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ORTHOGONALITY PRESERVING TRANSFORMATIONS ON THE SET OF n-DIMENSIONAL SUBSPACES OF A HILBERT SPACE

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ABSTRACT. We characterize bijective transformations on the set of all n-dimensional subspaces of a Hilbert space that preserve orthogonality in both directions. This extends Uhlhorn's improvement of Wigner's classical theorem on symmetry transformations.

1. Introduction and statement of the results

Throughout this paper, H will be an infinite-dimensional (real or complex) Hilbert space. We denote by $B(H)$ the algebra of all bounded linear operators on H. By a projection we mean a self-adjoint idempotent in $B(H)$. For any $n \in \mathbb{N}$, $P_n(H)$ denotes the set of all rank-n projections on H, and $P_\infty(H)$ stands for the set of all infinite rank projections.

Wigner's unitary-antiunitary theorem plays a fundamental role in quantum mechanics. It states that every quantum mechanical invariance transformation can be represented by a unitary or antiunitary operator on a complex Hilbert space. Reformulated in mathematical language, it states that every bijective transformation ϕ on the set of all one-dimensional linear subspaces of a complex Hilbert space H preserving the angle between every pair of such subspaces (which is the transition probability in the language of quantum mechanics) is induced by a unitary or antiunitary operator. Uhlhorn [3] improved this result by requiring only that ϕ preserves orthogonality between the one-dimensional subspaces of H. This is a significant generalization, since Uhlhorn's transformation preserves only the logical structure of the quantum mechanical system in question, while Wigner's transformation preserves its complete probabilistic structure.

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Recently, Molnár [2] improved Wigner's theorem in a different way. He characterized bijective transformations on the set of all n-dimensional subspaces of a Hilbert space preserving principal angles between such subspaces. In fact, he also obtained the general form of nonbijective transformations preserving principal angles, but we will consider the nonbijective case later. The concept of principal angles is a natural generalization of the usual notion of angles between one-dimensional subspaces.

The aim of this paper is to extend and unify the above results by characterizing bijective transformations on the set of n-dimensional subspaces of a Hilbert space preserving orthogonality. Of course, every n-dimensional subspace of H can be identified with the projection onto this subspace. If P and Q are two projections with the range spaces $\mathcal{R}(P)$ and $\mathcal{R}(Q)$, respectively, then the spaces $\mathcal{R}(P)$ and $\mathcal{R}(Q)$ are orthogonal if and only if $PQ = 0$, which is equivalent to $QP = 0$.

THEOREM 1.1. *Let $n \in \mathbb{N}$ and let H be a real or complex infinite-dimensional Hilbert space. Suppose that $\phi : P_n(H) \to P_n(H)$ is a bijective transformation such that*

$$PQ = 0 \iff \phi(P)\phi(Q) = 0, \quad P, Q \in P_n(H).$$

Then there exists a unitary or antiunitary operator U on H such that

$$\phi(P) = UPU^*, \quad P \in P_n(H).$$

Molnár [2] characterized maps $\phi : P_n(H) \to P_n(H)$ satisfying the much stronger assumption that $\angle(P, Q) = \angle(\phi(P), \phi(Q))$ for every pair $P, Q \in P_n(H)$. Here, $\angle(P, Q)$ denotes the system of all principal angles between $\mathcal{R}(P)$ and $\mathcal{R}(Q)$. However, he did not need the bijectivity assumption for his characterization. Having his result in mind, it would be tempting to conjecture that our result also holds without the bijectivity assumption, with the obvious difference that then the operator U in the conclusion does not need to be unitary or antiunitary, but just a (not necessarily surjective) linear or conjugate-linear isometry on H. This conjecture turns out to be wrong. To see this, take any nonsurjective linear isometry $U : H \to H$ and choose and fix some $R \in P_n(H)$. Then U^* is surjective and because the null space of U^* is the orthogonal complement of $\mathcal{R}(U)$, we can find a subspace $W \subset H$ of dimension n such that $W \not\subset \mathcal{R}(U)$ and $U^*W = \mathcal{R}(R)$. Let S be the projection onto W. Define $\phi : P_n(H) \to P_n(H)$ by $\phi(P) = UPU^*$ whenever $P \neq R$, and $\phi(R) = S$. Then ϕ is not of the form $P \mapsto VPV^*$ with V being a linear or conjugate-linear isometry. Indeed, if ϕ were of this form, then $VPV^* = UPU^*$ whenever $P \neq R$ would imply that $\mathcal{R}(V) = \mathcal{R}(U)$, which in turn would yield that $W = \mathcal{R}(S) = \mathcal{R}(VRV^*) \subset \mathcal{R}(U)$, a contradiction. But ϕ preserves orthogonality in both directions. Indeed, it is clear that $PQ = 0 \iff \phi(P)\phi(Q) = 0$ whenever $P \neq R$ and $Q \neq R$. If $P = R$

and $Q \neq R$, then $QR = 0$ if and only if $QU^*W = \{0\}$, which is equivalent to $\phi(Q)W = \{0\}$, and this is true if and only if $\phi(Q)\phi(R) = 0$. So, the bijectivity assumption is indispensable in our theorem.

For $P, Q \in P_n(H)$ we have $\angle(P,Q) = \angle(\phi(P), \phi(Q))$ if and only if the positive operators QPQ and $\phi(Q)\phi(P)\phi(Q)$ are unitarily equivalent. This observation enabled Molnár to also extend Wigner's theorem to the case of infinite rank projections [2]. We thus have the natural question whether every bijective transformation $\phi : P_\infty(H) \to P_\infty(H)$ satisfying $PQ = 0 \iff \phi(P)\phi(Q) = 0$, $P, Q \in P_\infty(H)$, is of the form $\phi(P) = UPU^*$, $P \in P_\infty(H)$, for some unitary or antiunitary operator U on H. It is easy to see that the answer to this question is negative. We write $P_\infty(H)$ as a disjoint union $P_\infty(H) = P^\infty(H) \cup \mathcal{F}$, where $P^\infty(H)$ denotes the set of all projections on H with infinite-dimensional range space and infinite-dimensional null space, while $\mathcal{F} \subset P_\infty(H)$ denotes the subset of all projections whose range spaces are of finite codimension in H. If $\varphi : \mathcal{F} \to \mathcal{F}$ is any bijective map, then the map $\phi : P_\infty(H) \to P_\infty(H)$ defined by

$$\phi(P) = \begin{cases} \mathrm{P} & \text{if } \ P \in P^\infty(H), \\ \varphi(P) & \text{if } \ P \in \mathcal{F}, \end{cases}$$

is bijective and preserves orthogonality in both directions. So, nothing can be said about the behaviour of bijective orthogonality preserving transformations on the subset $\mathcal{F}$. But if we consider such maps on $P^\infty(H)$, then we get the expected result.

Theorem 1.2. *Let H be a real or complex infinite-dimensional Hilbert space. Suppose that $\phi : P^\infty(H) \to P^\infty(H)$ is a bijective transformation such that*

$$PQ = 0 \iff \phi(P)\phi(Q) = 0, \quad P, Q \in P^\infty(H).$$

Then there exists a unitary or antiunitary operator U on H such that

$$\phi(P) = UPU^*, \quad P \in P^\infty(H).$$

The bijectivity assumption is indispensable in this result. To see this we use the same idea as in the case of Theorem 1.1.

2. Proofs

Before proving Theorem 1.1, we introduce some more notation. Let H be a real or complex infinite-dimensional Hilbert space and P, Q two projections on H. We write $P \perp Q$ if the range spaces $\mathcal{R}(P)$ and $\mathcal{R}(Q)$ are orthogonal. We further write $P \leq Q$ if $\mathcal{R}(P) \subset \mathcal{R}(Q)$, or equivalently, $PQ = QP = P$. For a positive integer n we denote by $P_{\geq n}(H)$ the set of all finite rank projections on H whose rank is at least n, i.e., $P_{\geq n}(H) = \bigcup_{k \geq n} P_k(H)$.

Proof of Theorem 1.1. Let $\mathcal{S}$ be any subset of $P_n(H)$. We denote by $\mathcal{S}^\perp$ the set of all projections $P \in P_n(H)$ satisfying $P \perp Q$ for every $Q \in \mathcal{S}$. For $P, Q \in P_n(H)$ we write $P \sim Q$ if $P \neq Q$ and for every $R \in P_n(H) \setminus \{P,Q\}^\perp$ we have $\#(\{R\} \cup \{P,Q\}^\perp)^\perp \leq 1$. Here, the symbol $\#$ stands for the cardinality of the set. Since $\phi : P_n(H) \to P_n(H)$ is a bijective map preserving orthogonality in both directions, we have $\phi(\mathcal{S}^\perp) = \phi(\mathcal{S})^\perp$ for every subset $\mathcal{S} \subset P_n(H)$, and consequently,

$$P \sim Q \iff \phi(P) \sim \phi(Q), \quad P, Q \in P_n(H).$$

Next, we will prove that for $P, Q \in P_n(H)$ with $P \neq Q$ we have $P \sim Q$ if and only if there exists $T \in P_{n+1}(H)$ such that $P \leq T$ and $Q \leq T$. Assume first that $P, Q \in P_n(H)$ satisfy $P \neq Q$ and that there exists $T \in P_{n+1}(H)$ such that $P \leq T$ and $Q \leq T$. Because $P \neq Q$ we have $\dim(\mathcal{R}(P) + \mathcal{R}(Q)) \geq n+1$. Hence, $\mathcal{R}(T) = \mathcal{R}(P) + \mathcal{R}(Q)$, and thus, $\{P,Q\}^\perp = \{S \in P_n(H) \,:\, S \leq I - T\}$. Choose any $R \in P_n(H) \setminus \{P,Q\}^\perp$. Then $\mathcal{R}(R) \not\subset \mathcal{R}(I-T)$. If $S \in (\{R\} \cup \{P,Q\}^\perp)^\perp$, then $\mathcal{R}(S) \perp (\mathcal{R}(R) + \mathcal{R}(I-T))$. But the codimension of $\mathcal{R}(R) + \mathcal{R}(I-T)$ is at most n. In the case when $\operatorname{codim}(\mathcal{R}(R) + \mathcal{R}(I-T)) = n$ there exists exactly one $S \in P_n(H)$ whose range is orthogonal to $\mathcal{R}(R) + \mathcal{R}(I-T)$, while there is no such projection $S \in P_n(H)$ when $\operatorname{codim}(\mathcal{R}(R) + \mathcal{R}(I-T)) < n$. It follows that $P \sim Q$. To prove the converse, assume that for a given pair $P, Q \in P_n(H)$ there is no $T \in P_{n+1}(H)$ with $P \leq T$ and $Q \leq T$. Denote by S the projection onto $\mathcal{R}(P) + \mathcal{R}(Q)$. We have $S \in P_{\geq n+2}(H)$. Therefore, we can find $R \in P_n(H) \setminus \{P,Q\}^\perp$ such that $\operatorname{codim}(\mathcal{R}(R) + \mathcal{R}(I-S)) \geq n+1$. Thus, there are infinitely many projections $T \in P_n(H)$ satisfying $\mathcal{R}(T) \perp (\mathcal{R}(R) + \mathcal{R}(I-S))$, or equivalently, $T \in (\{R\} \cup \{P,Q\}^\perp)^\perp$. Consequently, $P \not\sim Q$, as desired.

We will define now a new map $\varphi : P_{n+1}(H) \to P_{n+1}(H)$. For every $T \in P_{n+1}(H)$ we can find $P, Q \in P_n(H)$ with $P \neq Q$ and $P \leq T$ and $Q \leq T$. So, $P \sim Q$ and therefore $\phi(P) \sim \phi(Q)$. Thus, by the above observation there exists $R \in P_{n+1}(H)$ such that $\phi(P) \leq R$ and $\phi(Q) \leq R$. Obviously, such an R is uniquely determined. We define $\varphi(T) = R$. To see that φ is well-defined we choose $P_1, Q_1 \in P_n(H)$ with $P_1 \neq Q_1$ and $P_1 \leq T$ and $Q_1 \leq T$. As before there is a unique $R_1 \in P_{n+1}(H)$ with $\phi(P_1) \leq R_1$ and $\phi(Q_1) \leq R_1$. We have to show that $R = R_1$. We have $\{S \in P_n(H) \,:\, S \leq I - R\} = \{\phi(P), \phi(Q)\}^\perp = \phi(\{P,Q\}^\perp) = \phi(\{P_1, Q_1\}^\perp) = \{S \in P_n(H) \,:\, S \leq I - R_1\}$, and consequently, $R = R_1$, as desired. Clearly, φ is bijective, and for $P \in P_n(H)$ and $Q \in P_{n+1}(H)$ we have $P \leq Q$ if and only if $\phi(P) \leq \varphi(Q)$. It follows easily that for $P, Q \in P_{n+1}(H)$ we have $P \perp Q$ if and only if $\varphi(P) \perp \varphi(Q)$.

We extend $\phi : P_n(H) \to P_n(H)$ to a map from $P_n(H) \cup P_{n+1}(H)$ onto $P_n(H) \cup P_{n+1}(H)$ by defining $\phi(P) = \varphi(P)$ whenever $P \in P_{n+1}(H)$. Then, in the same way as we extended ϕ from $P_n(H)$ to $P_n(H) \cup P_{n+1}(H)$, we extend ϕ further to a bijective map from the set $P_n(H) \cup P_{n+1}(H) \cup P_{n+2}(H)$ onto itself. Proceeding inductively, we extend ϕ to a bijective map $\phi : P_{\geq n}(H) \to P_{\geq n}(H)$

satisfying

$$P \perp Q \iff \phi(P) \perp \phi(Q), \quad P, Q \in P_{\geq n}(H),$$

and

$$P \leq Q \iff \phi(P) \leq \phi(Q), \quad P, Q \in P_{\geq n}(H).$$

In the next step we will define yet another map $\psi : P_1(H) \to P_1(H)$ with the following properties:

- ψ is bijective,
- $P \perp Q$ if and only if $\psi(P) \perp \psi(Q)$ for every pair $P, Q \in P_1(H)$, and
- $P \leq Q$ if and only if $\psi(P) \leq \phi(Q)$ for every pair $P \in P_1(H)$, $Q \in P_n(H)$.

Let $P \in P_1(H)$. Then we can find $Q \in P_{n+1}(H)$ and $R \in P_n(H)$ such that $P = Q - R$. Then, of course, $P \perp R$. Set $\psi(P) = \phi(Q) - \phi(R)$.

We first have to show that ψ is well-defined. In order to do this suppose that we have $P = Q_1 - R_1$ for another pair of projections $Q_1 \in P_{n+1}(H)$ and $R_1 \in P_n(H)$. The first possibility we will treat is that $R_1 \perp R$. Set $T = P + R_1 + R$. This is the orthogonal sum. In particular, $Q \perp R_1$ and $Q_1 \perp R$. Hence, $\phi(R) \leq \phi(Q) \leq \phi(T)$, $\phi(R_1) \leq \phi(Q_1) \leq \phi(T)$, $\phi(Q) \perp \phi(R_1)$, $\phi(Q_1) \perp \phi(R)$, and $\phi(R), \phi(R_1) \in P_n(H)$, $\phi(Q), \phi(Q_1) \in P_{n+1}(H)$, and $\phi(T) \in P_{2n+1}(H)$. It follows easily that $\phi(Q) - \phi(R) = \phi(Q_1) - \phi(R_1) = \phi(T) - (\phi(R) + \phi(R_1))$. In the case when $R_1 \not\perp R$ we can find $R_2 \in P_n(H)$ and $Q_2 \in P_{n+1}(H)$ such that $P = Q_2 - R_2$ and $R_2 \perp R$ and $R_2 \perp R_1$. Then, by the previous step, we have $\phi(Q) - \phi(R) = \phi(Q_2) - \phi(R_2)$ as well as $\phi(Q_1) - \phi(R_1) = \phi(Q_2) - \phi(R_2)$. So, we have $\phi(Q) - \phi(R) = \phi(Q_1) - \phi(R_1)$ in this case as well.

Hence, ψ is well-defined and it is easy to see that it has all of the above-mentioned properties. One can now complete the proof using Uhlhorn's theorem. However, for the sake of completeness we present here a short argument leading to the desired conclusion.

We denote by $\mathbb{P}H$ the projective space over H, $\mathbb{P}H = \{[x] : x \in H \setminus \{0\}\}$. Here, $[x]$ denotes the one-dimensional subspace of H spanned by x. Now, ψ induces in a natural way a bijective map ξ from $\mathbb{P}H$ onto itself. If for $x \in H \setminus \{0\}$ we denote by P_x the projection onto the linear span of x and if $\psi(P_x) = P_y$, then we define $\xi([x]) = [y]$. Clearly, if $[x] \subset [y] + [z]$ for some nonzero $x, y, z \in H$, then every $Q \in P_1(H)$ orthogonal to P_y and P_z is also orthogonal to P_x, and therefore any $R \in P_1(H)$ orthogonal to $P_{y'}$ and $P_{z'}$ must be orthogonal to $P_{x'}$. Here, $x' \in \xi([x])$, $y' \in \xi([y])$, and $z' \in \xi([z])$ are nonzero vectors. Thus, $\xi([x]) \subset \xi([y]) + \xi([z])$. The same is true for the inverse of ξ, and consequently, by the fundamental theorem of projective geometry, there exists a bijective semilinear map $U : H \to H$ such that $\xi([x]) = [Ux]$, $x \in H \setminus \{0\}$. Here we have to distinguish between the real and the complex case. The real case is much easier, since in this case every semilinear map is linear. Therefore, we will only consider the complex case. Let $f : \mathbb{C} \to \mathbb{C}$ be the automorphism of the field of complex numbers corresponding to U.

Choose any $x, y \in H$ and $\lambda \in \mathbb{C}$ with $x \perp y$ and $\|x\| = \|y\| = 1$. Then $[\lambda x + y] \perp [x - \overline{\lambda} y]$ and hence, $[f(\lambda)Ux + Uy] \perp [Ux - f(\overline{\lambda})Uy]$, $\lambda \in \mathbb{C}$. We use $[Ux] \perp [Uy]$ and $f(1) = 1$ to conclude first that $\|Ux\| = \|Uy\|$ and then that $f(\lambda) = \overline{f(\overline{\lambda})}$, $\lambda \in \mathbb{C}$. Thus, $f(\mathbb{R}) \subset \mathbb{R}$. The only nonzero endomorphism of the real field is the identity and therefore $f : \mathbb{C} \to \mathbb{C}$ is either the identity, or the complex conjugation. So, U is either linear, or conjugate-linear. It maps the orthogonal complement of a nonzero element $x \in H$ onto the orthogonal complement of Ux in H. By [1, Lemma 3], U is a bounded linear or conjugate-linear operator on H. It also preserves orthogonality. Thus, $\langle U^*Ux, y\rangle = 0$ whenever $\langle x, y\rangle = 0$, $x, y \in H$. It follows that $U^*U = cI$ for some positive real number c. Absorbing the constant we may assume that U is a unitary or antiunitary operator. Thus we have $\psi(P) = UPU^*$, $P \in P_1(H)$. The third property of the map ψ given above yields now easily that $\phi(P) = UPU^*$, $P \in P_n(H)$. This completes the proof. □

Remark. We have assumed that $\dim H = \infty$. We obviously get the same conclusion with the same proof when $\dim H$ is finite, but large enough.

Proof of Theorem 1.2. We first observe that ϕ preserves the order in both directions, that is, $P \le Q$ if and only if $\phi(P) \le \phi(Q)$ for every pair of projections $P, Q \in P^\infty(H)$. Indeed, we have $P \le Q$ if and only if $RQ = 0$ implies $RP = 0$ for every $R \in P^\infty(H)$. Moreover, if $P \le Q$ and if the codimension of $\mathcal{R}(P)$ in $\mathcal{R}(Q)$ is one, then the same is true for $\phi(P)$ and $\phi(Q)$. Namely, for projections $P, Q \in P^\infty(H)$ satisfying $P \le Q$ the codimension of $\mathcal{R}(P)$ in $\mathcal{R}(Q)$ is one if and only if $P \ne Q$ and every $R \in P^\infty(H)$ satisfying $P \le R \le Q$ is either equal to P, or equal to Q. As in the proof of the previous theorem we define a new map $\psi : P_1(H) \to P_1(H)$ in the following way. For $P \in P_1(H)$ choose $Q, R \in P^\infty(H)$ with $P = Q - R$. This, of course, is equivalent to the fact that $\mathcal{R}(Q)$ is the orthogonal direct sum of $\mathcal{R}(P)$ and $\mathcal{R}(R)$. Then we define $\psi(P) = \phi(Q) - \phi(R)$.

Assume for a moment that we have already proved that ψ is well-defined. Then, obviously, $P \le Q$, $P \in P_1(H)$, $Q \in P^\infty(H)$, yields that $\psi(P) \le \phi(Q)$. It follows easily that $P \perp Q$ implies $\psi(P) \perp \psi(Q)$, $P, Q \in P_1(H)$. It is also not difficult to see that ψ is a bijective map whose inverse also preserves orthogonality. One can then complete the proof using a similar approach as in the previous theorem.

So, it remains to prove that ψ is well-defined. First note that if $P, Q \in P^\infty(H)$ with $P \perp Q$ and $P + Q \in P^\infty(H)$, then we have $\phi(P + Q) = \phi(P) + \phi(Q)$. This is true because $P, Q \le P + Q$ and $\{P, Q\}^\perp = \{P + Q\}^\perp$. Here, for $\mathcal{S} \subset P^\infty(H)$, $\mathcal{S}^\perp$ denotes the set of all projections from $P^\infty(H)$ that are orthogonal to every member of $\mathcal{S}$. Now, if $Q - R = Q_1 - R_1$ with $R \le Q$, $R_1 \le Q_1$, the codimension of $\mathcal{R}(R)$ in $\mathcal{R}(Q)$ is one, the codimension of $\mathcal{R}(R_1)$ in $\mathcal{R}(Q_1)$ is one, $R \perp R_1$, and $R + R_1 \in P^\infty(H)$, that is, the null space of

$R+R_1$ is infinite-dimensional, then we use the same arguments as in the proof of our first theorem to show that $\phi(Q)-\phi(R)=\phi(Q_1)-\phi(R_1)$.

Take now $Q, Q_1, R, R_1 \in P^\infty(H)$ with $Q-R=Q_1-R_1 \in P_1(H)$ and $R_1 \leq R$ with $\mathcal{R}(R_1)$ having infinite codimension in $\mathcal{R}(R)$. Then the set $\{T \in P^\infty(H) : T \leq Q \ \text{ and } \ T \perp Q_1\} = \{T \in P^\infty(H) : T \leq R \ \text{ and } \ T \perp R_1\}$ is mapped onto the set $\{S \in P^\infty(H) : S \leq \phi(Q) \ \text{ and } \ S \perp \phi(Q_1)\}$ and this set is equal to $\{S \in P^\infty(H) : S \leq \phi(R) \ \text{ and } \ S \perp \phi(R_1)\}$. It follows easily that $\phi(Q)-\phi(R)=\phi(Q_1)-\phi(R_1)$ also in this case.

Finally, let $Q, Q_1, R, R_1 \in P^\infty(H)$ be any projections with $P=Q-R=Q_1-R_1 \in P_1(H)$. We choose inductively a sequence of orthonormal vectors $\{e_j, f_j, g_j, h_j : j=1,2,\ldots\}$ with $\{e_j, f_j : j=1,2,\ldots\} \subset \mathcal{R}(R)$ and $\{g_j, h_j : j=1,2,\ldots\} \subset \mathcal{R}(R_1)$. We denote by R_2 and R_3 the orthogonal projections on the closed linear spans of $\{e_j : j=1,2,\ldots\}$ and $\{g_j : j=1,2,\ldots\}$, respectively. The range space of the projection R_2+R_3 has infinite codimension because the orthonormal set $\{f_j : j=1,2,\ldots\}$ is contained in the null space of R_2+R_3. Set $Q_j=R_j+P$, $j=2,3$. By the previous two steps we have $\phi(Q)-\phi(R)=\phi(Q_2)-\phi(R_2)$, $\phi(Q_1)-\phi(R_1)=\phi(Q_3)-\phi(R_3)$, and $\phi(Q_2)-\phi(R_2)=\phi(Q_3)-\phi(R_3)$. Hence, $\phi(Q)-\phi(R)=\phi(Q_1)-\phi(R_1)$, as desired. This completes the proof. $\square$

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