Illinois Journal of Mathematics
Volume 51, Number 2, Summer 2007, Pages 379–395
S 0019-2082

AN IRREGULARITY IN THE CLASS OF WEAK HILBERT SPACES

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Abstract. In the regular class of weak Hilbert spaces we exhibit a complex space which is not isomorphic to its complex conjugate. Thus there exists a real weak Hilbert space with at least two non-isomorphic complex structures.

1. Introduction and preliminaries

Using random techniques, several infinite dimensional Banach spaces with some interesting irregularities were constructed in the late 80's:

(1) a real Banach space with at least two non-isomorphic complex structures (J. Bourgain [B], with a variant by S. Szarek [S1]),
(2) a real Banach space which does not admit a complex structure (S. Szarek [S1]),
(3) a Banach space with a finite dimensional decomposition which does not have a basis (S. Szarek [S2]).

P. Mankiewicz and N. Tomczak-Jaegermann [MT-J] showed later, also by random methods, that these phenomena can be found in a general situation, namely as infinite dimensional quotients of subspaces of $l_2(X)$, for every non-Hilbertian X. (In [MT-J] condition (3) is replaced by

(3') a Banach space without a basis.

Nevertheless, in many cases the infinite dimensional quotients obtained satisfy also (3).) In addition, they produced some other results which provide strong evidence towards the following conjecture.

Conjecture ([MT-J]). *If X is not a weak Hilbert space, there is an infinite dimensional quotient of a subspace of X which satisfies* (3') *(respectively* (1)*, respectively* (2)*).*

Received July 19, 2004; received in final form August 5, 2007.
2000 *Mathematics Subject Classification.* Primary 46B20.
Supported in part by an NSERC Postdoctoral Fellowship and by NSERC Grant 312594-05.

If we now pass to the regular class of weak Hilbert spaces (which were introduced by G. Pisier in [P1]; see also [P2]), it is interesting to see if such irregularities still survive in this context. In fact, an important open problem in the area asks whether or not there exists a weak Hilbert space without a basis ([C], [P2]).

In this paper, we will show that the phenomenon (1) mentioned above can be observed in the class of weak Hilbert spaces. Namely, we will exhibit a complex weak Hilbert space X not isomorphic to its complex conjugate $\overline{X}$ (which is the Banach space with the same elements and norm as X, the same addition of vectors, while the multiplication by scalars is given by $\lambda \odot x = \overline{\lambda}\, x$, for $\lambda \in \mathbf{C}$ and $x \in X$). In particular, this is a stronger irregularity than being without an unconditional basis (such examples were obtained by R. Komorowski [Ko], R. Komorowski and N. Tomczak-Jaegermann [KoT-J]).

For the actual construction, the random methods are no longer suitable in the context of weak Hilbert spaces. We will employ some intuition from our previous work [A] and from [K].

A *weak Hilbert spaces* X is characterized by the property that every finite dimensional subspace contains a further subspace of fixed proportional dimension which is uniformly Euclidean and uniformly complemented: there exist constants $C > 0$ and $0 < \delta < 1$ such that for every finite dimensional $E \subset X$ there is $F \subset E$, with $\dim F \geq \delta \dim E$, and there is a projection $P : X \to F$ satisfying $d(F, \ell_2^{\dim F}) \leq C$ and $\|P\| \leq C$ (here d stands for the Banach-Mazur distance). The original definition is different and the characterization mentioned above is chosen out of many equivalent properties proved by Pisier. Weak Hilbert spaces are stable under passing to subspaces, dual spaces and quotient spaces. The canonical example of a weak Hilbert space which is not a Hilbert space is the 2-convexification of Tsirelson's space (most precisely, of the space T introduced by T. Figiel and W. B. Johnson in [FJ]), whose construction and properties are presented bellow.

Throughout this paper we write $E \leq F$ (respectively $E < F$) when E and F are subsets of the natural numbers and $\max E \leq \min F$ (respectively, $\max E < \min F$). If n is a positive integer, we write $n \leq F$ if $\{n\} \leq E$. Let c_{00} be the space of all sequences of complex numbers which are eventually zero and let $\{t_n\}_{n=1}^{\infty}$ be the unit vector basis of c_{00}. For $x = \sum_{n=1}^{\infty} a_n t_n \in c_{00}$ and a subset E of the positive integers, we put $Ex = \sum_{n\in E} a_n t_n$.

For $0 < \theta < 1$, T_θ is the completion of c_{00} with respect to the norm $\|\cdot\| := \lim_m \|\cdot\|_m$, where $\{\|\cdot\|_m\}_m$ is the monotone sequence of norms on c_{00} given by (for $x = \sum_n a_n t_n \in c_{00}$)

$$\|x\|_0 = \max_n |a_n|,$$

$$\|x\|_{m+1} = \max\left\{ \|x\|_m,\, \theta \max \sum_{j=1}^{k} \|E_j x\|_m \right\} \quad (\text{for } m \geq 0),$$

where the inner maximum is taken over all choices $k \leq E_1 < E_2 < \cdots < E_k$ and $k = 1, 2, \ldots$.

In this way we obtain a family $\{T_\theta\}$ of totally incomparable "Tsirelson-like" spaces (see [CS], Chapter X.A). When $\theta = 2^{-1}$, T_θ is the space which appears in [FJ]. Next we consider the 2-convexification of each of the spaces T_θ, following the procedure introduced in [J] (see also [CS], Chapter X.E). Namely, for a fixed $0 < \theta < 1$, we define the space X_θ (this notation is consistent with the one used by Pisier in [P2], Chapter 13) as the set of all $x = \sum_{n=1}^{\infty} a_n t_n$ with $\sum_n |a_n|^2 t_n \in T_\theta$, endowed with the norm

$$\|x\|_{X_\theta} = \Big\| \sum_n |a_n|^2 t_n \Big\|_{T_\theta}^{1/2}.$$

We will maintain the notation $\{t_n\}_n$ for the unit vector basis in all the spaces X_θ. It is easy to see that $\{t_n\}_n$ is 1-unconditional in each T_θ and X_θ.

From the known properties of T_θ (see [CJT], [CS]) we can easily deduce the following results about X_θ, which will be useful to the sequel.

PROPOSITION 1.1. *Let $0 < \theta < 1$ be fixed.*

(i) *For all $x \in X_\theta$, $\|x\|_{X_\theta} \leq \|x\|_{l_2}$.*
(ii) *If $\theta < \lambda < 1$, then $\|x\|_{X_\theta} \leq \|x\|_{X_\lambda}$, for all $x \in X_\lambda$.*
(iii) *If $\{k_n\}_n$ and $\{j_n\}_n$ are two increasing sequences of positive integers such that $k_n \leq j_n$ for all n, then*

$$\Big\| \sum_n a_n t_{k_n} \Big\|_{X_\theta} \leq \Big\| \sum_n a_n t_{j_n} \Big\|_{X_\theta}.$$

(iv) *For every increasing sequence of positive integers $\{k_n\}_n$ and any choice of scalars $\{a_n\}_n$ we have*

$$\Big\| \sum_n a_n t_{k_n} \Big\|_{X_\theta} \leq \Big\| \sum_n a_n t_{k_{2n}} \Big\|_{X_\theta} \leq 3^{1/2} \Big\| \sum_n a_n t_{k_n} \Big\|_{X_\theta}.$$

(v) *If $\{k_n\}_n$ and $\{j_n\}_n$ are two increasing sequences of positive integers such that $k_n < j_n < k_{n+1}$ for all n, then*

$$\Big\| \sum_n a_n t_{k_n} \Big\|_{X_\theta} \leq \Big\| \sum_n a_n t_{j_n} \Big\|_{X_\theta} \leq 3^{1/2} \Big\| \sum_n a_n t_{k_n} \Big\|_{X_\theta}.$$

(vi) *Let $y_n = \sum_{j=p_n+1}^{p_{n+1}} a_j t_j$, for $n \geq 1$, be a normalized block basis of $\{t_n\}_{n\geq 1}$ in X_θ. Then for every choice of natural numbers $p_n < k_n \leq p_{n+1}$, for $n \geq 1$, and scalars $\{b_n\}_{n\geq 1}$ we have*

$$3^{-1/2} \Big\| \sum_{n\geq 1} b_n t_{k_n} \Big\|_{X_\theta} \leq \Big\| \sum_{n\geq 1} b_n y_n \Big\|_{X_\theta} \leq 18^{1/2} \Big\| \sum_{n\geq 1} b_n t_{k_n} \Big\|_{X_\theta}.$$

(vii) *Let* $\{t_{k_n}\}_{n\geq 1}$ *be a subsequence of* $\{t_n\}_{n\geq 1}$. *For every integer* $m \geq 3$ *there exists* $x = \sum_{n\geq 1} a_n t_{k_n}$, *with* $\{a_n\}_{n\geq 1}$ *real scalars, such that*

$$\lambda^m \leq \|x\|_{X_\lambda} \leq m\,\lambda^m, \qquad \forall\ \theta \leq \lambda < 1.$$

It is not hard to verify that, for any $0 < \theta < 1$ and $x \in X_\theta$,

$$\|x\|_{X_\theta} = \max\left\{\|x\|_0,\, \theta^{1/2} \sup\left(\sum_{j=1}^{k} \|E_j x\|_{X_\theta}^2\right)^{1/2}\right\},$$

where the sup is taken over all $k \leq E_1 < E_2 < \cdots < E_k$ and $k = 1, 2, \ldots$.

We will conclude the introduction by mentioning that N. J. Nielsen and N. Tomczak-Jaegermann [NT-J] showed that every separable weak Hilbert space which is a Banach lattice is, in terms of tail behaviour, very much like X_θ.

2. Preliminary construction

We will work with 5-tuples $\eta = (\theta_1, \ldots, \theta_5)$ satisfying $0 < \theta_1 < \cdots < \theta_5 < 1$ and $\theta_1/\theta_2 = \cdots = \theta_4/\theta_5 = \alpha$ for some $\alpha \in (0,1)$.

For every η as above and every $N \in \mathbf{N}$ we construct a Banach space $Y_{N,\eta}$ as follows: we define 2-dimensional subspaces Z_k of $X_{\theta_1} \oplus_2 \cdots \oplus_2 X_{\theta_5}$ which will form an unconditional decomposition for $Y_{N,\eta} = \overline{\operatorname{span}}\,\{Z_k\}_k$. Namely, if we denote by $\{t_{j,k}\}_k$ the unit vector basis of each X_{θ_j}, for $j = 1, \ldots, 5$, define $Z_k \subset X_{\theta_1} \oplus_2 \cdots \oplus_2 X_{\theta_5}$ as being spanned by x_k and y_k, where

$$\begin{aligned} x_k &= t_{1,k} && + \gamma_1 t_{3,k} && + \gamma_2 t_{4,k} && + \gamma_3 t_{5,k}, \\ y_k &= \qquad t_{2,k} && && + \gamma_2 t_{4,k} && + i\gamma_3 t_{5,k}, \end{aligned} \tag{1}$$

with $\gamma_1 = \alpha^{4N}$, $\gamma_2 = \alpha^{10N}$ and $\gamma_3 = \alpha^{22N}$.

The decomposition $\{Z_k\}_{k\geq 1}$ is clearly 1-unconditional, while $x_1, y_1, x_2, y_2, \ldots$ form a Schauder basis in $Y_{N,\eta}$.

We will now explore the behavior of the (complex) linear operators acting between $Y_{N,\eta}$ and $\overline{Y}_{N,\eta}$. In fact, we will concentrate on such operators which are block-diagonal with respect to the 2-dimensional decompositions of $Y_{N,\eta}$ and $\overline{Y}_{N,\eta}$. If W, V are Banach spaces having finite-dimensional decompositions $\{W_k\}_k$ and $\{V_j\}_j$ respectively, we say that a bounded linear operator $T : W \to V$ is *block-diagonal* with respect to $\{W_k\}_k$ and $\{V_j\}_j$ if there exist finite sets $\{B_k\}_k$ such that

$$\begin{cases} \max B_k < \min B_l & \text{if } k < l, \\ \operatorname{supp} T w_k \subset B_k & \forall w_k \in W_k, \end{cases} \tag{2}$$

where $\operatorname{supp} T w_k$ is taken with respect to the decomposition $\{V_j\}_j$.

Proposition 2.1. *Let* $\eta = (\theta_1, \ldots, \theta_5)$ *be as above and* N *be a positive integer. Let* $I \subset \{1, 2, \ldots\}$ *be an infinite set and let* Y *be the subspace of* $Y_{N,\eta}$ *defined by* $Y = \overline{\operatorname{span}}\,\{Z_k\}_{k\in I}$. *If* $T : Y \longrightarrow \overline{Y}_{N,\eta}$ *is a block-diagonal* *operator (with respect to* $\{Z_k\}_{k\in I}$ *and* $\{\overline{Z_k}\}_{k\geq 1}$*), with* $\|T\|\leq 1$*, then there is an infinite subset* $J\subset I$ *such that*

$$\|Tx_k\|\leq 250N\alpha^N, \quad \textit{for all } k\in J.$$

Proof. For $k\in I$, let $B_k\subset\{1,2,\dots\}$ be a finite set and $u_k=(u_k(j))_j$, $v_k=(v_k(j))_j$, $w_k=(w_k(j))_j$, $s_k=(s_k(j))_j$ be sequences of scalars so that

$$\begin{cases} \max B_k<\min B_l, \quad \forall k,l\in I \text{ with } k<l, \\ Tx_k=\sum_{j\in B_k}\left(u_k(j)x_j+v_k(j)y_j\right), \\ Ty_k=\sum_{j\in B_k}\left(w_k(j)x_j+s_k(j)y_j\right). \end{cases} \tag{3}$$

By passing to a subsequence $\tilde{I}$ of I (and disregarding the elements not belonging to $\tilde{I}$) we may assume that $\{B_k\}_{k\in\tilde{I}}$ satisfy, besides (3), $\max B_k\geq k$ for all $k\in\tilde{I}$. Indeed, if $I=\{k_1,k_2,\dots,k_n,\dots\}$, then choose $\tilde{I}=\{k_1,k_{k_1},k_{k_{k_1}},\dots\}$.

Taking into account (1) we have, for every $k\in\tilde{I}$

$$\begin{aligned} Tx_k=&\sum_{j\in B_k}u_k(j)t_{1,j}+\sum_{j\in B_k}v_k(j)t_{2,j}+\sum_{j\in B_k}\gamma_1u_k(j)t_{3,j} \\ &+\sum_{j\in B_k}\gamma_2\left(u_k(j)+v_k(j)\right)t_{4,j}+\sum_{j\in B_k}\gamma_3\left(u_k(j)+iv_k(j)\right)t_{5,j}. \end{aligned} \tag{4}$$

For $1\leq l\leq 5$ let $Q_l:\overline{X}_{\theta_1}\oplus_2\cdots\oplus_2\overline{X}_{\theta_5}\to\overline{X}_{\theta_l}$ be the natural projection (which coincides with the natural projection from $X_{\theta_1}\oplus_2\cdots\oplus_2 X_{\theta_5}$ onto X_{θ_l}).

(I) We first show that there exists an infinite set $J_1\subset\tilde{I}$ such that

$$\|Q_2Tx_k\|=\Big\|\sum_{j\in B_k}v_k(j)t_{2,j}\Big\|_{X_{\theta_2}}\leq 8N\alpha^N, \quad \text{for all } k\in J_1. \tag{5}$$

Indeed, let A_1 be the set of all $k\in\tilde{I}$ such that

$$\|Q_2Tx_k\|=\Big\|\sum_{j\in B_k}v_k(j)t_{2,j}\Big\|_{X_{\theta_2}}>8N\alpha^N.$$

The conclusion will follow once we show that A_1 is finite.

If A_1 is infinite, take (Proposition 1.1(vii)) real scalars $\{a_k\}_{k\in A_1}$ such that

$$\begin{cases} \theta_1^N\leq\left\|\sum_{k\in A_1}a_kt_{1,k}\right\|_{X_{\theta_1}}\leq N\theta_1^N, \\ \vdots \\ \theta_5^N\leq\left\|\sum_{k\in A_1}a_kt_{5,k}\right\|_{X_{\theta_5}}\leq N\theta_5^N. \end{cases} \tag{6}$$

Letting $x = \sum_{k \in A_1} a_k x_k$ and taking into account the definition of x_k, we get

$$
\begin{aligned}
\|x\|_Y \leq & \Big\| \sum_{k \in A_1} a_k t_{1,k} \Big\|_{X_{\theta_1}} + \gamma_1 \Big\| \sum_{k \in A_1} a_k t_{3,k} \Big\|_{X_{\theta_3}} + \gamma_2 \Big\| \sum_{k \in A_1} a_k t_{4,k} \Big\|_{X_{\theta_4}} + \\
& \qquad + \gamma_3 \Big\| \sum_{k \in A_1} a_k t_{5,k} \Big\|_{X_{\theta_5}} \\
\leq & \, N\theta_1^N + \alpha^{4N} N\theta_3^N + \alpha^{10N} N\theta_4^N + \alpha^{22N} N\theta_5^N \qquad \text{(by (6))} \\
\leq & \, 4N\theta_1^N \qquad \text{(by definition of } \alpha).
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
\|Tx\| \geq & \Big\| Q_2 T\Big(\sum_{k \in A_1} a_k x_k \Big) \Big\|_{X_{\theta_2}} = \Big\| \sum_{k \in A_1} \|Q_2 T x_k\| \, a_k \, Q_2 T x_k / \|Q_2 T x_k\| \, \Big\|_{X_{\theta_2}} \\
& > 8N\alpha^N \Big\| \sum_{k \in A_1} a_k \, Q_2 T x_k / \|Q_2 T x_k\| \, \Big\|_{X_{\theta_2}} \qquad \text{(by unconditionality)} \\
& \geq 8N\alpha^N \, 3^{-1/2} \Big\| \sum_{k \in A_1} a_k t_{2, \max B_k} \Big\|_{X_{\theta_2}} \qquad \text{(by Proposition 1.1(vi))} \\
& \geq 4N\alpha^N \Big\| \sum_{k \in A_1} a_k t_{2,k} \Big\|_{X_{\theta_2}} \qquad \text{(by Proposition 1.1(iii))} \\
& \geq 4N\alpha^N \theta_2^N \geq 4N\theta_1^N \qquad \text{(by (6) and definition of } \alpha).
\end{aligned}
$$

Note that we have used above the fact that $\{\operatorname{supp} Tx_k\}_{k \in \tilde{I}}$ are successive and $\max B_k \geq k$, for all $k \in \tilde{I}$.

The above calculations show that we have obtained a contradiction, since $\|T\| \leq 1$. Thus the set A_1 must be finite.

In a similar manner we can get infinite sets $J_4 \subset J_3 \subset J_2 \subset J_1$ such that

$$
\begin{aligned}
\|Q_3 T x_k\| \leq 8N\alpha^N, & \quad \text{for all } k \in J_2, \\
\|Q_4 T x_k\| \leq 8N\alpha^N, & \quad \text{for all } k \in J_3, \\
\|Q_5 T x_k\| \leq 8N\alpha^N, & \quad \text{for all } k \in J_4,
\end{aligned}
$$

and hence we get by (4) that

$$
\|Tx_k\| \leq \Big\| \sum_{j \in B_k} u_k(j) t_{1,j} \Big\|_{X_{\theta_1}} + 32N\alpha^N, \quad \forall k \in J_4. \tag{7}
$$

(II) We show that there exists an infinite set $J_5 \subset J_4$ such that

$$
\Big\| \sum_{j \in B_k} w_k(j) t_{3,j} \Big\|_{X_{\theta_3}} \leq 30N\alpha^N, \quad \text{for all } k \in J_5. \tag{8}
$$

Let A_2 be the set of all $k \in J_4$ such that

$$\Big\| \sum_{j \in B_k} w_k(j) t_{3,j} \Big\| > 30 N \alpha^N.$$

If A_2 is an infinite set, pick real scalars $\{a_k\}_{k \in A_2}$ such that

$$\tag{9} \begin{cases} \theta_1^M \le \left\| \sum_{k \in A_2} a_k t_{1,k} \right\|_{X_{\theta_1}} \le M \theta_1^M, \\ \vdots \\ \theta_5^M \le \left\| \sum_{k \in A_2} a_k t_{5,k} \right\|_{X_{\theta_5}} \le M \theta_5^M, \end{cases}$$

where $M = 5N$. Let $y = \sum_{k \in A_2} a_k y_k$. Then

$$\begin{aligned} \|y\|_Y &\le \Big\| \sum_{k \in A_2} a_k t_{2,k} \Big\|_{X_{\theta_2}} + \gamma_2 \Big\| \sum_{k \in A_2} a_k t_{4,k} \Big\|_{X_{\theta_4}} + \gamma_3 \Big\| \sum_{k \in A_2} a_k t_{5,k} \Big\|_{X_{\theta_5}} \\ &\le M \theta_2^M + \alpha^{10N} M \theta_4^M + \alpha^{22N} M \theta_5^M \\ &\le M \theta_2^M + \alpha^{2M} M \theta_4^M + \alpha^{3M} M \theta_5^M = 3 M \theta_2^M, \end{aligned}$$

while

$$\begin{aligned} \|Ty\| \ge \|Q_3 Ty\| = \Big\| \sum_{k \in A_2} a_k Q_3 T y_k \Big\|_{X_{\theta_3}} &= \gamma_1 \Big\| \sum_{k \in A_2} a_k \sum_{j \in B_k} w_k(j) t_{3,j} \Big\|_{X_{\theta_3}} \\ &> 30 N \alpha^N \gamma_1 \Big\| \sum_{k \in A_2} a_k \Big(\sum_{j \in B_k} w_k(j) t_{3,j} \Big) \Big/ \Big\| \sum_{j \in B_k} w_k(j) t_{3,j} \Big\| \Big\|_{X_{\theta_3}} \quad \text{(uncond.)} \\ &\ge 30 N \alpha^N \gamma_1 \, 3^{-1/2} \Big\| \sum_{k \in A_2} a_k t_{3,k} \Big\|_{X_{\theta_3}} \quad \text{(by Prop 1.1(vi) and (iii))} \\ &\ge 15 N \alpha^{5N} \theta_3^M = 3 M \alpha^M \theta_3^M = 3 M \theta_2^M, \end{aligned}$$

which is a contradiction. Hence A_2 is a finite set and we obtain (8). By Prop 1.1(ii)

$$\tag{10} \Big\| \sum_{j \in B_k} w_k(j) t_{1,j} \Big\|_{X_{\theta_1}} \le 30 N \alpha^N, \quad \forall k \in J_5.$$

(III) We show that there exists an infinite set $J_6 \subset J_5$ such that

$$\tag{11} \Big\| \sum_{j \in B_k} \left(u_k(j) + v_k(j) - w_k(j) - s_k(j) \right) t_{4,j} \Big\|_{X_{\theta_4}} \le 100 N \alpha^N, \quad \forall k \in J_6.$$

Take $A_3 \subset J_5$ an infinite set with the property that

$$\Big\| \sum_{j \in B_k} (u_k(j) + v_k(j) - w_k(j) - s_k(j)) t_{4,j} \Big\|_{X_{\theta_4}} > 100 N \alpha^N$$

for all $k \in A_3$, and pick real scalars $\{a_k\}_{k\in A_3}$ similarly as in (9), with $M = 11N$. Let $z = \sum_{k\in A_3} a_k(x_k - y_k)$. The contradiction is obtained by estimating $\|z\|$ from above and $\|Tz\|(\geq \|Q_4Tz\|)$ from bellow.

Once (11) is shown, by Proposition 1.1(ii) we get

$$\left\| \sum_{j\in B_k} (u_k(j) + v_k(j) - w_k(j) - s_k(j))\, t_{1,j} \right\|_{X_{\theta_1}} \leq 100N\alpha^N, \quad \forall k \in J_6. \tag{12}$$

(IV) Finally, there exists an infinite set $J_7 \subset J_6$ such that

$$\left\| \sum_{j\in B_k} (u_k(j) + s_k(j) + iv_k(j) - iw_k(j))\, t_{5,j} \right\|_{X_{\theta_5}} \leq 230N\alpha^N, \quad \forall k \in J_7,$$

and thus

$$\left\| \sum_{j\in B_k} (u_k(j) + s_k(j) + iv_k(j) - iw_k(j))\, t_{1,j} \right\|_{X_{\theta_1}} \leq 230N\alpha^N, \ \forall k \in J_7. \tag{13}$$

This is proved as above by considering elements of the form $z = \sum_{k\in A_4} a_k(x_k + iy_k)$, where $\{a_k\}_{k\in A_4}$ are chosen similarly as in (9) (take now $M = 23N$). Notice that

$$Tz = \sum_{k\in A_4} a_k(Tx_k + i \odot Ty_k) = \sum_{k\in A_4} a_k(Tx_k - iTy_k),$$

since $\operatorname{Range} T \subset \overline{Y}_{N,\eta}$.

This finishes the proof of Proposition 2.1 since we can conclude, by combining (5), (10), (12) and (13), that

$$\left\| \sum_{j\in B_k} u_k(j) t_{1,j} \right\|_{X_{\theta_1}} \leq 210N\alpha^N, \quad \text{for all } k \in J_7.$$

Together with (7) this gives the announced result. □

Remark 2.2. If N is chosen large enough, e.g., $N \geq 250$, as it will be the case later in the arguments, then the conclusion of Proposition 2.1 can be restated as follows: there is an infinite subset J such that

$$\|Tx_k\| \leq N^2\alpha^N, \quad \text{for all } k \in J$$

Remark 2.3. If $T : Y_{N,\eta} \to \overline{Y}_{N,\eta}$ is a bounded operator, then, by a classical gliding hump argument, we can find a subspace $Y = \overline{\operatorname{span}}\{Z_k\}_{k\in I}$ such that a perturbation of $T_{|Y} : Y \longrightarrow \overline{Y}_{N,\eta}$ is a block-diagonal operator. This is due to the fact that the basis $x_1, y_1, x_2, y_2, \ldots$ of $Y_{N,\eta}$ is shrinking, and therefore w-null. Indeed, if the basis is not shrinking, we can find $\delta > 0$ and normalized blocks $\{w_l\}_l$ (with respect to the decomposition $\{Z_k\}_k$) such that for $\{a_l\}_l \in c_{00}$ we have $\|\sum_l a_l w_l\| \geq \delta \sum_l |a_l|$, This is a contradiction, by Prop 1.1(i).

Using Proposition 2.1 we can prove a similar result about the behavior of block-diagonal operators with respect to some blocks of the basis in $Y_{N,\eta}$. For the sake of clarity of the exposition, we will present the proof of this fact at the end of the paper.

Proposition 2.4. *Let $\eta = (\theta_1, \dots, \theta_5)$ be as in Proposition 2.1 and let $N \geq 250$. Let I be an infinite set of positive integers and let Y be the subspace of $Y_{N,\eta}$ defined by $Y = \overline{\text{span}}\,\{Z_k\}_{k\in I}$. Let $T : Y \longrightarrow \overline{Y}_{N,\eta}$ be a block-diagonal operator (with respect to $\{Z_k\}_{k\in I}$ and $\{\overline{Z_k}\}_{k\geq 1}$) with $\|T\| \leq 1$.*

There exist $I_0 \subset I$ and real scalars $\{\beta_k\}_{k\in I_0}$ such that

$$1/2\ \theta^N \leq \Big\|\sum_{k\in I_0} \beta_k t_k\Big\|_{X_\theta} \leq 2N\theta^N \tag{14}$$

for all $\theta \in \{\theta_1, \dots, \theta_5\}$, and

$$\Big\|T\Big(\sum_{k\in I_0} \beta_k x_k\Big)\Big\|_{\overline{Y}_{N,\eta}} \leq N^4\alpha^N\theta_1^N.$$

Moreover, given a finite set Θ of numbers from $(0,1)$, we can choose $\{\beta_k\}_{k\in I_0}$ so that (14) is also satisfied for every $\theta \in \Theta$.

3. A weak Hilbert space

In this section we exhibit a weak Hilbert space which will be later used for the construction which is the object of this paper. As before, we will use the notation $\{t_n\}_n$ for the unit vector basis in all the spaces X_θ.

Proposition 3.1. *There is an absolute constant $C > 0$ such that, for all $1/2 \leq \theta < 1$ and $n \geq 1$, whenever $E \subset \overline{\text{span}}\,\{t_j\}_{j\geq n}$ is a subspace of X_θ with $\dim\ E \leq n$, then*

$$d\left(E, \ell_2^{\dim\ E}\right) \leq C.$$

Proof. The proof is similar to the one for the case $\theta = 1/2$ (see [CS], Proposition Ab.2) and it uses the fact that X_θ is C-isomorphic to its "modified" version, which is obtained by considering disjoint sets instead of successive ones in the definition of X_θ ([Be], [CO]). □

We now require the introduction of some new norms on X_θ, for each $0 < \theta < 1$. Namely, for every positive integer $p \geq 1$, we define the Banach space S_θ^p exactly as X_θ except that in the inner maximum of

$$\|x\|_{p,m+1} = \max\left\{\|x\|_{p,m},\ \theta^{1/2}\ \max\left(\sum_{j=1}^{2^p k}\|E_j x\|_{p,m}^2\right)^{1/2}\right\}$$

we allow the finite sets $\{E_j\}_{j=1}^{2^p k}$ to satisfy $k \leq E_1 < E_2 < \cdots < E_{2^p k}$.

It is easy to verify that for every $p \geq 1$ and every choice of scalars $\{a_n\}_n$

$$\left\|\sum_n a_n t_n\right\|_{S_\theta^p} = \left\|\sum_n a_n t_{2^p n}\right\|_{X_\theta}. \tag{15}$$

Inductively, it follows from Proposition 1.1(iv) that for all $p \geq 1$ and $x \in X_\theta$

$$\|x\|_{X_\theta} \leq \|x\|_{S_\theta^p} \leq 3^{p/2}\|x\|_{X_\theta}. \tag{16}$$

As a consequence of (15) and Proposition 3.1 we get:

Proposition 3.2. *There is an absolute constant $C > 0$ such that, for all $1/2 \leq \theta < 1$ and all positive integers $p, n \geq 1$, whenever $E \subset \overline{\text{span}}\,\{t_j\}_{j\geq n}$ is a subspace of S_θ^p with* $\dim\ E \leq 2^p n$, *then*

$$d\left(E, \ell_2^{\dim\ E}\right) \leq C.$$

This enables us to construct a class of weak Hilbert spaces, which are close in spirit to the ones used in [Ko].

Proposition 3.3. *Let $\{\theta_p\}_{p\geq 1}$ be a sequence in $[1/2, 1)$. Then the space $S = (\sum_{p\geq 1} \oplus S_{\theta_p}^p)_{l_2}$ is a weak Hilbert space.*

Proof. For a positive integer n let $J_n = \{n+1, n+2, \dots\}$. For simplicity, denote $\overline{\text{span}}\,\{t_j\}_{j\in J_n}$ in $S_{\theta_p}^p$ by $S_{\theta_p\,|\,J_n}^p$.

For every $m = 2^r$ $(r = 1, 2, \dots)$ let S_m be the following $(m-1)$-codimensional subspace of S:

$$S_m = S^1_{\theta_1\,|\,J_{m/2}} \oplus_2 S^2_{\theta_2\,|\,J_{m/2^2}} \oplus_2 \cdots \oplus_2 S^r_{\theta_r\,|\,J_{m/2^r}} \oplus_2 (\sum_{p>r} \oplus S_{\theta_p}^p)_{l_2}.$$

By Proposition 3.2, every m-dimensional subspace $E \subset S_m$ has the property $d(E, \ell_2^m) \leq C$. In a manner similar to the proof of Lemma 13.5 of [P2], it follows that S is of weak cotype 2, and since clearly S is 2-convex as a Banach lattice it is of type 2 by a result of Maurey. Hence S is a weak Hilbert space. $\square$

4. A weak Hilbert space non-isomorphic to its complex conjugate

For $\eta = (\theta_1, \dots, \theta_5)$ satisfying $0 < \theta_1 < \cdots < \theta_5 < 1$ and $\theta_1/\theta_2 = \cdots = \theta_4/\theta_5 = \alpha$, for some $\alpha \in (0,1)$, and any positive integer $N \geq 1$ we defined in Section 2 the Banach space $Y_{N,\eta}$ as a subspace of $X_{\theta_1} \oplus_2 \cdots \oplus_2 X_{\theta_5}$. In the view of (16), $Y_{N,\eta}$ can also be seen as a subspace of $S_{\theta_1}^p \oplus_2 \cdots \oplus_2 S_{\theta_5}^p$, for every $p \geq 1$. In this case $Y_{N,\eta}$ is the same vector space as before endowed with an equivalent norm.

We can now proceed and construct a weak Hilbert space which is not isomorphic to its complex conjugate.

THEOREM 4.1. *There exists a complex weak Hilbert space Y which is not isomorphic to its complex conjugate. Thus Y, treated as a real space, has at least two non- isomorphic complex structures.*

Proof. Pick an increasing sequence $\{\theta_k\}_{k\geq 5}$ of numbers from the interval $[1/2, 1)$ such that, for all $m = 1, 2, \ldots$,

$$\frac{\theta_{5m}}{\theta_{5m+1}} = \frac{\theta_{5m+1}}{\theta_{5m+2}} = \cdots = \frac{\theta_{5m+4}}{\theta_{5m+5}} =: \alpha_m \tag{17}$$

for some $\alpha_m \in (0, 1)$. In particular, the 5-tuple $\eta_m := (\theta_{5m+1}, \ldots, \theta_{5m+5})$ is of the form considered in Section 2, for all $m = 1, 2, \ldots$. For each $m = 1, 2, \ldots$ let N_m be a positive integer satisfying

$$\begin{cases} N_m^3 \geq 400(m-1), \\ (1/N_m)^4 (1/\alpha_m)^{N_m} \geq 16m3^m. \end{cases} \tag{18}$$

Let $Y_m = Y_{N_m,\eta_m}$ be the space defined in Section 2, treated as a subspace of $S^m_{\theta_{5m+1}} \oplus_2 \cdots \oplus_2 S^m_{\theta_{5m+5}}$, as we discussed before. Let $Y = (\sum_{m\geq 1} \oplus Y_m)_{\ell_2}$. By Proposition 3.3, Y is a weak Hilbert space. We will show that Y is not isomorphic to its complex conjugate $\overline{Y} = (\sum_{m\geq 1} \oplus \overline{Y}_m)_{\ell_2}$.

Assume that there exists an isomorphism $T : Y \to \overline{Y}$ such that $\|T\| \leq 1/4$ and let $a = \|T^{-1}\|$. Let $m \geq 2$ be arbitrarily fixed. We will show that $a \geq m$, which will clearly imply the contradiction.

To this end we will concentrate on $T_{|Y_m}$. Recall that $Y_m = Y_{N_m,\eta_m} = \overline{\operatorname{span}}\{Z_k\}_k$, where each Z_k is spanned by x_k and y_k given in (1).

(I) Denote by $R_m : \overline{Y} \to (\sum_{j>m} \oplus \overline{Y}_j)_{l_2}$ the natural projection. We will show that there exists a subsequence I such that, after some perturbations, we get an operator (denoted again by) $T : \overline{\operatorname{span}}\{Z_k\}_{k\in I} \to \overline{Y}$ such that

$$\begin{cases} R_m T = 0, \\ \frac{1}{4a}\|x\| \leq \|Tx\| \leq \|x\|, \quad \text{for all } x \in \overline{\operatorname{span}}\{Z_k\}_{k\in I}. \end{cases} \tag{19}$$

Let $P_j : \overline{Y} \to \overline{Y}_j$ be the canonical projection of $\overline{Y}$ onto its j-th term.

Let $s \geq m+1$. We first show that for every infinite set $L \subset \{1, 2, \ldots\}$ and every $\epsilon_s > 0$ there is $k \in L$ such that

$$\|P_s T z_k\| < \epsilon_s \|z_k\|, \qquad \forall z_k \in Z_k \tag{20}$$

Otherwise we can find $\epsilon_s > 0$, an infinite set $\{k_j\}_j$ and, for each $j \geq 1$, normalized elements $z_j \in Z_{k_j}$ such that

$$\epsilon_s \leq \|P_s T z_j\| \ \left(\leq \|Q_{1,s} P_s T z_j\| + \cdots + \|Q_{5,s} P_s T z_j\|\right),$$

where, similarly as in Section 2, we denote by $Q_{t,s} : S^s_{5s+1} \oplus_2 \cdots \oplus_2 S^s_{5s+5} \to S^s_{5s+t}$ the canonical projection ($t = 1, \ldots, 5$). By passing to a subsequence of

j's and perturbing the operator P_sT (see Remark 2.3), we may assume that $(P_sTz_j)_j$ are successive blocks in $\overline{Y}_s$ and also

$$\|Q_{t,s}P_sTz_j\| \geq \epsilon_s/10, \quad \text{for all } j \geq 1$$

for some $t \in \{1, \dots, 5\}$. A similar (but less delicate) argument to the one repeatedly used in the proof of Proposition 2.1 contradicts the fact that P_sT is continuous. The important fact here is that $\theta_{5m+5} < \theta_{5s+1} < \cdots < \theta_{5s+5}$.

Using (20), by recursion and a standard diagonal argument we obtain that for every $\epsilon_s \searrow 0$ there exists $\tilde{I} = \{k_{m+1}, k_{m+2}, \dots\}$ such that

$$\|P_sT_{\mid \overline{\operatorname{span}}\{Z_{k_j}\}_{j\geq s}}\| < \epsilon_s, \quad \text{for all } s \geq m+1.$$

Therefore, after a perturbation, we get an operator (denoted again by) $T : \overline{\operatorname{span}}\{Z_k\}_{k\in\tilde{I}} \to \overline{Y}$ satisfying

$$\begin{cases} P_sT_{\mid \overline{\operatorname{span}}\{Z_{k_j}\}_{j\geq s}} = 0, & \text{for all } s \geq m+1, \\ \frac{1}{2a}\|x\| \leq \|Tx\| \leq \frac{1}{2}\|x\|, & \text{for all } x \in \overline{\operatorname{span}}\{Z_k\}_{k\in\tilde{I}}. \end{cases} \tag{21}$$

In order to show (19), it is now enough to prove that for all $\delta > 0$ and every infinite set $L \subset \tilde{I}$ there is $k \in L$ so that

$$\|R_mTz_l\| \leq \delta\|z_l\|, \quad \text{for all } z_l \in Z_l. \tag{22}$$

Then (19) follows by recursion and a perturbation argument. If (22) does not hold true, then we can find $\delta > 0$, an infinite set $L \subset \tilde{I}$ and, for each $l \in L$, a normalized element $z_l \in Z_l$ such that $\|R_mTz_l\| > \delta$. If $L = \{l_1, l_2, \dots\}$ with $l_1 < l_2 < \cdots$, then, by (21), we have $\operatorname{supp} R_mTz_{l_1} \supset \operatorname{supp} R_mTz_{l_2} \supset \dots$, where the supports are considered with respect to the decomposition $\{\overline{Y}_s\}_{s\geq m+1}$. After a gliding hump argument we may assume that $(R_mTz_l)_{l\in L}$ are successive blocks in $(\sum_{s\geq m+1} \oplus \overline{Y}_s)_{l_2}$. We can now take real scalars $\{a_l\}_{l\in L}$ such that $\sum_{l\in L} |a_l|^2 = \infty$, while $\sum_{l\in L} a_lt_l$ is convergent in $X_{\theta_{5m+5}}$ (and therefore also in $X_{\theta_{5m+1}}, \dots, X_{\theta_{5m+4}}$). Thus $z = \sum_{l\in L} a_lz_l$ is convergent in $X_{\theta_{5m+1}} \oplus_2 \cdots \oplus_2 X_{\theta_{5m+5}}$ (and hence also in $S^m_{\theta_{5m+1}} \oplus_2 \cdots \oplus_2 S^m_{\theta_{5m+5}}$), while $R_mTz = \sum_{l\in L} a_lR_mTz_l$ is divergent in $\overline{Y}$. This shows that the above assumption is false and completes the first stage of the proof.

(II) We may assume (see Remark 2.3) that, in addition to (19), each of the operators $P_1T : \overline{\operatorname{span}}\{Z_k\}_{k\in I} \longrightarrow \overline{Y}_1, \dots, P_mT : \overline{\operatorname{span}}\{Z_k\}_{k\in I} \longrightarrow \overline{Y}_m$ is block-diagonal. In fact we may assume that there are finite sets of integers $\{B_k\}_{k\in I}$ such that

$$\begin{cases} \max B_k < \min B_l & \text{if } k < l, \\ \operatorname{supp} P_sTz_k \subset B_k & \forall z_k \in Z_k, \ \forall k \in I, \end{cases}$$

for every $s = 1, \dots, m$ (we consider $\operatorname{supp} P_sTz_k$ with respect to the 2-dimensional decomposition of $\overline{Y}_s$). After some further passing to a subsequence of

I we may also assume that we also have

$$k < \max B_k < l \qquad \text{if } k < l. \tag{23}$$

We will now use Proposition 2.4 for P_mT. Thus we can find $I_0 \subset I$ and real scalars $\{\beta_k\}_{k\in I_0}$ such that

$$1/2\theta^{N^m} \leq \Big\| \sum_{k\in I_0} \beta_k t_k \Big\|_{X_\theta} \leq 2N_m \theta^{N^m} \tag{24}$$

for all $\theta \in \{\theta_j \,:\, 5 \leq j \leq 5m+5\}$ and also, if we let $x = \sum_{k\in I_0} \beta_k x_k \in Y_m$, in the view of (16),

$$\|P_mTx\| \leq 3^m N_m^4 \alpha_m^{N_m} \theta_{5m+1}^{N_m}. \tag{25}$$

Now there exists $q \in \{1,\dots,m-1\}$ such that

$$\begin{aligned}
\|P_qTx\| &\geq \frac{1}{m-1}\|(P_1+\cdots+P_{m-1})Tx\| \geq \frac{1}{m-1}(\|Tx\| - \|P_mTx\|)\\
&\geq \frac{1}{m-1}\left(\frac{1}{4a}\|x\|_{Y_m} - 3^m N_m^4 \alpha_m^{N_m}\theta_{5m+1}^{N_m}\right) \quad \text{(by (19), (25))}\\
&\geq \frac{1}{m-1}\left(\frac{1}{8a} - 3^m N_m^4 \alpha_m^{N_m}\right)\theta_{5m+1}^{N_m} \quad \text{(by (16), (1), (24))}.
\end{aligned}$$

Assuming that $1/(16a) - 3^m N_m^4 \alpha_m^{N_m} \geq 0$ we get a contradiction. Indeed,

$$\begin{aligned}
\|P_qTx\| &\geq \frac{1}{m-1}\,\frac{1}{16a}\theta_{5m+1}^{N_m}\\
&\geq \frac{1}{m-1}3^m N_m^4 \alpha_m^{N_m}\theta_{5m+1}^{N_m}\\
&\geq 400\cdot 3^m N_m \theta_{5m}^{N_m} \quad \text{(by (13), (14))},
\end{aligned}$$

while, on the other hand, in $\overline{Y}_q \subset \overline{S}^q_{\theta_{5q+1}} \oplus_2 \cdots \oplus_2 \overline{S}^q_{\theta_{5q+5}}$ we can write by (16)

$$\begin{aligned}
\|P_qTx\| &= \Big\|\sum_{k\in I_0}\beta_k P_qTx_k\Big\| \\
&\le \Big\|\sum_{k\in I_0}\beta_k Q_{1,q}P_qTx_k\Big\| + \cdots + \Big\|\sum_{k\in I_0}\beta_k Q_{5,q}P_qTx_k\Big\| \\
&\le 3^q\left(\Big\|\sum_{k\in I_0}\beta_k Q_{1,q}P_qTx_k\Big\|_{X_{\theta_{5q+1}}} + \cdots + \Big\|\sum_{k\in I_0}\beta_k Q_{5,q}P_qTx_k\Big\|_{X_{\theta_{5q+5}}}\right) \\
&\le 4\cdot 3^q\left(\Big\|\sum_{k\in I_0}\beta_k \frac{Q_{1,q}P_qTx_k}{\|Q_{1,q}P_qTx_k\|}\Big\|_{X_{\theta_{5q+1}}} + \cdots +\right. \\
&\qquad\qquad \left. + \Big\|\sum_{k\in I_0}\beta_k \frac{Q_{5,q}P_qTx_k}{\|Q_{5,q}P_qTx_k\|}\Big\|_{X_{\theta_{5q+5}}}\right) \\
&\le 4\cdot 3^q\cdot 18^{1/2}\left(\Big\|\sum_{k\in I_0}\beta_k t_{\max B_k}\Big\|_{X_{\theta_{5q+1}}} + \cdots + \Big\|\sum_{k\in I_0}\beta_k t_{\max B_k}\Big\|_{X_{\theta_{5q+5}}}\right) \\
&\le 20\cdot 3^q\cdot 3^{1/2}\left(\Big\|\sum_{k\in I_0}\beta_k t_k\Big\|_{X_{\theta_{5q+1}}} + \cdots + \Big\|\sum_{k\in I_0}\beta_k t_k\Big\|_{X_{\theta_{5q+5}}}\right) \text{ (Prop 1.1(v))} \\
&\le 400\cdot 3^q N_m \theta_{5q+5}^{N_m} \quad \text{(by (24))}.
\end{aligned}$$

Notice that in the third line of inequalities we have used

$$\|Q_{t,q}P_qTx_k\|_{X_{\theta_{5q+t}}} \le \|Q_{t,q}P_qTx_k\|_{S^q_{\theta_{5q+t}}} \le \|Tx_k\| \le \|x_k\|_{Y_m} \le 4,$$

for each $t = 1, \ldots, 5$.

Since we have obtained a contradiction, we must have

$$a \ge 1/(16\cdot 3^m)(1/N_m)^4(1/\alpha_m)^{N_m} \ge m \quad \text{(by (18))}. \qquad \square$$

5. Additional proofs

Proof of Proposition 2.4. We will use the notations from Section 2.

Let $\{B_k\}_{k\in I}$ be finite sets of integers such that

$$\begin{cases} \max B_k < \min B_l & \text{if } k < l, \\ \operatorname{supp} Tz_k \subset B_k & \forall z_k \in Z_k,\ \forall k \in I. \end{cases}$$

After passing to a subsequence of I if necessary, we may assume that $\{B_k\}_{k\in I}$ also satisfy

$$k < \max B_k < l \qquad \text{for all } k, l \in I \text{ with } k < l \tag{26}$$

and, by Proposition 2.1 and Remark 2.2,

$$\|Tx_k\| \le N^2\alpha^N, \quad \text{for all } k \in I. \tag{27}$$

Using Proposition 1.1(vii) and a standard gliding hump procedure we can find subsets $I_1 < I_2 < \cdots$ of I and sequences $\{\beta_1(k)\}_{k\in I_1}, \{\beta_2(k)\}_{k\in I_2}, \ldots$ of real numbers such that, for every $m \geq 1$,

$$1/2\ \theta^N \leq \Big\|\sum_{k\in I_m} \beta_m(k)t_k\Big\|_{X_\theta} \leq 2N\theta^N \qquad \forall\theta \in \Theta \cup \{\theta_1, \ldots, \theta_5\}. \tag{28}$$

Let $\tilde{x}_m = \sum_{k\in I_m} \beta_m(k)x_k$, for $m \geq 1$. We will show that we can find m_0 so that $\|T\tilde{x}_{m_0}\| \leq N^4\alpha^N\theta_1^N$, thus proving Proposition 2.4. For each $m \geq 1$

$$\begin{aligned}
\|Q_1T\tilde{x}_m\| &= \Big\|\sum_{k\in I_m} \beta_m(k)Q_1Tx_k\Big\| \\
&\leq N^2\alpha^N \Big\|\sum_{k\in I_m} \beta_m(k)\frac{Q_1Tx_k}{\|Q_1Tx_k\|}\Big\| \quad \text{(by (27))} \\
&\leq 18^{1/2}N^2\alpha^N \Big\|\sum_{k\in I_m} \beta_m(k)t_{\max B_k}\Big\|_{X_{\theta_1}} \quad \text{(by Prop. 1.1(vi))} \\
&\leq 54^{1/2}N^2\alpha^N \Big\|\sum_{k\in I_m} \beta_m(k)t_k\Big\|_{X_{\theta_1}} \quad \text{(by Prop. 1.1(v) and (26))} \\
&\leq 16N^3\alpha^N\theta_1^N \quad \text{(by (28))}.
\end{aligned}$$

We will show now that there is an infinite set $M \subset \{1, 2, \ldots\}$ such that

$$\|Q_2T\tilde{x}_m\| \leq 80N^2\alpha^N\theta_1^N, \qquad \forall m \in M. \tag{29}$$

Indeed, let M_1 be the set of all m such that $\|Q_2T\tilde{x}_m\| > 80N^2\alpha^N\theta_1^N$. The conclusion will follow once we prove that M_1 is finite. If M_1 is infinite, pick real scalars $\{a_m\}_{m\in M_1}$ such that

$$\begin{cases} \theta_1^N \leq \Big\|\sum_{m\in M_1} a_m t_{\max I_m}\Big\|_{X_{\theta_1}} \leq N\theta_1^N, \\ \vdots \\ \theta_5^N \leq \Big\|\sum_{m\in M_1} a_m t_{\max I_m}\Big\|_{X_{\theta_5}} \leq N\theta_5^N. \end{cases} \tag{30}$$

Let $x = \sum_{m\in M_1} a_m\tilde{x}_m$. Taking into account (1) we get

$$\begin{aligned}
\|x\| \leq{}& \Big\|\sum_{m\in M_1} a_m \sum_{k\in I_m} \beta_m(k)t_k\Big\|_{X_{\theta_1}} + \gamma_1\Big\|\sum_{m\in M_1} a_m \sum_{k\in I_m} \beta_m(k)t_k\Big\|_{X_{\theta_3}} + \\
&+ \gamma_2\Big\|\sum_{m\in M_1} a_m \sum_{k\in I_m} \beta_m(k)t_k\Big\|_{X_{\theta_4}} + \gamma_3\Big\|\sum_{m\in M_1} a_m \sum_{k\in I_m} \beta_m(k)t_k\Big\|_{X_{\theta_5}}.
\end{aligned}$$

By unconditionality, Proposition 1.1(vi), (28) and (30) we get

$$\begin{aligned}\Big\|\sum_{m\in M_1} a_m \sum_{k\in I_m} \beta_m(k) t_k\Big\|_{X_{\theta_1}} &\le 2N\theta_1^N\, 18^{1/2} \Big\|\sum_{m\in M_1} a_m t_{\max I_m}\Big\|_{X_{\theta_1}}\\ &\le 10N^2\theta_1^{2N}.\end{aligned}$$

We can argue similarly in $X_{\theta_3},\dots,X_{\theta_5}$ and hence obtain

$$\begin{aligned}\|x\| &\le 10N^2\theta_1^{2N} + \gamma_1\cdot 10N^2\theta_3^{2N} + \gamma_2\cdot 10N^2\theta_4^{2N} + \gamma_3\cdot 10N^2\theta_5^{2N}\\ &= 10N^2\theta_1^{2N}(1+\gamma_1\alpha^{-4N}+\gamma_2\alpha^{-6N}+\gamma_3\alpha^{-8N}) \le 40N^2\theta_1^{2N}.\end{aligned}$$

On the other hand,

$$\begin{aligned}\|Tx\| \ge \|Q_2Tx\| &= \Big\|\sum_{m\in M_1} a_m Q_2 T\tilde{x}_m\Big\|\\ &> 80N^2\alpha^N\theta_1^N\cdot 3^{-1/2}\Big\|\sum_{m\in M_1} a_m t_{\max\operatorname{supp} T\tilde{x}_m}\Big\|_{X_{\theta_2}} \quad \text{(Prop. 1.1 (vi))}\\ &\ge 40N^2\alpha^N\theta_1^N\Big\|\sum_{m\in M_1} a_m t_{\max I_m}\Big\|_{X_{\theta_2}} \quad \text{(by Prop. 1.1(iii) and (26))}\\ &\ge 40N^2\theta_1^N\alpha^N\theta_2^N = 40N^2\theta_1^{2N},\end{aligned}$$

which is a contradiction with $\|T\|\le 1$. Hence (29) must hold.

Similarly we can find an infinite set $\tilde{M}\subset M$ such that $80N^2\alpha^N\theta_1^N$ is an upper bound for each of $\|Q_3T\tilde{x}_m\|, \|Q_4T\tilde{x}_m\|$ and $\|Q_5T\tilde{x}_m\|$, for all $m\in\tilde{M}$. This ends the proof of Proposition 2.4. □

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