

## FRAME WAVELETS WITH FRAME SET SUPPORT IN THE FREQUENCY DOMAIN

XINGDE DAI, YUANAN DIAO, AND QING GU

ABSTRACT. In this paper, we study a special class of frame wavelets. The supports of the Fourier transforms of these frame wavelets are the so called frame wavelet sets. Our results provide ways to construct frame wavelets not known previously. In particular, for any given positive number  $\alpha$ , we show that there exist various frame wavelets whose Fourier transforms are supported by a set of measure  $\alpha$ .

### 1. Introduction

Let  $\mathbb{H}$  be a Hilbert space. A set of elements  $\{e_i\}$  is called a *frame* of  $\mathbb{H}$  if there exist two positive constants  $0 < A \leq B$  such that

$$(1.1) \quad A\|f\|^2 \leq \sum_i |\langle f, e_i \rangle|^2 \leq B\|f\|^2$$

for all  $f \in \mathbb{H}$ . The supremum of all such numbers  $A$  and the infimum of all such numbers  $B$  are called the *frame bounds* of the frame and are denoted by  $A_0$  and  $B_0$ , respectively. A frame  $\{e_i\}$  is called a *tight* frame when  $A_0 = B_0$ , and a *normalized tight* frame when  $A_0 = B_0 = 1$ . Frames can be regarded as generalizations of orthogonal bases of Hilbert spaces. Although the concept of frames was introduced a long time ago ([12], [22]), it is only in recent years that they have been studied extensively. For example, among those frames that have been widely studied lately are the Weyl-Heisenberg frames (or Gabor frames). See [2], [3], [4], [5], [7], [8], [13] and [21] for some of the work on these subjects. Frames in the space  $L^2(\mathbb{R})$  (or  $L^2(\mathbb{R}^d)$ ) generated by a single function through dilations and translations are of special interest due to their close relation to wavelets and will be our focus in this paper. Interested readers may also see [1], [6], [16], [17], [18], [19] and [20] for more references on these frames, or on frames in more general settings. The literature on this subject is large, and the references given here are far from complete.

---

Received June 23, 2003; received in final form January 30, 2004.  
2000 *Mathematics Subject Classification.* 42C99.

©2004 University of Illinois

Let  $D$  and  $T$  be the standard dilation and translation operators, respectively, on  $L^2(\mathbb{R})$ , defined by  $(Df)(x) = \sqrt{2}f(2x)$  and  $(T)(x) = f(x - 1)$  for any  $f \in L^2(\mathbb{R})$ . A function  $\psi \in L^2(\mathbb{R})$  is called a *frame wavelet* for  $L^2(\mathbb{R})$  if

$$(1.2) \quad \{\psi_{n,\ell}(x)\} = \{2^{n/2}\psi(2^n x - \ell) : n, \ell \in \mathbb{Z}\} = \{D^n T^\ell \psi : n, \ell \in \mathbb{Z}\}$$

is a frame of  $L^2(\mathbb{R})$ , i.e., if there exist two positive constants  $0 < A \leq B$  such that

$$(1.3) \quad A\|f\|^2 \leq \sum_{n,\ell \in \mathbb{Z}} |\langle f, D^n T^\ell \psi \rangle|^2 \leq B\|f\|^2$$

for all  $f \in L^2(\mathbb{R})$ .  $\psi$  is called a *tight* frame wavelet if this frame is tight. Similarly,  $\psi$  is called a *normalized tight* frame wavelet if this frame is a normalized tight frame.

Let  $E$  be a Lebesgue measurable set of finite measure. Define  $\psi_0 \in L^2(\mathbb{R})$  by  $\widehat{\psi_0} = \frac{1}{\sqrt{2\pi}}\chi_E$ , where  $\widehat{\psi_0}(s) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-i \cdot s \cdot t} \psi_0(t) dt$  is the Fourier transform of  $\psi_0$ . If the function  $\psi_0$  so defined is a frame wavelet for  $L^2(\mathbb{R})$ , then the set  $E$  is called a *frame wavelet set*, or just a frame set for short. The function  $\psi_0$  is called an *s-elementary frame wavelet*. Similarly,  $E$  is called a (normalized) *tight frame wavelet set* if  $\psi_0$  is a (normalized) tight frame wavelet, and a wavelet set if  $\psi_0$  is a wavelet. To simplify the names of these sets, let us call a frame set an *f*-set, a tight frame wavelet set a *t*-set, a normalized tight frame wavelet set an *n*-set and a wavelet set a *w*-set. If we denote the collection of all *f*-sets by  $\mathbf{F}$ , the collection of all *t*-sets by  $\mathbf{T}$ , the collection of all *n*-sets by  $\mathbf{N}$  and the collection of all *w*-sets by  $\mathbf{W}$ , then we have  $\mathbf{W} \subset \mathbf{N} \subset \mathbf{T} \subset \mathbf{F}$ . We should point out here that  $\mathbf{F}$  is a much richer set than  $\mathbf{T}$ . This can be seen from the characterization of *t*-sets (given in Section 2) and the result from Section 3.

A characterization of *n*-sets was obtained in [19]. This also follows from a characterization of normalized tight frame wavelets obtained in [20]. In [9], a complete characterization of *t*-sets was obtained, together with some necessary or sufficient conditions for a set  $E$  to be an *f*-set. Let  $\psi \in L^2(\mathbb{R})$  and let  $E_{\widehat{\psi}} = \text{supp}(\widehat{\psi})$ . It turns out that if  $E_{\widehat{\psi}}$  is a *t*-set, then its Lebesgue measure is at most  $2\pi$ . Furthermore, the result in [20] regarding frame wavelets can be readily applied to determine whether  $\psi$  is a frame wavelet. This implies that we are unlikely to obtain new results if we only concentrate on frame wavelets whose Fourier transforms are supported by *t*-sets. However, as we will show in Section 3, the *f*-sets are quite different and have a much richer structure even though we do not fully understand them at this time. It is the main purpose of this paper to investigate frame wavelets whose Fourier transforms are supported by *f*-sets. This turns out to be quite fruitful. For example, we will show that for certain *f*-sets  $E$ , any  $L^2(\mathbb{R})$  function  $\psi$  with  $E_{\widehat{\psi}} = E$  is a frame wavelet, provided that  $\widehat{\psi}$  is bounded and  $|\widehat{\psi}| \geq a > 0$  for

some constant  $a$  on a particular subset of  $E$ . We also show that in general, if  $E_{\widehat{\psi}}$  is an  $f$ -set,  $\widehat{\psi}$  is bounded,  $|\widehat{\psi}| \geq a > 0$  for some constant  $a$  on  $E$ , and  $\widehat{\psi}$  is the restriction of a 2-dilation periodic function to  $E_{\widehat{\psi}}$ , then  $\psi$  is a frame wavelet. Since for any given positive number  $\alpha$ , there exist frame sets of measure  $\alpha$ , this enables us to construct various frequency domain frame wavelets with support of measure  $\alpha$ . The study of  $f$ -sets appears to play a very important role here. There are many related questions for which we do not have an answer at this point. For example, we know that the condition that  $E_{\widehat{\psi}}$  is an  $f$ -set alone is neither sufficient nor necessary for  $\psi$  to be a frame wavelet. However, we do not know whether there is a necessary and sufficient condition for  $\psi$  to be a frame wavelet given that  $E_{\widehat{\psi}}$  is an  $f$ -set in general. We will discuss some of these questions in the last section.

In the next section, we will recall, for the convenience of the reader, some known results about  $f$ -sets that will be needed in later sections. In Section 3, we will extend the sufficient condition for an  $f$ -set obtained in [9]. This result reveals a much larger class of  $f$ -sets than was previously known, and it gives us the tool to construct various examples, as we will do in the later sections. In Section 4, we prove our main results, which provide new ways for creating frame wavelets. In Section 5, we construct some frame wavelets using the results obtained in Section 4 and compare them with some known results.

### 2. Basic concepts and results on frame wavelet sets

In this section, we first introduce some basic concepts and terms concerning  $f$ -sets (frame sets). We then briefly go over some known results about  $f$ -sets and  $t$ -sets (tight frame sets). For a more detailed treatment of these materials, we refer our reader to [9].

Let  $E$  be a measurable sets. Two elements  $x, y \in E$  are  $\overset{\delta}{\sim}$  equivalent if  $x = 2^n y$  for some integer  $n$ . The  $\delta$ -index of a point  $x$  in  $E$  is the number of elements in its  $\overset{\delta}{\sim}$  equivalent class and is denoted by  $\delta_E(x)$ . Let  $E(\delta, k) = \{x \in E : \delta_E(x) = k\}$ . Then  $E$  is the disjoint union of the sets  $E(\delta, k)$ . Two sets  $E$  and  $F$  with  $E = E(\delta, 1)$  and  $F = F(\delta, 1)$  are said to be 2-dilation equivalent (also denoted by  $E \overset{\delta}{\sim} F$ ) if every point in  $E$  is  $\overset{\delta}{\sim}$  to a point in  $F$  and vice versa.

LEMMA 2.1. *If  $E$  is a Lebesgue measurable set, then each  $E(\delta, k)$  ( $k \geq 1$ ) is also Lebesgue measurable. Furthermore, each  $E(\delta, k)$  is a disjoint union of  $k$  measurable sets  $\{E^{(j)}(\delta, k)\}$ ,  $1 \leq j \leq k$ , such that  $E^{(j)}(\delta, k) \overset{\delta}{\sim} E^{(j')}(\delta, k)$  for any  $1 \leq j, j' \leq k$ .*

If we let  $\Delta(E) = \bigcup_{k \in \mathbb{Z}} E^{(1)}(\delta, k)$ , then every point in it has  $\delta$ -index one (within  $\Delta(E)$  itself). Furthermore, we have  $\bigcup_{k \in \mathbb{Z}} 2^k E = \bigcup_{k \in \mathbb{Z}} 2^k \Delta(E)$ . A set

$E$  is called a *2-dilation generator* of  $\mathbb{R}$  if  $E = E(\delta, 1)$  and  $\bigcup_{k \in \mathbb{Z}} 2^k E = \mathbb{R}$ . A 2-dilation generator for a subset of  $\mathbb{R}$  that is invariant under 2-dilation can be similarly defined.

In the case of translation, we say that  $x, y \in E$  are  $2\pi$ -translation equivalent, denoted by  $x \sim y$ , if  $x = y + 2n\pi$  for some integer  $n$ . The  $\tau$ -index of a point  $x$  in  $E$  is the number of elements in its  $\sim$  equivalent class and is denoted by  $\tau_E(x)$ . Let  $E(\tau, k) = \{x \in E : \tau_E(x) = k\}$ . Then  $E$  is the disjoint union of the sets  $E(\tau, k)$ . Define  $\tau(E) = \bigcup_{n \in \mathbb{Z}} (E \cap ([2n\pi, 2(n+1)\pi) - 2n\pi))$ . This is a disjoint union if and only if  $E = E(\tau, 1)$ . Two sets  $E$  and  $F$  with  $E = E(\tau, 1)$  and  $F = F(\tau, 1)$  are said to be  $2\pi$ -translation equivalent (and denoted by  $E \sim F$ ) if every point in  $E$  is  $\sim$  to a point in  $F$  and vice versa.

LEMMA 2.2. *If  $E$  is a Lebesgue measurable set, then each  $E(\tau, k)$  ( $k \geq 1$ ) is also Lebesgue measurable. Furthermore, each  $E(\tau, k)$  is a disjoint union of  $k$  measurable sets  $\{E^{(j)}(\tau, k)\}$ ,  $1 \leq j \leq k$ , such that  $E^{(j)}(\tau, k) \sim E^{(j')}(\tau, k)$  for any  $1 \leq j, j' \leq k$ .*

REMARK 2.3. The decompositions of  $E(\delta, k)$  (resp.  $E(\tau, k)$ ) into  $E^{(j)}(\delta, k)$  (resp.  $E^{(j)}(\tau, k)$ ) are not unique in general. However, the existence of one of such decomposition is guaranteed by the construction given in [10], and the set  $E(\tau, 1)$  is uniquely determined by  $E$ . Since  $E(\tau, 1)$  consists of all points in  $E$  that are  $2\pi$  translation redundancy free, we also denote it by  $T_{r,f}(E)$ .

DEFINITION 2.4. A set  $E$  is called a *basic set* if there exists a constant  $M > 0$  such that  $\mu(E(\delta, m)) = 0$  and  $\mu(E(\tau, m)) = 0$  for any  $m > M$ .

DEFINITION 2.5. Let  $\mathcal{F}$  be the Fourier transformation and define  $\widehat{D} = \mathcal{F}D\mathcal{F}^{-1}$  and  $\widehat{T} = \mathcal{F}T\mathcal{F}^{-1}$ , where  $D$  and  $T$  are the standard dilation and translation operators defined in the last section. It is left to the reader to verify that  $\widehat{D}(f)(\xi) = D^{-1}(f)(\xi) = (1/\sqrt{2})f(\xi/2)$  and  $\widehat{T}(f)(\xi) = e^{-i\xi}f(\xi)$ .

Now let  $E$  be a Lebesgue measurable set with finite measure. For any  $f \in L^2(\mathbb{R})$ , let  $H_E f$  be the following formal summation:

$$(2.1) \quad (H_E f)(\xi) = \sum_{n, \ell \in \mathbb{Z}} \langle f, \widehat{D}^n \widehat{T}^\ell \frac{1}{\sqrt{2\pi}} \chi_E \rangle \widehat{D}^n \widehat{T}^\ell \frac{1}{\sqrt{2\pi}} \chi_E(\xi).$$

Notice that if  $H_E f$  converges to a function in  $L^2(\mathbb{R})$  under the  $L^2(\mathbb{R})$  norm, then (1.3) (with  $\widehat{\psi} = \frac{1}{\sqrt{2\pi}} \chi_E$ ) is equivalent to

$$(2.2) \quad A\|f\|^2 \leq \langle H_E f, f \rangle \leq B\|f\|^2, \quad \forall f \in L^2(\mathbb{R}).$$

When  $H_E f$  converges to a function in  $L^2(\mathbb{R})$  under the  $L^2(\mathbb{R})$  norm for all  $f \in L^2(\mathbb{R})$  such that  $\|H_E f\| \leq b\|f\|$  for some constant  $b > 0$ ,  $H_E$  defines a bounded linear operator on  $L^2(\mathbb{R})$ . We say that  $E$  is a *Bessel set* in this case.

We now state some known results concerning the basic sets, Bessel sets and  $f$ -sets from [9].

**THEOREM 2.6.** *Let  $E$  be a Lebesgue measurable set with finite measure. Then the following statements are equivalent:*

- (i)  $E$  is a Bessel set.
- (ii)  $E$  is a basic set.
- (iii) There exists a constant  $B > 0$  such that  $\sum_{n,\ell \in \mathbb{Z}} |\langle f, \widehat{D}^n \widehat{T}^\ell \frac{1}{\sqrt{2\pi}} \chi_E \rangle|^2 \leq B \|f\|^2$  for all  $f \in L^2(\mathbb{R})$ .

In fact, when  $E$  is a basic set, more can be said about  $H_E$ , as shown in the following lemma.

**LEMMA 2.7.** *If  $E$  is a basic set such that  $E(\delta, m) = E(\tau, m) = \emptyset$  for all  $m > M$ , where  $M > 0$  is a positive integer, then for any  $f, g \in L^2(\mathbb{R})$  we have*

$$|\langle H_E(f), g \rangle| \leq M^{5/2} \cdot \|f\| \cdot \|g\|.$$

The operator  $H_E$  also has the property given in the following lemma.

**LEMMA 2.8.** *Let  $E$  and  $F$  be two translation disjoint basic sets. Then for any  $f \in L^2(\mathbb{R})$  we have  $H_{E \cup F}(f) = H_E(f) + H_F(f)$ .*

The following theorem gives a sufficient condition for  $E$  to be an  $f$ -set (recall that  $T_{r,f}(E) = E(\tau, 1)$ ).

**THEOREM 2.9.** *Let  $E$  be a Lebesgue measurable set with finite measure. Then  $E$  is an  $f$ -set if  $E$  is a basic set and  $\bigcup_{n \in \mathbb{Z}} 2^n T_{r,f}(E) = \mathbb{R}$ .*

On the other hand, the following theorem gives a simple necessary condition for  $E$  to be an  $f$ -set.

**THEOREM 2.10.** *Let  $E$  be a Lebesgue measurable set with finite measure. If  $E$  is an  $f$ -set, then  $E$  must be a basic set with  $\bigcup_{n \in \mathbb{Z}} 2^n E = \mathbb{R}$ .*

### 3. A sufficient condition for $f$ -sets

The main result in this section is a generalization of Theorem 2.9 from the last section. This result will allow us to construct  $f$ -sets that will be used in later sections. We need some lemmas first. The following lemma is Proposition 1 from [9] in an equivalent but slightly different form.

LEMMA 3.1. *Let  $E$  be a basic set. Then for any  $f \in L^2(\mathbb{R})$   $H_E(f)$  converges to a function in  $L^2(\mathbb{R})$  under the  $L^2(\mathbb{R})$  norm topology. Furthermore,*

$$(3.1) \quad \begin{aligned} H_E^k(f) &= \sum_{j \in \mathbb{Z}} (f \cdot \chi_{2^k E})(\xi + 2^k \cdot 2\pi j) \\ &= \sum_{j \in \mathbb{Z}} f(\xi + 2^k \cdot 2\pi j) \cdot \chi_{2^k E}(\xi + 2^k \cdot 2\pi j) \end{aligned}$$

also converges to a function in  $L^2(\mathbb{R})$  under the  $L^2(\mathbb{R})$  norm topology for each  $k$  and  $\sum_{k \in \mathbb{Z}} H_E^k(f)$  converges to  $H_E(f)$  under the  $L^2(\mathbb{R})$  norm topology.

LEMMA 3.2. *Let  $E$  be a basic set and  $E_1, E_2$  be subsets of  $E$  such that  $E_1 \cup E_2 = E$  and  $\Omega_1 = \bigcup_{k \in \mathbb{Z}} 2^k E_1$  and  $\Omega_2 = \bigcup_{k \in \mathbb{Z}} 2^k E_2$  are disjoint. Then for any functions  $f$  and  $g$  such that  $\text{supp}(f) \subset \Omega_1$  and  $\text{supp}(g) \subset \Omega_2$  we have  $\langle H_E(f), f \rangle = \langle H_{E_1}(f), f \rangle$  and  $\langle H_E(g), g \rangle = \langle H_{E_2}(g), g \rangle$ .*

*Proof.* For each  $\xi \in 2^k E_1$ , if  $\xi + 2^k \cdot 2\pi j \notin 2^k E_1$ , then  $\xi + 2^k \cdot 2\pi j \in 2^k E_2$  or  $\xi + 2^k \cdot 2\pi j \notin 2^k E$ . We have  $(f \cdot \chi_{2^k E})(\xi + 2^k \cdot 2\pi j) = 0$  in either case. Therefore

$$\sum_{j \in \mathbb{Z}} (f \cdot \chi_{2^k E})(\xi + 2^k \cdot 2\pi j) = \sum_{j \in \mathbb{Z}} (f \cdot \chi_{2^k E_1})(\xi + 2^k \cdot 2\pi j), \quad \forall \xi \in 2^k E_1.$$

Hence,

$$\chi_{2^k E_1}(\xi) \sum_{j \in \mathbb{Z}} (f \cdot \chi_{2^k E})(\xi + 2^k \cdot 2\pi j) = \sum_{j \in \mathbb{Z}} (f \cdot \chi_{2^k E_1})(\xi + 2^k \cdot 2\pi j),$$

i.e.,  $\chi_{2^k E_1} H_E^k(f) = H_{E_1}^k(f)$ . Noticing that  $\Omega_1 \cap 2^k E = 2^k E_1$  for any  $k$ , we get

$$\begin{aligned} \langle H_E(f), f \rangle &= \sum_{k \in \mathbb{Z}} \langle H_E^k(f), f \rangle = \sum_{k \in \mathbb{Z}} \langle H_E^k(f), f \cdot \chi_{\Omega_1} \rangle \\ &= \sum_{k \in \mathbb{Z}} \langle H_E^k(f) \cdot \chi_{\Omega_1}, f \rangle = \sum_{k \in \mathbb{Z}} \langle H_{E_1}^k(f), f \rangle = \langle H_{E_1}(f), f \rangle. \end{aligned}$$

Similarly, we have  $\langle H_E(g), g \rangle = \langle H_{E_2}(g), g \rangle$ . □

The following lemma is a natural extension of Theorem 2 in [9] to a subset of  $\mathbb{R}$  that is invariant under 2-dilation. We refer to [9] and [10] for its proof.

LEMMA 3.3. *Let  $E$  be a basic set with the property that  $\bigcup_{k \in \mathbb{Z}} 2^k T_{r_f}(E) = \bigcup_{k \in \mathbb{Z}} 2^k E = \Omega$ . Then for any  $f$  with support in  $\Omega$  we have  $\langle H_E f, f \rangle \geq \|f\|^2$ .*

Let  $E$  be a basic set and let  $\Omega = \bigcup_{k \in \mathbb{Z}} 2^k E$ . Set  $E_1 = E \cap (\bigcup_{k \in \mathbb{Z}} 2^k T_{r_f}(E))$ ,  $\bar{E}_1 = E \setminus E_1$ ,  $E_2 = \bar{E}_1 \cap (\bigcup_{k \in \mathbb{Z}} 2^k T_{r_f}(\bar{E}_1))$ ,  $\bar{E}_2 = \bar{E}_1 \setminus E_2$ ,  $E_3 = \bar{E}_2 \cap (\bigcup_{k \in \mathbb{Z}} 2^k T_{r_f}(\bar{E}_2))$ ; in general, once  $E_n$  is defined, set  $\bar{E}_n = \bar{E}_{n-1} \setminus E_n$  and define  $E_{n+1} = \bar{E}_n \cap (\bigcup_{k \in \mathbb{Z}} 2^k T_{r_f}(\bar{E}_n))$ . Let  $\Omega_j = \bigcup_{k \in \mathbb{Z}} 2^k E_j$ . By the definition,  $\Omega_i = \bigcup_{k \in \mathbb{Z}} 2^k T_{r_f}(\bar{E}_{i-1})$  and  $\Omega_i \cap \Omega_j = \emptyset$  if  $i \neq j$ .

**THEOREM 3.4.** *Let  $E$  be a basic set such that  $E(\delta, m) = E(\tau, m) = \emptyset$  for  $m > M$ , and let  $\Omega = \bigcup_{1 \leq j \leq n} \Omega_j$  for some  $n \geq 1$ . Then for any  $f$  with support on  $\Omega$  we have*

$$\langle H_E(f), f \rangle \geq a_n \|f\|^2,$$

where  $a_n$  is the  $n$ -th term of the sequence of positive numbers defined recursively by  $a_k = a_{k-1}^2 / (1 + 4M^{5/2})^2$  and  $a_1 = 1 / ((1 + 2M^{5/2})^2 + 1)$ . In other words,  $E$  is a frame set on  $\Omega$  such that the frame wavelet  $\psi$  defined by  $\hat{\psi} = (1/\sqrt{2\pi})\chi_E$  has a lower frame bound at least  $a_n$ . In particular, if  $\Omega = \mathbb{R}$ , then  $E$  is an  $f$ -set.

*Proof.* Since  $M \geq 1$ , the sequence  $\{a_n\}$  is strictly decreasing and  $1 - a_n > 1/2$  for any  $n$ . We will prove this theorem by induction. No proof is needed if  $n = 1$  due to Lemma 3.3. Consider the case  $n = 2$ . Notice that it is necessary that  $M \geq 2$  in this case. Let  $f \in L^2(\mathbb{R}) \cdot \chi_\Omega = L^2(\Omega)$  and let  $f_1 = f \cdot \chi_{\Omega_1}$ ,  $f_2 = f \cdot \chi_{\Omega_2}$ . It is obvious that  $\|f\|^2 = \|f_1\|^2 + \|f_2\|^2$ . Also,

$$\begin{aligned} \langle H_E f, f \rangle &= \langle H_{T_{r_f}(E)} f, f \rangle + \langle H_{E \setminus T_{r_f}(E)} f, f \rangle \\ &\geq \langle H_{T_{r_f}(E)} f, f \rangle = \langle H_{T_{r_f}(E)} f_1, f_1 \rangle \geq \|f_1\|^2 \end{aligned}$$

by Lemmas 2.8 and 3.3. So if  $\|f_1\|^2 \geq a_1 \cdot \|f\|^2 = (1 / ((1 + 2M^{5/2})^2 + 1)) \|f\|^2$ , we are done. Now assume that  $\|f_1\|^2 < (1 / ((1 + 2M^{5/2})^2 + 1)) \|f\|^2$ . This implies that

$$((1 + 2M^{5/2})^2 + 1) \|f_1\|^2 < \|f\|^2 = \|f_1\|^2 + \|f_2\|^2.$$

It follows that

$$\|f_1\| < \frac{1}{1 + 2M^{5/2}} \|f_2\|.$$

By Lemmas 2.7, 3.2 and 3.3 we have

$$\begin{aligned} \langle H_E f, f \rangle &= \langle H_E f_1, f_1 \rangle + \langle H_E f_1, f_2 \rangle + \langle H_E f_2, f_1 \rangle + \langle H_E f_2, f_2 \rangle \\ &= \langle H_{E_1} f_1, f_1 \rangle + \langle H_E f_1, f_2 \rangle + \langle H_E f_2, f_1 \rangle + \langle H_{E_2} f_2, f_2 \rangle \\ &\geq \|f_1\|^2 + \langle H_E f_1, f_2 \rangle + \langle H_E f_2, f_1 \rangle + \|f_2\|^2 \\ &\geq \|f_1\|^2 - 2M^{5/2} \|f_1\| \|f_2\| + \|f_2\|^2 \\ &\geq \|f_1\|^2 - \frac{2M^{5/2}}{1 + 2M^{5/2}} \|f_2\|^2 + \|f_2\|^2 \\ &= \|f_1\|^2 + \frac{1}{1 + 2M^{5/2}} \|f_2\|^2 \\ &\geq \frac{1}{1 + 2M^{5/2}} (\|f_1\|^2 + \|f_2\|^2) = a_1 \|f\|^2. \end{aligned}$$

This proves the case  $n = 2$ . Assume the result is true for  $n$  and consider the case  $n + 1$ . Let  $f \in L^2(\mathbb{R}) \cdot \chi_\Omega$  and consider  $f_1 = f \cdot \chi_{\Omega_1}$ ,  $f_2 = f \cdot \chi_{\Omega \setminus \Omega_1}$ . Notice that the induction assumption now applies to  $f_2$  since  $\Omega \setminus \Omega_1 = \bigcup_{2 \leq j \leq n+1} \Omega_j$

is generated by  $\overline{E}_1$  and all conditions in the induction hypothesis are satisfied. ( $\overline{E}_1$  is still a basic set with the property that  $\overline{E}_1(\delta, m) = \overline{E}_1(\tau, m) = \emptyset$  for  $m \geq M$ .) Therefore, we have

$$\langle H_{\overline{E}_1} f_2, f_2 \rangle \geq a_n \|f_2\|^2.$$

Since  $E_1$  and  $\overline{E}_1$  satisfy the conditions in Lemma 3.2, we again have

$$\langle H_E f, f \rangle \geq \langle H_{T_{r,f}(E)} f, f \rangle = \langle H_{T_{r,f}(E)} f_1, f_1 \rangle \geq \|f_1\|^2$$

and

$$\begin{aligned} \langle H_E f, f \rangle &= \langle H_E f_1, f_1 \rangle + \langle H_E f_1, f_2 \rangle + \langle H_E f_2, f_1 \rangle + \langle H_E f_2, f_2 \rangle \\ &= \langle H_{E_1} f_1, f_1 \rangle + \langle H_E f_1, f_2 \rangle + \langle H_E f_2, f_1 \rangle + \langle H_{\overline{E}_1} f_2, f_2 \rangle \\ &\geq \|f_1\|^2 - 2M^{5/2} \|f_1\| \|f_2\| + a_n \|f_2\|^2. \end{aligned}$$

So, if  $\|f_1\|^2 \geq a_{n+1} \|f\|^2$ , we are done. If  $\|f_1\|^2 < a_{n+1} \|f\|^2 = a_{n+1} (\|f_1\|^2 + \|f_2\|^2)$ , then

$$\|f_1\| < \sqrt{\frac{a_{n+1}}{1 - a_{n+1}}} \|f_2\|.$$

It follows that

$$\begin{aligned} \langle H_E f, f \rangle &\geq \|f_1\|^2 - 2M^{5/2} \sqrt{\frac{a_{n+1}}{1 - a_{n+1}}} \|f_2\|^2 + a_n \|f_2\|^2 \\ &\geq \|f_1\|^2 - 4M^{5/2} \frac{a_n}{1 + 4M^{5/2}} \|f_2\|^2 + a_n \|f_2\|^2 \\ &= \|f_1\|^2 + \frac{a_n}{1 + 4M^{5/2}} \|f_2\|^2 \\ &\geq \frac{a_n}{1 + 4M^{5/2}} \|f\|^2 > a_{n+1} \|f\|^2. \end{aligned}$$

This finishes our proof. □

We end this section with a few examples of  $f$ -sets.

EXAMPLE 3.5. Let  $E = [-\frac{3\pi}{2}, -\frac{\pi}{2}] \cup [\frac{\pi}{4}, \pi)$ . Then  $T_{r,f}(E) = [-\pi, -\frac{\pi}{4}] \cup [\frac{\pi}{4}, \frac{\pi}{2})$ . Clearly,  $\bigcup_{k \in \mathbb{Z}} 2^k T_{r,f}(E) = \mathbb{R}$ . So  $E$  is a frame set.

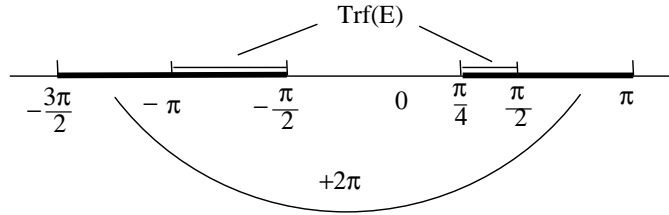


FIGURE 1.  $T_{r,f}(E)$  is a 2-dilation generator of  $\mathbb{R}$ .



EXAMPLE 3.6. Let  $E = [-3\pi, -\pi) \cup [\pi, 2\pi)$ . Then  $E$  does not satisfy the conditions in Theorem 2.9 since  $T_{r,f}(E) = [-2\pi, -\pi)$ ,  $\Omega_1 = \bigcup_{n \in \mathbb{Z}} 2^n E_1 = \mathbb{R}^-$ . So  $E_2 = [\pi, 2\pi)$  and  $\Omega_2 = \bigcup_{n \in \mathbb{Z}} 2^n E_2 = \mathbb{R}^+$ . Thus  $E$  is a frame set as well.

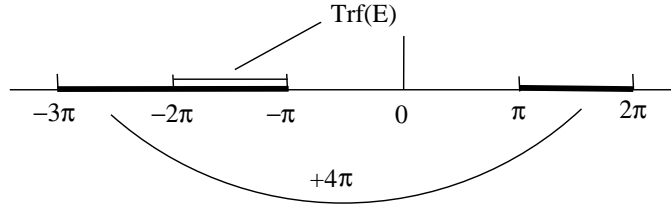


FIGURE 2.  $T_{r,f}(E)$  is a 2-dilation generator of  $\mathbb{R}^-$  and  $T_{r,f}(E_1)$  is a 2-dilation generator of  $\mathbb{R}^+$ .

EXAMPLE 3.7. The set  $E = [-13\pi/8, -3\pi/8) \cup [\pi/4, 13\pi/8)$  is a frame set. One can show that  $T_{r,f}(E) = [\pi/4, 3\pi/8)$ , so it only generates part of  $\mathbb{R}^+$  under 2-dilation. But  $T_{r,f}(E_1)$  contains a 2-dilation generator of  $\mathbb{R}^-$  and  $E_2 = T_{r,f}(E_2)$  contains a 2-dilation generator of the rest of  $\mathbb{R}^+$ . We leave the details of this example to the reader.

**4. Frame wavelets with  $f$ -set support in the frequency domain**

In this section, we will address some of the questions raised in Section 1. Let  $\psi \in L^2(\mathbb{R})$  and let  $E_{\hat{\psi}}$  be the support of  $\hat{\psi}$ . Following the  $H_E$  notation introduced in Section 2, we write

$$(4.1) \quad H_{\hat{\psi}}(f) = \sum_{n, \ell \in \mathbb{Z}} \langle f, \hat{D}^n \hat{T}^\ell \hat{\psi} \rangle \hat{D}^n \hat{T}^\ell \hat{\psi}.$$

Again, in the case when  $H_{\hat{\psi}}$  defines a bounded linear operator, (1.3) is equivalent to

$$(4.2) \quad A\|f\|^2 \leq \langle H_{\hat{\psi}}(f), f \rangle \leq B\|f\|^2$$

for all  $f \in L^2(\mathbb{R})$ . When  $E_{\hat{\psi}}$  is a basic set, we have the following lemma, which is a natural generalization of Lemma 3.1.

LEMMA 4.1. *If  $E_{\hat{\psi}}$  is a basic set and  $|\hat{\psi}| \leq b$  for some constant  $b > 0$  on  $E_{\hat{\psi}}$ , then  $H_{\hat{\psi}}$  defines a bounded linear operator. Furthermore, we have*

$$(4.3) \quad H_{\hat{\psi}}(f) = \sum_{k \in \mathbb{Z}} H_{\hat{\psi}}^k(f), \quad \forall f \in L^2(\mathbb{R}),$$

where

$$(4.4) \quad H_{\widehat{\psi}}^k(f) = \widehat{\psi} \left( \frac{\xi}{2^k} \right) \sum_{j \in \mathbb{Z}} f(\xi + 2^k 2\pi j) \overline{\widehat{\psi}} \left( \frac{\xi}{2^k} + 2\pi j \right)$$

and the above sums all converge under the  $L^2(\mathbb{R})$  norm topology.

The proof of this lemma is straightforward by applying the formulas given in Lemma 3.1 and is left to the reader. Let  $E$  be an  $f$ -set satisfying the conditions of Theorem 3.4, i.e.,  $E(\delta, m) = E(\tau, m) = \emptyset$  for  $m > M$ , and  $\mathbb{R} = \bigcup_{1 \leq j \leq n} \Omega_j$  for some  $n$ ,  $\Omega_j = \bigcup_{k \in \mathbb{Z}} 2^k T_{rf}(\overline{E}_{j-1})$ . Recall the definition of the set  $\Delta(F)$  for a basic set  $F$  in Section 2 and define

$$(4.5) \quad S(E) = \bigcup_{1 \leq j \leq n} \Delta(T_{rf}(\overline{E}_{j-1}))$$

(where  $\overline{E}_0 = E$ ). Notice that  $S(E)$  is a 2-dilation generator of  $\mathbf{R}$  and is not unique in general. We call  $S(E)$  a *core* of  $E$ . The theorem below is our main result of this section.

**THEOREM 4.2.** *Let  $E$  be an  $f$ -set satisfying the conditions of Theorem 3.4, i.e.,  $E(\delta, m) = E(\tau, m) = \emptyset$  for  $m > M$ , and  $\mathbb{R} = \bigcup_{1 \leq j \leq n} \Omega_j$  for some  $n$ ,  $\Omega_j = \bigcup_{k \in \mathbb{Z}} 2^k T_{rf}(\overline{E}_{j-1})$ . If the support of  $\widehat{\psi}$  is contained in  $E$  and there exists a constant  $a > 0$  such that  $|\widehat{\psi}(\xi)| \geq a$  a.e. on a core of  $E$ , then  $\psi$  is a frame wavelet.*

Before we proceed to prove Theorem 4.2, we would like to use the following example to show that it is necessary in general to add certain conditions on the bound for  $|\widehat{\psi}|$  to ensure that  $\psi$  is a frame wavelet. On the other hand, the above theorem tells us that we do not have to require  $|\widehat{\psi}(\xi)| \geq a$  on the whole set  $E$ .

**EXAMPLE 4.3.** Let  $E = [-2\pi, -\pi) \cup [\pi, 2\pi)$  and define

$$\widehat{\psi}(s) = \begin{cases} \frac{1}{n+1} & \text{if } s \in E_n, \ n = 0, 1, \dots, \\ 0 & \text{otherwise,} \end{cases}$$

where  $E_n = [-\pi - \frac{1}{2^n}\pi, -\pi - \frac{1}{2^{n+1}}\pi) \cup [\pi + \frac{1}{2^{n+1}}\pi, \pi + \frac{1}{2^n}\pi)$ . Then  $\text{supp}(\widehat{\psi}) = E$ . Since  $E$  is actually a wavelet set, its core is itself and for any function  $f$  with  $\text{supp}(f) \subset E$  we have

$$H_{\widehat{\psi}}(f) = \sum_{n \geq 0} \frac{1}{(n+1)^2} f \cdot \chi_{E_n}.$$

So if we let  $f_n = \chi_{E_n}$ , then

$$\lim_{n \rightarrow \infty} \frac{\langle H_{\widehat{\psi}}(f_n), f_n \rangle}{\|f_n\|^2} = 0.$$

Therefore  $\psi$  is not a frame wavelet. This shows that merely requiring that  $E_{\widehat{\psi}}$  is an  $f$ -set is not enough to guarantee that  $\psi$  is a frame wavelet.

The following two lemmas are analogous to Lemma 3.2 and Lemma 3.3.

LEMMA 4.4. *Let  $g$  be a bounded function in  $L^2(\mathbb{R})$  and  $g = g_1 + g_2$  with  $\text{supp}(g_1) = E_1$ ,  $\text{supp}(g_2) = E_2$  such that  $E = E_1 \cup E_2$  is a basic set and  $\Omega_1 = \bigcup_{k \in \mathbb{Z}} 2^k E_1$  and  $\Omega_2 = \bigcup_{k \in \mathbb{Z}} 2^k E_2$  are disjoint. Then for any functions  $f_1$  and  $f_2$  such that  $\text{supp}(f_1) \subset \Omega_1$  and  $\text{supp}(f_2) \subset \Omega_2$  we have  $\langle H_g(f_1), f_1 \rangle = \langle H_{g_1}(f_1), f_1 \rangle$  and  $\langle H_g(f_2), f_2 \rangle = \langle H_{g_2}(f_2), f_2 \rangle$ .*

*Proof.* Notice that

$$\begin{aligned} \langle H_g(f_1), f_1 \rangle &= \sum_{k \in \mathbb{Z}} \langle H_g^k(f_1), f_1 \rangle \\ &= \sum_{k \in \mathbb{Z}} \langle H_g^k(f_1), f_1 \cdot \chi_{\Omega_1} \rangle \\ &= \sum_{k \in \mathbb{Z}} \langle H_g^k(f_1) \cdot \chi_{\Omega_1}, f_1 \rangle. \end{aligned}$$

On the other hand, by Lemma 4.1 we have

$$H_g^k(f_1) = g\left(\frac{\xi}{2^k}\right) \sum_{j \in \mathbb{Z}} f_1(\xi + 2^k 2\pi j) \overline{g}\left(\frac{\xi}{2^k} + 2\pi j\right).$$

Therefore,

$$H_g^k(f_1) \cdot \chi_{\Omega_1} = g\left(\frac{\xi}{2^k}\right) \cdot \chi_{\Omega_1} \sum_{j \in \mathbb{Z}} f_1(\xi + 2^k 2\pi j) \overline{g}\left(\frac{\xi}{2^k} + 2\pi j\right).$$

For each non-vanishing term in the sum  $\sum_{j \in \mathbb{Z}} f_1(\xi + 2^k 2\pi j) \overline{g}((\xi/2^k) + 2\pi j)$  we must have  $\xi + 2^k 2\pi j \in \Omega_1$ . It follows that  $\xi/2^k + 2\pi j \in \Omega_1$  as well. Therefore,  $(\xi/2^k) + 2\pi j \in E_1$  since  $\Omega_1$  and  $\Omega_2$  are disjoint and  $E_2$  is contained in  $\Omega_2$ . This leads to

$$\sum_{j \in \mathbb{Z}} f_1(\xi + 2^k 2\pi j) \overline{g}\left(\frac{\xi}{2^k} + 2\pi j\right) = \sum_{j \in \mathbb{Z}} f_1(\xi + 2^k 2\pi j) \overline{g_1}\left(\frac{\xi}{2^k} + 2\pi j\right).$$

Finally, it is obvious that  $g(\xi/2^k) \cdot \chi_{\Omega_1}(\xi) = g_1(\xi/2^k)$ . Combining the above equalities, we obtain  $\langle H_g(f_1), f_1 \rangle = \langle H_{g_1}(f_1), f_1 \rangle$ . Similarly, we obtain  $\langle H_g(f_2), f_2 \rangle = \langle H_{g_2}(f_2), f_2 \rangle$ .  $\square$

LEMMA 4.5. *Let  $E$  be a basic set with the property that  $\bigcup_{k \in \mathbb{Z}} 2^k T_{r,f}(E) = \bigcup_{k \in \mathbb{Z}} 2^k E = \Omega$  and let  $\Delta(E)$  be a core of  $E$ . If  $g$  is a bounded function with its support in  $E$  such that  $|g| \geq a > 0$  for some constant  $a$  on  $\Delta(E)$ , then for any  $f$  with support in  $\Omega$  we have  $\langle H_g(f), f \rangle \geq a^2 \|f\|^2$ . In other words,  $g$  is a frequency frame wavelet in  $L^2(\Omega)$ .*

*Proof.* Let  $g_1 = g \cdot \chi_{\Delta(E)}$  and  $g_2 = g \cdot \chi_{E \setminus \Delta(E)}$ . For any  $f$  with  $\text{supp}(f) \subset \Omega$  we have

$$\begin{aligned} H_g^k(f) &= g \left( \frac{\xi}{2^k} \right) \sum_{j \in \mathbb{Z}} f(\xi + 2^k 2\pi j) \bar{g} \left( \frac{\xi}{2^k} + 2\pi j \right) \\ &= g_1 \left( \frac{\xi}{2^k} \right) \sum_{j \in \mathbb{Z}} f(\xi + 2^k 2\pi j) \bar{g} \left( \frac{\xi}{2^k} + 2\pi j \right) \\ &\quad + g_2 \left( \frac{\xi}{2^k} \right) \sum_{j \in \mathbb{Z}} f(\xi + 2^k 2\pi j) \bar{g} \left( \frac{\xi}{2^k} + 2\pi j \right). \end{aligned}$$

If  $g_1(\xi/2^k) \neq 0$ , then  $\xi/2^k \in \Delta(E)$ , and hence  $(\xi/2^k) + 2\pi j \notin E$  for any  $j \neq 0$ , since  $\Delta(E) \subset T_{r_f}(E)$ . It follows that

$$g_1 \left( \frac{\xi}{2^k} \right) \sum_{j \in \mathbb{Z}} f(\xi + 2^k 2\pi j) \bar{g} \left( \frac{\xi}{2^k} + 2\pi j \right) = \left| g_1 \left( \frac{\xi}{2^k} \right) \right|^2 \cdot f(\xi).$$

Similarly, if  $g_2(\xi/2^k) \neq 0$ , then  $\xi/2^k \in E \setminus \Delta(E)$ , and hence  $(\xi/2^k) + 2\pi j \notin \Delta(E)$  for any  $j \neq 0$ . It follows that

$$\begin{aligned} &g_2 \left( \frac{\xi}{2^k} \right) \sum_{j \in \mathbb{Z}} f(\xi + 2^k 2\pi j) \bar{g} \left( \frac{\xi}{2^k} + 2\pi j \right) \\ &= g_2 \left( \frac{\xi}{2^k} \right) \sum_{j \in \mathbb{Z}} f(\xi + 2^k 2\pi j) \bar{g}_2 \left( \frac{\xi}{2^k} + 2\pi j \right). \end{aligned}$$

In other words, we have

$$H_g^k(f) = H_{g_1}^k(f) + H_{g_2}^k(f) = \left| g_1 \left( \frac{\xi}{2^k} \right) \right|^2 \cdot f + H_{g_2}^k(f)$$

and

$$H_g(f) = H_{g_1}(f) + H_{g_2}(f) = \sum_{k \in \mathbb{Z}} \left( \left| g_1 \left( \frac{\xi}{2^k} \right) \right|^2 \cdot f + H_{g_2}^k(f) \right).$$

It follows that

$$\langle H_g(f), f \rangle \geq a^2 \|f\|^2$$

since  $|g_1(\xi/2^k)| \geq a$  for any  $\xi \in \Delta(E)$ ,  $\sum_{k \in \mathbb{Z}} \chi_{\Delta(E)}(\xi/2^k) = \chi_{\Omega}$  and  $\langle H_{g_2}(f), f \rangle \geq 0$ .  $\square$

Now Theorem 4.2 can be proved by induction in a way that is similar to the proof of Theorem 3.4. The details are left to the reader.

**REMARK 4.6.** Theorem 4.2 has a potential drawback: it cannot be applied to a frame set without a core, although at this time we do not know whether there exist coreless frame sets. On the other hand, the following theorem

ensures that we can still construct frame wavelets with a certain level of flexibility on an arbitrary frame set with or without a core.

**THEOREM 4.7.** *Let  $E$  be a frame set. Then  $\psi \in L^2(\mathbb{R})$  is a frame wavelet if  $\widehat{\psi}$  is bounded,  $\text{supp}(\widehat{\psi}) = E$ ,  $|\widehat{\psi}| \geq a > 0$  on  $E$  for some constant  $a > 0$ , and  $\widehat{\psi}(s) = \widehat{\psi}(2^k s)$  whenever  $s$  and  $2^k s$  are both in  $E$  for any integer  $k$ .*

*Proof.* Since  $E$  is a frame set, there exists a positive constant  $c > 0$  such that  $\langle H_E(f), f \rangle \geq c\|f\|^2$  for any  $f \in L^2(\mathbb{R})$ . Furthermore, by Lemma 4.1, for any  $f \in L^2(\mathbb{R})$  we have

$$\begin{aligned} H_{\widehat{\psi}}^k(f) &= \widehat{\psi}\left(\frac{\xi}{2^k}\right) \sum_{j \in \mathbb{Z}} f(\xi + 2^k 2\pi j) \overline{\widehat{\psi}\left(\frac{\xi}{2^k} + 2\pi j\right)} \\ &= \widehat{\psi}(\xi) \sum_{j \in \mathbb{Z}} f(\xi + 2^k 2\pi j) \overline{\widehat{\psi}(\xi + 2^k 2\pi j)} \\ &= \widehat{\psi} \cdot H_E^k(f \cdot \overline{\widehat{\psi}}). \end{aligned}$$

Therefore,  $H_{\widehat{\psi}}(f) = \widehat{\psi} \cdot H_E(f \cdot \overline{\widehat{\psi}})$  and it follows that

$$\langle H_{\widehat{\psi}}(f), f \rangle = \langle \widehat{\psi} H_E(f \cdot \overline{\widehat{\psi}}), f \rangle = \langle H_E(f \cdot \overline{\widehat{\psi}}), f \cdot \overline{\widehat{\psi}} \rangle \geq c\|f \cdot \overline{\widehat{\psi}}\|^2 \geq ca^2\|f\|^2.$$

This proves that  $\widehat{\psi}$  is a frequency frame wavelet. Therefore,  $\psi$  is a frame wavelet. □

The last result of this section concerns the measure of the support of the frequency domain frame wavelets. This is stated as the following theorem.

**THEOREM 4.8.** *Let  $\alpha > 0$  be any given constant. Then there exist frequency domain frame wavelets with support of measure  $\alpha$ .*

*Proof.* In light of Theorems 4.2 and 4.7, it suffices to show that for any given  $\alpha > 0$  there exist frame sets (with or without a core) of measure  $\alpha$ . If  $\alpha \leq 2\pi$ , we can simply let  $E = [-\alpha, -\frac{\alpha}{2}] \cup [\frac{\alpha}{2}, \alpha]$ . In this case  $E$  is actually an  $n$ -set. If  $\alpha > 2\pi$ , then choose an integer  $m > 0$  large enough such that  $\beta = \frac{\alpha - \pi}{m} < \frac{\pi}{2}$ . Define

$$E = \left( \bigcup_{k=1}^m [2k\pi, 2k\pi + \beta] \right) \cup [-\pi, -\frac{\pi}{2}] \cup [\frac{\pi}{2}, \pi).$$

One can easily check that  $E(\delta, 1) = [-\pi, -\frac{\pi}{2}] \cup [\frac{\pi}{2}, \pi)$  is the core of  $E$  (so  $E$  is an  $f$ -set) and the measure of  $E$  is  $\alpha$ . □

**5. Examples and comparisons with existing results**

In this section, we will give some examples of frame wavelets constructed using Theorem 4.2. As we pointed out in Section 1, we will also compare these examples to some known results.

Define

$$t_m(\xi) = \sum_{j=0}^{\infty} \widehat{\psi}(2^j \xi) \overline{\widehat{\psi}(2^j(\xi + 2m\pi))}, \quad \xi \in \mathbb{R}, \quad m \in \mathbb{Z},$$

and

$$S(\xi) = \sum_{j \in \mathbb{Z}} |\widehat{\psi}(2^j \xi)|^2, \quad \xi \in \mathbb{R}.$$

Also define

$$\underline{S}_\psi = \operatorname{ess\,inf}_{\xi \in \mathbb{R}} S(\xi), \quad \overline{S}_\psi = \operatorname{ess\,sup}_{\xi \in \mathbb{R}} S(\xi)$$

and

$$\beta_\psi(m) = \operatorname{ess\,sup}_{\xi \in \mathbb{R}} \sum_{k \in \mathbb{Z}} |t_m(2^k \xi)|.$$

The following theorem (see [20]) gives the best sufficient condition (known to the authors) for a frame wavelet.

**THEOREM 5.1.** *Let  $\psi \in L^2(\mathbb{R})$  be such that*

$$A_\psi = \underline{S}_\psi - \sum_{q \in 2\mathbb{Z}+1} (\beta_\psi(q)\beta_\psi(-q))^{1/2} > 0,$$

and

$$B_\psi = \overline{S}_\psi + \sum_{q \in 2\mathbb{Z}+1} (\beta_\psi(q)\beta_\psi(-q))^{1/2} < \infty.$$

*Then  $\psi$  is a frame wavelet with frame bounds  $A_\psi$  and  $B_\psi$ .*

When a frame set  $E$  satisfies the condition  $E = T_{r,f}(E)$ , then  $t_m(\xi) = 0$  for any function  $\psi$  with  $\operatorname{supp}(\widehat{\psi}) = E$  as one can easily check. It follows that  $A_\psi = \underline{S}_\psi$  and  $B_\psi = \overline{S}_\psi$  always exist, and hence  $\psi$  is a frame wavelet by Theorem 5.1. Frame wavelets that satisfy the condition  $\operatorname{supp}(\widehat{\psi}) = E$  include the well known Frazier-Jawerth frame wavelets ([14], [15], [16], [17]), whose Fourier transforms are supported by the interval  $1/2 \leq |\xi| \leq 2$ . If  $E = T_{r,f}(E)$ , then Theorem 4.2 is covered by Theorem 5.1. However, when this is not the case, Theorem 5.1 becomes hard to apply since  $t_m(\xi)$ ,  $\beta_\psi(m)$  and  $\sum_{q \in 2\mathbb{Z}+1} (\beta_\psi(q)\beta_\psi(-q))^{1/2}$  are hard to calculate in general. Even when it is possible to calculate these quantities, it may be the case that  $A_\psi \leq 0$  or  $B_\psi = \infty$ . In such cases we will not be able to apply Theorem 5.1.

EXAMPLE 5.2. Let  $E = [-\frac{3\pi}{2}, -\frac{\pi}{2}] \cup [\frac{\pi}{4}, \pi)$  as we did in Example 3.5, so that  $T_{rf}(E) = [-\pi, -\frac{\pi}{4}] \cup [\frac{\pi}{4}, \frac{\pi}{2})$ . Define  $\psi$  by letting

$$\widehat{\psi} = \chi_{[-\pi, -\pi/2)} + 2\chi_{[-3\pi/2, -\pi)} + \chi_{[\pi/4, \pi/2)} + 2\chi_{[\pi/2, \pi)}.$$

For odd integers  $m$  we have

$$t_m(\xi) = \begin{cases} 4, & \text{if } \xi \in [-\frac{3\pi}{2}, -\pi) \text{ and } m = 1, \\ 4, & \text{if } \xi \in [\frac{\pi}{2}, \pi) \text{ and } m = -1, \\ 0, & \text{otherwise.} \end{cases}$$

It follows that  $\beta_\psi(m) = 4$  for  $m = \pm 1$  and  $\beta_\psi(m) = 0$  otherwise. On the other hand, one can easily calculate that  $\underline{S}_\psi = 1$ . So  $A_\psi = 1 - 4 = -3$  and Theorem 5.1 does not apply. However, according to Theorem 4.2,  $\psi$  is a frame wavelet. It is not hard to see from Lemma 2.7 and the proof of Theorem 4.2 that the lower frame bound for  $\psi$  is 1 and that the upper frame bound is at most  $16\sqrt{2}$ .

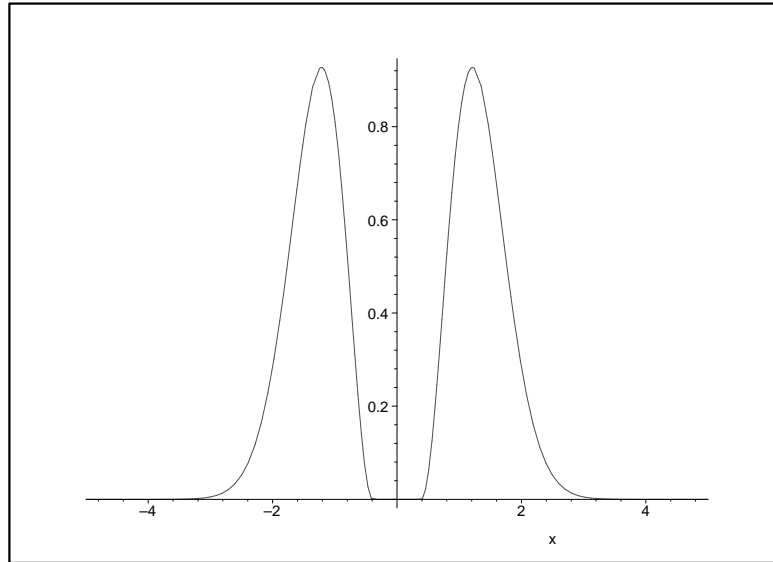
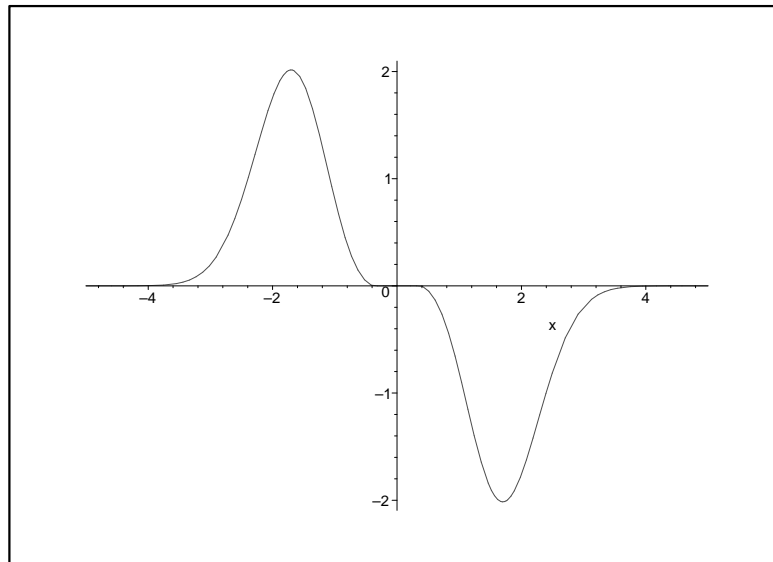
EXAMPLE 5.3. Let  $E = [-\frac{3\pi}{2}, -\frac{\pi}{16}] \cup [\frac{\pi}{16}, \frac{3\pi}{2})$ . It is easy to see that  $E$  is a frame set since  $T_{rf}(E) = [-\frac{\pi}{2}, -\frac{\pi}{16}] \cup [\frac{\pi}{16}, \frac{\pi}{2})$  actually contains three dilation generators of  $\mathbb{R}$ . On this set we can define one function that looks like the *Mexican hat* function (see [20, p. 415] or [11]), and another that is symmetric about the origin. Since both functions are bounded away from 0 on  $[-\frac{\pi}{2}, -\frac{\pi}{4}] \cup [\frac{\pi}{4}, \frac{\pi}{2})$  as one can check, they are both frame wavelets. Obviously, functions like these can be constructed in a rather arbitrary way.

$$\begin{aligned} \widehat{\psi}_1 &= \begin{cases} 6(|\xi| - \frac{\pi}{8})^2 \cdot e^{-\xi^2}, & \text{if } |\xi| \in [\frac{\pi}{8}, \frac{3\pi}{2}), \\ 0, & \text{otherwise,} \end{cases} \\ \widehat{\psi}_2 &= \begin{cases} -\xi(\xi^2 - \frac{\pi^2}{64})^2 \cdot e^{-(|\xi| - \pi/4)^2}, & \text{if } |\xi| \in [\frac{\pi}{8}, \frac{3\pi}{2}), \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

EXAMPLE 5.4. Define

$$\begin{aligned} \widehat{\psi}_3 &= \cos\left(x - \frac{\pi}{4}\right) \cdot \left(x + \frac{13\pi}{8}\right)^2 \left(x + \frac{3\pi}{8}\right)^4 \cdot e^x \cdot \chi_{E_1} \\ &\quad + \sin(x) \cdot \left(x - \frac{13\pi}{8}\right)^2 \left(x - \frac{\pi}{5}\right)^4 \cdot e^{-x} \cdot \chi_{E_2}, \end{aligned}$$

where  $E_1 = [-\frac{13\pi}{8}, -\frac{3\pi}{8})$  and  $E_2 = [\frac{\pi}{5}, \frac{13\pi}{8})$ . Consider the set  $E = E_1 \cup E_2 \cup [-\frac{9\pi}{5}, -\frac{7\pi}{4})$ . We leave it to our reader to prove that  $[-\pi, -\frac{3\pi}{2}) \cup [\frac{\pi}{4}, \frac{\pi}{2})$  is a core of  $E$ . It is obvious that  $|\widehat{\psi}| \geq a > 0$  for some constant  $a > 0$  on  $[-\pi, -\frac{3\pi}{2}) \cup [\frac{\pi}{4}, \frac{\pi}{2})$ . Therefore,  $\psi_3$  is a frame wavelet.

FIGURE 3.  $\widehat{\psi}_1$ : A Mexican hat like frequency frame wavelet.FIGURE 4.  $\widehat{\psi}_2$ : An odd frequency frame wavelet.



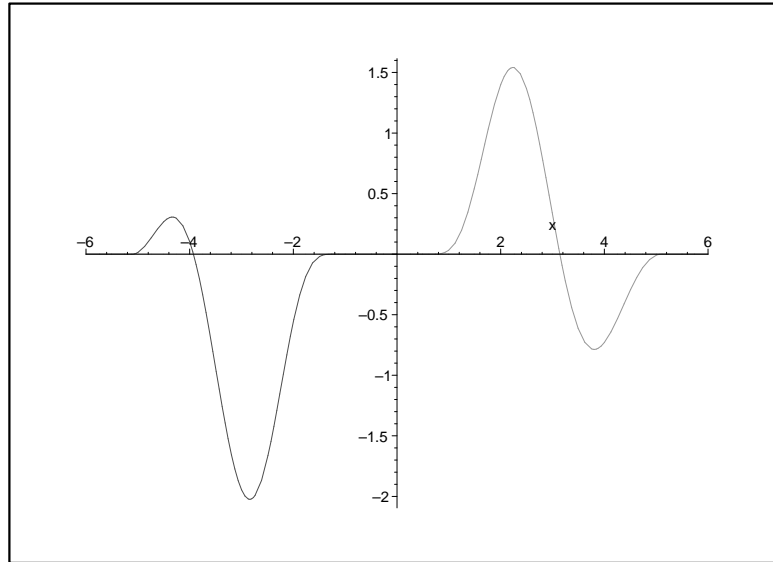


FIGURE 5.  $\widehat{\psi}_3$ : A not so obvious frequency frame wavelet.

REMARK 5.5. It would be interesting to know whether there are other approaches to verify that the functions defined in the above examples are frame wavelets.

REMARK 5.6. We need to point out that our method only provides a way to create frame wavelets, but does not give estimates for the frame bounds of the created frame wavelets. This is an area where further study is needed.

### 6. Further remarks

In this section, we will discuss the case when  $E_{\widehat{\psi}}$  is not an  $f$ -set. We begin with the following theorem, which gives a sufficient condition for a basic set  $E$  to be a non-frame set.

**THEOREM 6.1.** *Let  $E$  be a basic set such that  $E = E(\tau, 2) = E(\delta, 1)$ . Then  $E$  is not a frame set.*

*Proof.* Let  $f = \chi_{E^{(1)}(\tau,2)} - \chi_{E^{(2)}(\tau,2)}$ . We see that  $\|f\| > 0$ . However, we have  $H_E(f) = 0$  by Lemma 3.1 (the details are left to the reader). Therefore,  $E$  is not a frame set.  $\square$

One question here is whether there are non-trivial examples of frame wavelets whose Fourier transforms are supported by non-frame sets. One such example is the Mexican hat function ([11], [20]), since the support of its

Fourier transform is the entire set  $\mathbb{R}$ , which is not a frame set. One can also construct counterexamples in which the support of the frequency domain frame wavelet is not the entire set  $\mathbb{R}$  using Theorem 5.1. Another question is whether there exist non-frame sets  $E$  on which one cannot construct frequency frame wavelets (with their supports equal to  $E$ ). The following theorem gives a positive answer to this question.

**THEOREM 6.2.** *Let  $E$  be a basic set such that  $E = E(\tau, 2) = E(\delta, 1)$ . Then no function  $\psi \in L^2(\mathbb{R})$  with  $\text{supp}(\widehat{\psi}) = E$  can be a frame wavelet.*

*Proof.* Let  $\psi \in L^2(\mathbb{R})$  be a function such that  $\text{supp}(\widehat{\psi}) = E$ . Let  $\epsilon > 0$  be an arbitrary small number. Since  $\text{supp}(\widehat{\psi}) = E$ , there exists a positive number  $a > 0$  such that  $\mu(E'_1) < \epsilon$ , where  $E'_1$  is the subset of  $E$  consisting of all points  $\xi$  such that  $|\widehat{\psi}(\xi)| < a$  or  $|\widehat{\psi}(\xi)| > 1/a$ . Let  $E''_1 = E \cap (\bigcup_{j \in \mathbb{Z}} (E'_1 + 2\pi j))$  and  $E_1 = E \setminus E''_1$ . Then  $E_1 = E_1(\tau, 2)$  and  $\mu(E_1) < 2\epsilon$ . Let  $F_1 = E_1^{(1)}(\tau, 2)$  and  $F_2 = E_1^{(2)}(\tau, 2)$  be a partition of  $E_1$  as defined in Lemma 2.2. Define  $\widehat{\psi}_1 = \psi \cdot \chi_{E_1}$  and  $f = (\widehat{\psi})^{-1} \cdot \chi_{F_1} - (\widehat{\psi})^{-1} \cdot \chi_{F_2}$ . Since  $\text{supp}(f) \subset E_1$  and  $E_1 \subset E$  is 2-dilation redundancy free,  $\xi/2^k \notin E$  for any  $k \neq 0$  whenever  $\xi \in E_1$ . It follows that

$$\langle f, \widehat{D}^k \widehat{T}^\ell \widehat{\psi} \rangle = \langle f(\xi), e^{-i\ell\xi/2^k} \widehat{\psi}\left(\frac{\xi}{2^k}\right) \rangle = 0$$

whenever  $k \neq 0$ . For  $k = 0$ , the above is

$$\langle f, \widehat{T}^\ell \widehat{\psi} \rangle = \langle f(\xi), e^{-i\ell\xi} \widehat{\psi}(\xi) \rangle = \langle f(\xi), e^{-i\ell\xi} \widehat{\psi}_1(\xi) \rangle.$$

Since  $\psi_1$  is bounded, the above leads to

$$\sum_{k, \ell \in \mathbb{Z}} |\langle f, \widehat{D}^k \widehat{T}^\ell \widehat{\psi} \rangle|^2 = \langle f, H_{\psi_1}(f) \rangle.$$

But

$$\begin{aligned} H_{\psi_1}(f) &= \widehat{\psi}_1(\xi) \sum_{\ell \in \mathbb{Z}} f(\xi + 2\ell\pi) \overline{\widehat{\psi}_1(\xi + 2\ell\pi)} \\ &= \widehat{\psi}_1(\xi) (\chi_{E_1} - \chi_{E_1}) = 0. \end{aligned}$$

This shows that  $\psi$  is not a frame wavelet.  $\square$

#### REFERENCES

- [1] R. Balan, *Stability theorems for Fourier frames and wavelet Riesz bases*, J. Fourier Anal. Appl. **3** (1997), 499–504. MR **99f**:42060
- [2] K. Bittner and C. K. Chui, *Gabor frames with arbitrary windows*, Approximation theory, X (St. Louis, MO, 2001), Innov. Appl. Math., Vanderbilt Univ. Press, Nashville, TN, 2002, pp. 41–50. MR **2003h**:42055
- [3] P. G. Casazza and O. Christensen, *Weyl-Heisenberg frames for subspaces of  $L^2(\mathbb{R})$* , Proc. Amer. Math. Soc. **129** (2001), 145–154. MR **2001h**:42046

- [4] P. G. Casazza, O. Christensen, and A. J. E. M. Janssen, *Weyl-Heisenberg frames, translation invariant systems and the Walnut representation*, J. Funct. Anal. **180** (2001), 85–147. MR **2002b**:42042
- [5] P. G. Casazza and M. C. Lammers, *Analyzing the Weyl-Heisenberg frame identity*, Appl. Comput. Harmon. Anal. **12** (2002), 171–178. MR **2002k**:42068
- [6] O. Christensen, *A Paley-Wiener theorem for frames*, Proc. Amer. Math. Soc. **123** (1995), 2199–2201. MR **95i**:46027
- [7] ———, *Perturbation of frames and applications to Gabor frames*, Gabor analysis and algorithms, Appl. Numer. Harmon. Anal., Birkhäuser Boston, Boston, MA, 1998, pp. 193–209. MR **98j**:42017
- [8] O. Christensen, B. Deng, and C. Heil, *Density of Gabor frames*, Appl. Comput. Harmon. Anal. **7** (1999), 292–304. MR **2000j**:42043
- [9] X. Dai, Y. Diao, and Q. Gu, *Frame wavelet sets in  $\mathbb{R}$* , Proc. Amer. Math. Soc. **129** (2001), 2045–2055. MR **2002a**:46006
- [10] X. Dai, Y. Diao, and Q. Gu, *Subspaces with normalized tight frame wavelets in  $\mathbf{R}$* , Proc. Amer. Math. Soc. **130** (2002), 1661–1667. MR **2003b**:42050
- [11] I. Daubechies, *The wavelet transform, time-frequency localization and signal analysis*, IEEE Trans. Inform. Theory **36** (1990), 961–1005. MR **91e**:42038
- [12] R. J. Duffin and A. C. Schaeffer, *A class of nonharmonic Fourier series*, Trans. Amer. Math. Soc. **72** (1952), 341–366. MR 13,839a
- [13] H. G. Feichtinger and A. J. E. M. Janssen, *Validity of WH-frame bound conditions depends on lattice parameters*, Appl. Comput. Harmon. Anal. **8** (2000), 104–112. MR **2000j**:42044
- [14] M. Frazier and B. Jawerth, *Decomposition of Besov spaces*, Indiana Univ. Math. J. **34** (1985), 777–799. MR **87h**:46083
- [15] ———, *The  $\phi$ -transform and applications to distribution spaces*, Function spaces and applications (Lund, 1986), Lecture Notes in Math., vol. 1302, Springer, Berlin, 1988, pp. 223–246. MR **89g**:46064
- [16] ———, *A discrete transform and decompositions of distribution spaces*, J. Funct. Anal. **93** (1990), 34–170. MR **92a**:46042
- [17] M. Frazier, B. Jawerth, and G. Weiss, *Littlewood-Paley theory and the study of function spaces*, CBMS Regional Conference Series in Mathematics, vol. 79, Published for the Conference Board of the Mathematical Sciences, Washington, DC, 1991. MR **92m**:42021
- [18] K. Gröchenig, *Describing functions: atomic decompositions versus frames*, Monatsh. Math. **112** (1991), 1–42. MR **92m**:42035
- [19] D. Han and D. R. Larson, *Frames, bases and group representations*, Mem. Amer. Math. Soc. **147** (2000), x+94. MR **2001a**:47013
- [20] E. Hernández and G. Weiss, *A first course on wavelets*, Studies in Advanced Mathematics, CRC Press, Boca Raton, FL, 1996. MR **97i**:42015
- [21] W. Sun and X. Zhou, *On the stability of Gabor frames*, Adv. in Appl. Math. **26** (2001), 181–191. MR **2002c**:42043
- [22] B. de Sz. Nagy, *Expansion theorems of Paley-Wiener type*, Duke Math. J. **14** (1947), 975–978. MR 9,358c

XINGDE DAI, DEPARTMENT OF MATHEMATICS, UNIVERSITY OF NORTH CAROLINA AT CHARLOTTE, CHARLOTTE, NC 28223, USA  
*E-mail address:* [xdai@uncc.edu](mailto:xdai@uncc.edu)

YUANAN DIAO, DEPARTMENT OF MATHEMATICS, UNIVERSITY OF NORTH CAROLINA AT CHARLOTTE, CHARLOTTE, NC 28223, USA  
*E-mail address:* [ydiao@uncc.edu](mailto:ydiao@uncc.edu)

QING GU, DEPARTMENT OF MATHEMATICS, EAST CHINA NORMAL UNIVERSITY, SHANGHAI, 200062, PR CHINA  
*E-mail address:* [guqing2218@vip.sina.com](mailto:guqing2218@vip.sina.com)