplus (0 \oplus \cdots \oplus 0 \oplus 1).$$

Since $\mathcal{A}'(R)/\mathcal{J} = \mathcal{A}'(T^{(n)})/J_1 \oplus 0 = \mathcal{A} \oplus 0$, there exists a closed ideal $J \supseteq \mathcal{J}$ such that $\mathcal{A}'(R)/J = \mathcal{A}/\mathcal{J}_4 \oplus 0 = \mathcal{A}'(A_1^{(m_1)} \oplus B)/\ker\Phi \oplus 0$.

We consider $R = A_1^{(n)} \oplus S^{(n)} \oplus S$ and $J \supseteq \mathcal{J}$. There exists a closed ideal $\mathcal{J}''_1$ of $\mathcal{A}'(A_1^{(n)})$ such that

$$\mathcal{A}'(R)/J = \mathcal{A}'(A_1^{(n)})/\mathcal{J}''_1 \oplus 0 = \mathcal{A}'(A_1^{(m_1)} \oplus B)/\ker\Phi \oplus 0. \tag{5}$$

Let

$$\pi : \mathcal{A}'(R) \to \mathcal{A}'(R)/J,$$
$$\pi_1 : \mathcal{A}'({A_1}^{(n)}) \to \mathcal{A}'({A_1}^{(n)})/{\mathcal{J}_1}'',$$
$$\pi_2 : \mathcal{A}'(A_1^{(m_1)} \oplus B) \longrightarrow \mathcal{A}'(A_1^{(m_1)} \oplus B)/\ker\Phi$$

be quotient maps. We will prove

$$\pi_{1*}(K_0(\mathcal{A}'(A_1^{(n)}))) \cong \pi_*(K_0(\mathcal{A}'(R))) \cong \pi_{2*}(K_0(\mathcal{A}'(A_1^{(m_1)} \oplus B))).$$

Let $\alpha_* : \pi_{1*}(K_0(\mathcal{A}'(A_1^{(n)}))) \longrightarrow \pi_*(K_0(\mathcal{A}'(R)))$ be such that

$$\alpha_*(\pi_{1*}([e])) = \pi_*([e \oplus 0 \oplus \cdots \oplus 0]), \quad e \in \mathcal{P}_\infty(\mathcal{A}'(A_1^{(n)})).$$

First we show that α_* is injective. If $\pi_*([e \oplus 0 \oplus \cdots \oplus 0]) = 0$, then $\pi_*(e \oplus 0 \oplus \cdots \oplus 0) \sim_s 0$, so there exists r such that

$$\pi_*(e \oplus 0 \oplus \cdots \oplus 0) \oplus r \sim_a 0 \oplus r.$$

Then $\pi_{1*}(e) \oplus 0 \oplus \cdots \oplus 0 \oplus r \sim_a 0 \oplus r$, since, for all r, r is an idempotent and $0 \oplus r \sim_a r$. So if we set $r' = 0 \oplus \cdots \oplus 0 \oplus r$, we get

$$\pi_{1*}(e) \oplus r' \sim_a 0 \oplus r'.$$

Consequently, $\pi_{1*}(e) \sim_a 0$. Therefore $[\pi_{1*}(e)] = \pi_{1*}([e]) = 0$.

Next, we prove that α_* is surjective. For all $\beta \in K_0(\mathcal{A}'(R))$, by (5), there exists e such that $\pi_{1*}([e]) = \pi_*([\beta])$. For $(\beta_{ij})_{n\times n} \in M_n(K_0(\mathcal{A}'(R)))$, there exists e_{ij} such that $\pi_{1*}([e_{ij}]) = \pi_*([\beta_{ij}])$. In fact, by K-Theory, we have $[(\pi_*(e_{ij}) \oplus 0 \cdots \oplus 0)_{n\times n}]_0 = [\pi_*((e_{ij})_{n\times n}) \oplus 0]_0$. So α_* is surjective.

Similarly, we obtain $\pi_*(K_0(\mathcal{A}'(R))) \cong \pi_{2*}(K_0(\mathcal{A}'(A_1^{(m_1)} \oplus B)))$.

By Lemma 3.3, π_{1*} is injective. Therefore

$$\begin{aligned}\pi_{2*}(K_0(\mathcal{A}'(A_1^{(m_1)} \oplus B))) &\cong \pi_*(K_0(\mathcal{A}'(R))) \\ &\cong \pi_{1*}(K_0(\mathcal{A}'(A_1^{(n)}))) \cong K_0(\mathcal{A}'(A_1^{(n)})) \cong Z.\end{aligned}$$

On the other hand, Φ induces an isomorphism

$$\Psi : \mathcal{A}'(A_1^{(m_1)} \oplus B)/\ker\Phi \longrightarrow M_n(C) \oplus M_n(C).$$

By the property of the K_0 group, we get

$$\Phi_* = \Psi_* \cdot \pi_{2*} : K_0(\mathcal{A}'(A_1^{(m_1)} \oplus B)) \longrightarrow K_0(M_n(C) \oplus M_n(C)) = Z \oplus Z.$$

Since Ψ_* is an isomorphism, we have

$$\Phi_*(K_0(\mathcal{A}'(A_1^{(m_1)} \oplus B))) = \Psi_*(\pi_{2*}(K_0(\mathcal{A}'(A_1^{(m_1)} \oplus B)))) \cong Z. \tag{6}$$

Since

$$\Phi(1_{\mathcal{A}'(A_1^{(m_1)})} \oplus 0 \oplus \cdots \oplus 0) = (1 \oplus 0 \oplus \cdots \oplus 0) \oplus P$$

and

$$\Phi(0 \oplus \cdots \oplus 0 \oplus 1_{\mathcal{A}'(B_m)}) = 0 \oplus (0 \oplus \cdots \oplus 0 \oplus 1),$$

we get

$$\Phi_*([1_{\mathcal{A}'(A_1^{(m_1)})} \oplus 0 \oplus \cdots \oplus 0]) = [1 \oplus 0 \oplus \cdots \oplus 0] \oplus [P] = 1 \oplus [P],$$

$$\Phi_*([0 \oplus \cdots \oplus 0 \oplus 1_{\mathcal{A}'(B_m)}]) = [0] \oplus [0 \oplus \cdots \oplus 0 \oplus 1] = 0 \oplus 1.$$

By (6), there exists $n \in Z$ such that

$$\Phi_*([1_{\mathcal{A}'(A_1^{(m_1)})} \oplus 0 \oplus \cdots \oplus 0]) = n\Phi_*([0 \oplus \cdots \oplus 0 \oplus 1_{\mathcal{A}'(B_m)}]),$$

i.e., we have $1 \oplus P = n(0 \oplus 1) = 0 \oplus n \in Z \oplus Z$. This is a contradiction. Hence $m_i + m \leq n$ for $i = 1, 2$.

Claim 2. *$m_i + m = n$ for $i = 1, 2, \ldots, k$.*

By Claim 1, we only need to show that $m_i + m \geq n$ for $i = 1, 2, \ldots, k$.

By Lemma 3.9, each $\mathcal{A}'(B_i)/\operatorname{rad}\mathcal{A}'(B_i)$, $1 \leq i \leq m$, is commutative. Since $\mathcal{A} = \mathcal{A}'(T^{(n)})/\mathcal{J}_1 = \mathcal{A}'(A_1^{(n)})/\mathcal{J}_{11} \oplus 0$ is a homogeneous algebra and $\operatorname{diag}\mathcal{A} = (\mathcal{A}'(A_1^{(n)})/\mathcal{J}_{11}, \mathcal{A}'(B_1)/\mathcal{J}_{22}, \ldots, \mathcal{A}'(B_m)/\mathcal{J}_{1+m,1+m}, 0)$, for all $1 \leq j \leq m$, $\mathcal{A}'(B_j)/\mathcal{J}_{j+1,j+1}$ is essentially commutative. By Lemma 3.7, it follows that $\mathcal{A}/\mathcal{J}' \cong M_l(C)$, $l \leq m_1 + m$ for every maximal ideal $\mathcal{J}'$ of $\mathcal{A}$. Since $\mathcal{A}$ is an n-homogeneous algebra, we conclude $m_1 + m \geq n$.

Similarly, we obtain $m_i + m \geq n$ for $i = 2, 3, \ldots, k$.

Claim 3. *$m = 0$, i.e., we have $m_i = n$ for $1 \leq i \leq k$. Therefore $T^{(n)}$ has a unique (SI) decomposition up to similarly.*

Since $T = A_1 \oplus A_2 \oplus \cdots \oplus A_k$, and $\mathcal{A}'(A_i)/\operatorname{rad} \mathcal{A}'(A_i)$, $i = 1, 2$, are commutative, we have for all $\mathcal{J} \in m(\mathcal{A}'(T))$, $\mathcal{A}'(T)/\mathcal{J} \cong M_{r(\mathcal{J})}(C)$, where $1 \leq r(\mathcal{J}) \leq k$. Let

$$r_0 = \max\left\{r(\mathcal{J}) : \mathcal{A}'(T)/\mathcal{J} \cong M_{r(\mathcal{J})}(C), \quad \mathcal{J} \in m(\mathcal{A}'(T))\right\}.$$

Let $\hat{\mathcal{J}}$ be the maximal ideal of $\mathcal{A}'(T)$ such that $\mathcal{A}'(T)/\hat{\mathcal{J}} \cong M_{r_0}(C)$. If $r_0 = 1$, we are done. If $r_0 \geq 2$, then

$$\mathcal{A}'(T^{(n)})/M_n(\hat{\mathcal{J}}) \cong M_n(\mathcal{A}'(T))/\hat{\mathcal{J}}) \cong M_{nr_0}(C).$$

For $1 \leq i \leq k$ we have $m_i = n - m$ since

$$T^{(n)} \sim A_1^{(m_1)} \oplus A_2^{(m_2)} \oplus \cdots \oplus A_k^{(m_k)} \oplus B_1 \oplus \cdots \oplus B_m = T^{(n-m)} \oplus B_1 \oplus \cdots \oplus B_m.$$

Note that $\mathcal{A}'(T^{(n)})$ is an algebra, $\operatorname{diag}(\mathcal{A}'(T^{(n-m)}), \mathcal{A}'(B_1), \ldots, \mathcal{A}'(B_m))$. Therefore, for $\bar{\mathcal{J}} \in m(\mathcal{A}'(T^{(n)}))$, $\Phi_{11}(\bar{\mathcal{J}}) = \mathcal{A}'(T^{(n-m)})$ or $\Phi_{11}(\bar{\mathcal{J}})$ is a maximal ideal of $\mathcal{A}'(T^{(n-m)})$. So for arbitrary $\mathcal{J} \in m(\mathcal{A}'(T^{(n)}))$, $\mathcal{A}'(T^{(n)})/\mathcal{J} \cong M_s(C)$, where $s \leq (n-m)r_0 + m$. Hence $nr_0 \leq (n-m)r_0 + m$, i.e., $mr_0 \leq m$. Since $r_0 \geq 2$, we get $m = 0$. □

4. Main results

LEMMA 4.1. *Let $T = A_1^{(n_1)} \oplus A_2^{(n_2)}$, and suppose T and A_i, $i = 1, 2$, satisfy the following conditions:*

1. *$K_0(\mathcal{A}'(A_i)) = Z$ for $i = 1, 2$.*
2. *For any positive integer n and minimal idempotent $P \in \mathcal{A}'(T^{(n)})$, $\mathcal{A}'(T^{(n)}|_{PH})/\operatorname{rad}\mathcal{A}'(T^{(n)}|_{PH})$ is commutative.*

Then for arbitrary $\mathcal{J} \in m(\mathcal{A}'(T))$ we have

$$\mathcal{J} = \begin{bmatrix} \mathcal{J}_{11} & \ker \tau_{A_1^{(n_1)}, A_2^{(n_2)}} \\ \ker \tau_{A_2^{(n_2)}, A_1^{(n_1)}} & \mathcal{A}'(A_2^{(n_2)}) \end{bmatrix} \text{ or}$$

$$\mathcal{J} = \begin{bmatrix} \mathcal{A}'(A_1^{(n_1)}) & \ker \tau_{A_1^{(n_1)}, A_2^{(n_2)}} \\ \ker \tau_{A_2^{(n_2)}, A_1^{(n_1)}} & \mathcal{J}_{22} \end{bmatrix},$$

where $\mathcal{J}_{ii}$ is a maximal ideal of $\mathcal{A}'(A_i^{(n_i)})$ $(i = 1, 2)$.

Proof. First, we assume that $T = A_1 \oplus A_2$. Note that $\mathcal{A}'(A_1)/\operatorname{rad}\mathcal{A}'(A_1)$ and $\mathcal{A}'(A_2)/\operatorname{rad}\mathcal{A}'(A_2)$ are both commutative. If the result is not true, there exists

$$\mathcal{J} = \begin{bmatrix} \mathcal{J}_{11} & \mathcal{J}_{12} \\ \mathcal{J}_{21} & \mathcal{J}_{22} \end{bmatrix} \in m(\mathcal{A}'(T)),$$

where $\mathcal{J}_{12} \subset (\neq) \ker \tau_{A_1, A_2}$ and $\mathcal{J}_{21} \subset (\neq) \ker \tau_{A_2, A_1}$.

Let

$$\tilde{\mathcal{J}} = \mathcal{J} \dot{+} 1 = \begin{bmatrix} \mathcal{A}'(A_1) & \mathcal{J}_{12} \\ \mathcal{J}_{21} & \mathcal{A}'(A_2) \end{bmatrix}.$$

From the exact sequence

$$0 \longrightarrow \mathcal{J} \longrightarrow \mathcal{A}'(T) \longrightarrow \mathcal{A}'(T)/\mathcal{J} \longrightarrow 0,$$

using the property of the K_0-group, we obtain the following six-term exact sequence:

$$\begin{array}{ccccc} K_0(\mathcal{J}) & \xrightarrow{i_*} & K_0(\mathcal{A}'(T)) & \xrightarrow{\pi_*} & K_0(\mathcal{A}'(T)/\mathcal{J}) \\ \partial \uparrow & & & & \partial \downarrow \\ K_1(\mathcal{A}'(T)/\mathcal{J}) & \longleftarrow & K_1(\mathcal{A}'(T)) & \longleftarrow & K_1(\mathcal{J}) \end{array}$$

Since $\mathcal{A}'(T)/\mathcal{J} \cong M_2(C)$, we have $K_0(\mathcal{A}'(T)/\mathcal{J}) \cong Z$ and $K_1(\mathcal{A}'(T)/\mathcal{J}) \cong 0$. Moreover, we have $K_0(\mathcal{A}'(T)) \cong Z^{(2)}$. Since

$$\pi_* \begin{bmatrix} I_{\mathcal{A}'(A_1)} & 0 \\ 0 & 0 \end{bmatrix} = 1, \quad \pi_* \begin{bmatrix} 0 & 0 \\ 0 & I_{\mathcal{A}'(A_2)} \end{bmatrix} = 1,$$

$\pi_* : Z \oplus Z \longrightarrow Z$ is surjective. Therefore we obtain the following split exact sequence:

$$0 \longrightarrow K_0(\mathcal{J}) \xrightarrow{i_*} Z \oplus Z \overset{\pi_*}{\rightleftharpoons} Z \longrightarrow 0.$$

Since $\pi_*(K_0(\mathcal{A}'(T))) \cong Z$, we get $K_0(\mathcal{J}) \cong Z$.

From the split exact sequence

$$0 \longrightarrow \mathcal{J} \longrightarrow \mathcal{J} \dot{+} 1 \rightleftharpoons (\mathcal{J} \dot{+} 1)/\mathcal{J} \longrightarrow 0,$$

using the property of the K_0-group, we get following split exact sequence:

$$0 \longrightarrow K_0(\mathcal{J}) \longrightarrow K_0(\mathcal{J} \dot{+} 1) \rightleftharpoons K_0((\mathcal{J} \dot{+} 1)/\mathcal{J}) \longrightarrow 0.$$

Since $(\mathcal{J} \dot{+} 1)/\mathcal{J} \cong C \oplus C$, we have $K_0((\mathcal{J} \dot{+} 1)/\mathcal{J}) \cong Z^{(2)}$. Therefore $K_0(\mathcal{J} \dot{+} 1) \cong Z \oplus Z \oplus Z$, and there exist three minimal idempotents that are not similarity equivalent to one another in $M_\infty(\mathcal{J} \dot{+} 1)$. Let

$$P_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

be two minimal idempotents of $P_\infty(\mathcal{J} \dot{+} 1)$, and let the third one be P in $M_\infty(\mathcal{J} \dot{+} 1)$. Then $P \not\sim_a P_1$, $P \not\sim_a P_2$ in $M_\infty(\mathcal{J} \dot{+} 1)$.

Claim 4. *$I - P \sim_a P$ in $M_\infty(\mathcal{J} \dot{+} 1)$.*

Otherwise, $I - P \not\sim_a P$ in $M_\infty(\mathcal{J} \dot{+} 1)$. Then $I - P \sim_a P_1$ or $I - P \sim_a P_2$ in $M_\infty(\mathcal{J} \dot{+} 1)$. Therefore $P \sim_a P_2$ or $P \sim_a P_1$ in $M_\infty(\mathcal{J} \dot{+} 1)$. This is a contradiction.

Since $M_\infty(\mathcal{J} \dot{+} 1) \subset M_\infty(\mathcal{A}'(T))$, we have $P \sim_a P_1$ and $(I - P) \sim_a P_1$ (or $P \sim_a P_2$ and $(I - P) \sim_a P_2$) in $\mathcal{A}'(T)$. By Lemma 3.4, we have $T|_{\operatorname{ran} P} \sim T|_{\operatorname{ran}(I-P)}$. But $T|_{\operatorname{ran} P} \sim A_1$, $T|_{\operatorname{ran}(I-P)} \sim A_2$, so $A_1 \sim A_2$. This contradicts the relation $A_1 \not\sim A_2$.

Therefore $K_0(\mathcal{J} \dot{+} 1) \cong Z \oplus Z$, and $K_0(\mathcal{J}) = 0$. But we have already proved that $K_0(\mathcal{J}) \cong Z$, so this is impossible.

Next, assume $T = A_1^{(n_1)} \oplus A_2^{(n_2)}$. Let $\mathcal{J}$ be a maximal ideal of $\mathcal{A}'(T)$. If

$$\mathcal{J} = \begin{bmatrix} \begin{bmatrix} \mathcal{J}_{11} & \cdots & \mathcal{J}_{11} \\ \vdots & \cdots & \vdots \\ \mathcal{J}_{11} & \cdots & \mathcal{J}_{11} \end{bmatrix}_{n_1\times n_1} & \begin{bmatrix} \mathcal{J}_{12} & \cdots & \mathcal{J}_{12} \\ \vdots & \cdots & \vdots \\ \mathcal{J}_{12} & \cdots & \mathcal{J}_{12} \end{bmatrix}_{n_1\times n_2} \\ \begin{bmatrix} \mathcal{J}_{21} & \cdots & \mathcal{J}_{21} \\ \vdots & \cdots & \vdots \\ \mathcal{J}_{21} & \cdots & \mathcal{J}_{21} \end{bmatrix}_{n_2\times n_1} & \begin{bmatrix} \mathcal{J}_{22} & \cdots & \mathcal{J}_{22} \\ \vdots & \cdots & \vdots \\ \mathcal{J}_{22} & \cdots & \mathcal{J}_{22} \end{bmatrix}_{n_2\times n_2} \end{bmatrix},$$

where $\mathcal{J}_{12} \subset (\neq) \ker \tau_{A_1,A_2}$ and $\mathcal{J}_{21} \subset (\neq) \ker \tau_{A_2,A_1}$, then

$$\mathcal{J} = \begin{bmatrix} \begin{bmatrix} \mathcal{J}_{11} & \mathcal{J}_{12} \\ \mathcal{J}_{21} & \mathcal{J}_{22} \end{bmatrix} & * \\ * & \begin{bmatrix} \begin{bmatrix} \mathcal{J}_{11} & \cdots & \mathcal{J}_{11} \\ \vdots & \cdots & \vdots \\ \mathcal{J}_{11} & \cdots & \mathcal{J}_{11} \end{bmatrix}_{k\times k} & \begin{bmatrix} \mathcal{J}_{12} & \cdots & \mathcal{J}_{12} \\ \vdots & \cdots & \vdots \\ \mathcal{J}_{12} & \cdots & \mathcal{J}_{12} \end{bmatrix}_{k\times l} \\ \begin{bmatrix} \mathcal{J}_{21} & \cdots & \mathcal{J}_{21} \\ \vdots & \cdots & \vdots \\ \mathcal{J}_{21} & \cdots & \mathcal{J}_{21} \end{bmatrix}_{l\times k} & \begin{bmatrix} \mathcal{J}_{22} & \cdots & \mathcal{J}_{22} \\ \vdots & \cdots & \vdots \\ \mathcal{J}_{22} & \cdots & \mathcal{J}_{22} \end{bmatrix}_{l\times l} \end{bmatrix} \end{bmatrix}$$

is a maximal ideal of the commutant of $T = (A_1 \oplus A_2) \oplus (A_1^{(n_1-1)} \oplus A_2^{(n_2-1)})$, where $k = n_1 - 1, l = n_2 - 1$. Hence, by Lemma 3.6,

$$\begin{bmatrix} \mathcal{J}_{11} & \mathcal{J}_{12} \\ \mathcal{J}_{21} & \mathcal{J}_{22} \end{bmatrix}$$

is a maximal ideal of $\mathcal{A}'(A_1 \oplus A_2)$. This is a contradiction. $\square$

Corollary 4.2. *Let $T = A_1^{(n_1)} \oplus A_2^{(n_2)} \oplus \cdots \oplus A_k^{(n_k)}$, where $A_1, A_2, \ldots, A_k$ and T satisfy the conditions of Lemma 3.9. Then for arbitrary $B_{ij} = A_i^{(n_i)} \oplus A_j^{(n_j)}$, $i \neq j$, we have $\bigvee(\mathcal{A}'(B_{ij})) \cong N^{(2)}$, and for arbitrary $\mathcal{J} \in m(\mathcal{A}'(T))$ we have*

$$\mathcal{J} = \begin{bmatrix} \mathcal{J}_{11} & \mathcal{J}_{12} & \cdots & \mathcal{J}_{1k} \\ \mathcal{J}_{21} & \mathcal{J}_{22} & \cdots & \mathcal{J}_{2k} \\ \cdots & \cdots & \cdots & \cdots \\ \mathcal{J}_{k1} & \mathcal{J}_{k2} & \cdots & \mathcal{J}_{kk} \end{bmatrix},$$

where $\mathcal{J}_{ij} = \ker \tau_{A_i^{(n_i)}, A_j^{(n_j)}}$ when $i \neq j$, and there is a unique i such that $\mathcal{J}_{ii} \in m(\mathcal{A}'(A_i^{(n_i)}))$ and $\mathcal{J}_{jj} = \mathcal{A}'(A_j^{(n_j)})$ when $j \neq i$.

Corollary 4.3. *Let $T = A_1^{(n_1)} \oplus A_2^{(n_2)} \oplus \cdots \oplus A_k^{(n_k)}$, where $A_1, A_2, \ldots, A_k$ and T satisfy the conditions of Lemma 3.8. Then for arbitrary $B_{ij} = A_i^{(n_i)} \oplus A_j^{(n_j)}$, $i \neq j$, we have $\bigvee(\mathcal{A}'(B_{ij})) \cong N^{(2)}$, and for arbitrary $\mathcal{J} \in m(\mathcal{A}'(T))$,* *we have* $\mathcal{A}'(T)/\mathcal{J} \cong M_{n_i}(C)$ *for some* i. *Furthermore,* $\mathcal{A}'(T)/\mathcal{J} \cong M_{n_i}(C)$ *if and only if* $\mathcal{J}_{ii} \in m(\mathcal{A}'(A_i^{(n_i)}))$ *and* $\mathcal{J}_{jj} = \mathcal{A}'(A_j^{(n_j)})$ *when* $j \neq i$,

THEOREM 4.4. *Suppose* $A, B \in \mathcal{L}(\mathcal{H})$, *and*

$$A = A_1^{(n_1)} \oplus A_2^{(n_2)} \oplus \cdots \oplus A_k^{(n_k)},$$
$$B = B_1^{(m_1)} \oplus B_2^{(m_2)} \oplus \cdots \oplus B_l^{(m_l)},$$

where $A_i, B_j \in (SI)$ *for* $i = 1, 2, \ldots, k, j = 1, 2, \ldots, l$, A_i *and* B_j *are not similarity equivalent to each other, and* A, B *and* A_i, B_j $(i = 1, 2 \ldots, k,\ j = 1, 2, \ldots, l)$ *satisfy the following conditions:*

(1) $K_0(\mathcal{A}'(A_i)) = Z$, $K_0(\mathcal{A}'(B_j)) = Z$ *for* $i = 1, 2, \ldots, k$, $j = 1, 2, \ldots, l$.
(2) *For any positive integer* n *and minimal idempotent* $P \in \mathcal{A}'(T^{(n)})$, $\mathcal{A}'(T^{(n)}|_{P\mathcal{H}^{(n)}})/\operatorname{rad}\mathcal{A}'(T^{(n)}|_{P\mathcal{H}^{(n)}})$ *is commutative, where* $T \in \{A, B\}$.

Then $A \sim B$ *if and only if:*

(1) $(K_0(\mathcal{A}'(A \oplus B)), \bigvee(\mathcal{A}'(A \oplus B)), 1_{\mathcal{A}'(A\oplus B)}) \cong (Z^{(k)}, N^{(k)}, 1)$.
(2) *For all* $\mathcal{J} \in m(A \oplus B)$, *we have* $\mathcal{A}'(A \oplus B)/\mathcal{J} \cong M_m(C)$, $m \in (2n_1, 2n_2, \ldots, 2n_k)$.

Proof. "$\Leftarrow$": We assume that $B = B_1^{(s_1)} \oplus B_2^{(s_2)} \oplus \cdots \oplus B_m^{(s_m)}$, where $B_i \in (SI)$ for $i = 1, 2, \ldots, m$.

CLAIM 5. *For each* B_i, $i = 1, 2, \ldots, m$, *there exists* A_j, $j = 1, 2, \ldots, k$ *such that* $B_i \sim A_j$.

Otherwise, we may assume B_1 is not similar to any A_j, but any B_i for $i \neq 1$ is similar to some A_j. Then $(A \oplus B) \sim (A_1^{(t_1)} \oplus A_2^{(t_2)} \oplus \cdots \oplus A_k^{(t_k)} \oplus B_1^{(s_1)})$. By Lemma 3.8, we have $\bigvee(A_1^{(t_1)} \oplus A_2^{(t_2)} \oplus \cdots \oplus A_k^{(t_k)} \oplus B_1^{(s_1)}) \cong N^{(k+1)}$. This contradicts (1).

CLAIM 6. $m = k$.

By Claim 5, we have $m \leq k$. We assume $m < k$ and $B_1 \sim A_1, B_2 \sim A_2, \ldots, B_m \sim A_m$. Then $(A \oplus B) \sim (A_1^{(t_1)} \oplus A_2^{(t_2)} \oplus \cdots \oplus A_m^{(t_m)} \oplus A_{m+1}^{(n_{m+1})} \oplus \cdots \oplus A_k^{(n_k)})$. By Corollary 4.3 there exists a maximal ideal $\mathcal{J}$ such that $\mathcal{A}'(A \oplus B)/\mathcal{J} \cong M_{n_k}(C)$, but from (2), we have $\mathcal{A}'(A \oplus B)/\mathcal{J} \cong M_{2n_k}(C)$. This is a contradiction.

We may assume $B_1 \sim A_1, B_2 \sim A_2, \ldots, B_k \sim A_k$. Then

$$(A \oplus B) \sim (A_1^{(n_1+s_1)} \oplus A_2^{(n_2+s_2)} \oplus \cdots \oplus A_k^{(n_k+s_k)}).$$

CLAIM 7. $s_i = n_i$ *for* $i = 1, 2, \ldots, k$.

Otherwise, we can assume $s_1 \neq n_1$. By Corollary 4.3, there exists a maximal ideal $\mathcal{J}$ such that $\mathcal{A}'(A \oplus B)/\mathcal{J} \cong M_{n_1+s_1}(C)$. By (2), we have $\mathcal{A}'(A \oplus B)/\mathcal{J} \cong M_{2n_1}(C)$. This is a contradiction.

"$\Rightarrow$": Since $A \sim B$, we have $B = B_1^{(n_1)} \oplus B_2^{(n_2)} \oplus \cdots \oplus B_k^{(n_k)}$, where $B_i \in (SI)$ for $i = 1, 2, \ldots, k$ and $A_i \sim B_i$ for $i = 1, 2, \ldots, k$. By Lemma 3.2, we have $\bigvee(\mathcal{A}'(A \oplus B)) = \bigvee(\mathcal{A}'(\bigoplus_{i=1}^k A_i^{(t_k)}) = \bigvee(\mathcal{A}'(\bigoplus_{i=1}^k A_i) = N^{(k)}$. Thus (1) is true.

By Corollary 4.3, (2) is also true. □

C.L. Jiang and Z.Y. Wang proved the following theorem [JW, Chapter 3].

Lemma 4.5. *Every Cowen-Douglas operator can be written as the direct sum of finitely many strongly irreducible Cowen-Douglas operators.*

Thus we have the following corollary (cf. [JGJ]).

Corollary 4.6. *Let $A, B \in \mathcal{B}_n(\Omega)$. Suppose that*

$$A = A_1^{(n_1)} \oplus A_2^{(n_2)} \oplus \cdots \oplus A_k^{(n_k)},$$

where $0 \neq n_i \in N$, $A_i \in (SI)$ for $i = 1, 2, \ldots, k$ and $A_i \not\sim A_j$ for $i \neq j$. Then $A \sim B$ if and only if:

1. *$(K_0(\mathcal{A}'(A \oplus B)), \bigvee(\mathcal{A}'(A \oplus B)), I) \cong (Z^{(k)}, N^{(k)}, 1)$.*
2. *The isomorphism h from $\bigvee(\mathcal{A}'(A\oplus B))$ to $N^{(k)}$ sends $[I]$ to $(2n_1, 2n_2, \ldots 2n_k)$, i.e., $h([I]) = 2n_1e_1 + 2n_2e_2 + \cdots + 2n_ke_k$, where I is the unit of $\mathcal{A}'(A \oplus B)$ and $\{e_i\}_{i=1}^k$ are the generators of $N^{(k)}$.*

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