

MINIMAL FINE LIMITS ON TREES

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ABSTRACT. Let T be the set of vertices of a tree. We assume that the Green function is finite and $G(s, t) \rightarrow 0$ as $|s| \rightarrow \infty$ for each vertex t . For v positive superharmonic on T and E a subset of T , the reduced function of v on E is the pointwise infimum of the set of positive superharmonic functions that majorize v on E . We give an explicit formula for the reduced function in case E is finite as well as several applications of this formula. We define the minimal fine filter corresponding to each boundary point of the tree and prove a tree version of the Fatou-Naim-Doob limit theorem, which involves the existence of limits at boundary points following the minimal fine filter of the quotient of a positive superharmonic by a positive harmonic function. We deduce from this a radial limit theorem for such functions. We prove a growth result for positive superharmonic functions from which we deduce that, if the trees has transition probabilities all of which lie between δ and $1/2 - \delta$ for some $\delta \in (0, 1/2)$ (for example homogeneous trees with isotropic transition probabilities), then any real-valued function on T which has a limit at a boundary point following the minimal fine filter necessarily has a nontangential limit there. We give an example of a tree for which minimal fine limits do not imply nontangential limits, even for positive superharmonic functions. Motivated by work on potential theory on halfspaces and Brelot spaces, we define the harmonic fine filter corresponding to each boundary point of the tree. In contrast to the classical setting, we are able to show that it is the same as the minimal fine filter.

1. Introduction

Let D be an open subset of \mathbb{R}^n having a Green function and let Ω be the Martin boundary of D . Thus $D \cup \Omega$ is a compactification of D such that all positive harmonic functions on D have a unique integral representation with respect to a regular Borel measure on Ω supported on the subset known as the minimal Martin boundary. Corresponding to each minimal boundary point ω , Naim [N] defined a filter of neighbourhoods on D finer than the one given by the Martin topology, called the minimal fine filter. She proved that for each positive harmonic function u and potential p on D , the quotient p/u

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has a limit of 0 following the minimal fine filter at μ_u -almost every minimal boundary point, where μ_u is the representing measure of u .

Doob considered, more generally, the boundary behaviour of quotients v/u , where v is positive superharmonic on D . In [D1] he proved a theorem concerning the limits of v/u along Brownian paths, and in [D2] he gave a non-probabilistic proof that v/u has a limit following the minimal fine filter at μ_u -almost every minimal boundary point. GowriSankaran [G1], [G2] proved a result analogous to the latter in a more abstract setting, which in particular is valid in a Brelot harmonic space. This result is known as the Fatou-Naïm-Doob Theorem.

The non-probabilistic results of Doob can be specialized to an upper half plane in \mathbb{R}^n , $n \geq 2$, where the minimal Martin boundary is the topological boundary $\mathbb{R}^{n-1} \cup \{\infty\}$ and the Martin kernel is the Poisson kernel. In [BD] it is shown that the quotient h/u of two positive harmonic functions has a nontangential limit at every boundary point where it has a minimal fine limit, and in Theorem 1.XII.23 of [D3] the minimal fine theory is used to show that potentials have perpendicular limits of 0 at Lebesgue-almost every boundary point. Thus techniques based on the abstract minimal fine theory can be used to deduce the classical theorems of Fatou and Littlewood.

In this paper we consider some related ideas in the setting of infinite trees. In the next section we give precise statements of all definitions and results we shall use. Let T denote the vertices of an infinite tree. We fix a vertex, e , called the root. The boundary of T can be identified with the set of infinite geodesic paths $\omega = \{\omega_0, \omega_1, \dots\}$ beginning at the root together with the set of terminal vertices. Much of the basic potential theory was developed by Cartier in [C]. Assuming the existence of a Green function, he considered harmonic functions, superharmonic functions and Green potentials defined on T . He defined the Martin kernel on $T \times \Omega$, used it to give an integral representation of all positive harmonic functions u by associating a unique Borel measure μ_u , and proved a limit theorem for quotients v/u in the spirit of the probabilistic result of Doob.

We describe the latter in a bit more detail. Let W be the set of all infinite paths $s = \{s_0, s_1, \dots\}$ beginning at the root. For c a finite path beginning at e , let $W(c)$ be the cylinder consisting of paths in W beginning with the segment c . These cylinder sets form the base of a compact metrizable topology on W . Let u be positive harmonic on T . For every such u , Cartier proved the existence of a Borel measure Π^u on W supported by the set of paths whose vertices converge to a boundary point of the tree such that for every finite path c , $\Pi^u(W(c)) = p(c)u(e)$, where $p(c)$ is the product of the transition probabilities along the edges of c . It is a consequence of Corollary 3.1b, Theorem 3.1 and Theorem 3.2 in [C] that for any positive superharmonic function v on T , for μ_u -a.e. boundary point ω , the quotient $v(s_n)/u(s_n)$ has a limit along the vertices of the path s for Π^u -a.e. path s which converges to ω .

From this it is deduced in Theorem 3 of [KP] that such quotients have limits at μ_u -almost every boundary point if approach is restricted to nontangential regions provided the transition probabilities are uniformly bounded away from 0 and the Green function is finite.

We are, in the present work, more motivated by the non-probabilistic techniques of Doob and GowriSankaran. Accordingly we define and develop the properties of the minimal fine filter on trees and use them to study the boundary behaviour of the quotient of a positive superharmonic function by a positive harmonic function.

In [L], the minimal fine filter is considered on infinite networks using probabilistic ideas. There are many papers dealing with boundary behaviour of harmonic functions on trees using a geometric approach, for example, [C], [CCGS], [DB], [GS], [R], [SV].

In Section 2 we recall all of the definitions and results on trees which we shall use. We define the Green function $G(s, t)$, which we assume throughout is finite (see (9)). We usually assume in addition that for each $t \in T$, $G(s, t) \rightarrow 0$ as $|s| \rightarrow \infty$ (see (10)) and sometimes we assume that this limit tends to 0 uniformly exponentially fast (see (13)). This last condition is natural, as it holds in the case of homogeneous trees of degree at least 3 with isotropic transition probabilities. We also give examples which show that a Green function can satisfy (9) without (10) and it can satisfy (10) without (13).

In Section 3, assuming only that the Green function satisfies (9), we give upper and lower bounds for the ratio of values of a positive superharmonic function at neighbouring vertices. We deduce that among all positive superharmonic functions, the Martin kernel K_ω grows as quickly as possible along the vertices of ω and decreases as rapidly as possible outside the vertices of ω as we move away from the root.

For the remainder of the paper we assume the Green function satisfies (10). In Section 4 we give an explicit formula for $F^E f$, the solution of the Dirichlet problem on the complement of a finite subset, E , of T in terms of the inverse of the matrix determined by $G(s, t)$, $s, t \in E$. As an application, we prove that every function defined on a finite subset, E , of T can be written as the restriction of the difference of two positive potentials with support in E . For any subset E of T and positive superharmonic function v we define the reduced function R_v^E which is fundamental in studying the minimal fine filter, and which agrees with $F^E v$ in case E is finite. Using the explicit formula for $F^E v$ and certain limit results, we give very simple proofs of classically hard results such as the additivity of the reduced function, the fact that the reduction operation commutes with the integral representation of positive harmonic functions, and the domination principle.

In Section 5 we define the minimal thin sets and the minimal fine filter corresponding to each boundary point ω . A set of vertices, E , is said to be minimally thin at ω if there exists a potential which majorizes the Martin

kernel K_ω on E . We show how to generate examples of sets that are minimally thin at ω and examples of sets that are not minimally thin at ω . The set of complements of sets minimally thin at ω forms the minimal fine filter at ω . We show that a function has a limit following the minimal fine filter at a boundary point ω if and only if it has a limit as ω is approached in the tree topology outside of a set minimally thin at ω . We show that any function on T which has a minimal fine limit at ω necessarily has a radial limit there. In case the Green function satisfies (13), we show that the minimal fine filter is strictly coarser than the nontangential filter, so in particular a function with a minimal fine limit at ω necessarily has a nontangential limit at ω . This is in contrast with classical potential theory on a halfspace in \mathbb{R}^n where it is true for functions which are quotients of positive harmonic functions but not true in general. We give an example of a tree for which constants are harmonic and the Green function satisfies (10) but not (13) such that minimal fine limits are equivalent to radial limits, so the minimal fine filter is strictly coarser than the nontangential filter. On this tree we define a positive superharmonic function that has a minimal fine limit at every boundary point but which does not have a nontangential limit at a particular boundary point.

In Section 6 we prove Theorem 6.1 which establishes the existence of limits following the minimal fine filter of quotients of positive superharmonic functions by a positive harmonic function, thus extending the non-probabilistic result of Doob and the Fatou-Naim-Doob Theorem of GowriSankaran to trees.

In Section 7 we define a set, E , to be harmonic thin at a boundary point ω if there exists a harmonic function on T which majorizes the Martin kernel K_ω on E but not on T . The set of complements of such sets forms a filter, known as the harmonic fine filter at ω . This was introduced on \mathbb{R}^n by Aikawa [A] and by Zhang [Z] on Brelot spaces. In the latter, a limit theorem for quotients of positive harmonic functions following the harmonic filter was proved. In classical settings it is typically true that the harmonic fine filter is strictly coarser than the minimal fine filter, as all examples in [Z] show. However we show in Theorem 7.1 that the harmonic fine filter is the same as the minimal fine filter on trees.

In [BCCS] it is shown that one can provide a topology on the set \tilde{T} consisting of the vertices and edges of the tree and define functions on \tilde{T} so that it is given the structure of a Brelot space. A function on the vertices that is harmonic in the Cartier sense can be made harmonic in the Brelot sense by extending it “linearly” on the edges. Conversely the restriction to the vertices of a Brelot harmonic function is harmonic in the Cartier sense. Thus many results valid in a general Brelot space automatically hold when restricted to the vertices of a tree. We make no use of this fact in the present article since the techniques we use are more explicit and give more detailed results than

are available in the general setting. Furthermore our proof of the Fatou-Naïm-Doob Theorem (Theorem 6.1) is simpler than the one in [G2] which is valid in a general Brelot space.

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2. Preliminaries

In this section we recall several definitions and preliminary results. Good references for basic potential theory on trees are [C] and [KPT]. All of the results quoted in this section are proved there. A *tree* is a locally finite, connected graph with no loops. Let T denote the set of vertices of an infinite tree. If $s_1, s_2 \in T$, we write $s_1 \sim s_2$ provided there is an edge joining s_1 and s_2 in which case we say that s_1 and s_2 are *neighbours*. We assume each vertex has finitely many neighbours. A *path* from initial vertex s to terminal vertex t is a finite sequence of vertices $c = \{s_0, s_1, \dots, s_n\}$ such that $s_0 = s$, $s_n = t$, and $s_j \sim s_{j+1}$ for each j . The vertices s_1, \dots, s_{n-1} are called the *intermediate vertices* of the path. The *length* of c is n . We view $\{s\}$ as the unique path of length 0 from s to s . If $c = \{s_0, \dots, s_n\}$ and $c' = \{s'_0, \dots, s'_m\}$ are two paths in which $s_n = s'_0$, the product path cc' is the path from s_0 to s'_m whose first $n + 1$ vertices are the vertices of c and whose last $m + 1$ vertices are the vertices of c' . If s, t are two vertices, the unique path of minimal length joining s and t is called the *geodesic path* from s to t . We denote this path by $[s, t]$. The *distance* $d(s, t)$ is defined to be the length of the geodesic path from s to t . An *infinite path* is an infinite sequence $\{s_0, s_1, \dots\}$ of vertices such that consecutive vertices are neighbours. An infinite path is called an *infinite geodesic* or a *ray* if $s_{j-1} \neq s_{j+1}$ for every j .

We fix a vertex $e \in T$ and refer to it as the *root* of T . Let $s, t \in T$. The *length* of s , denoted by $|s|$, is defined to be $d(e, s)$. We say that s is *less or equal to* t , and write $s \leq t$, provided $s \in [e, t]$. Define the *sector* generated by s to be the set

$$T_s = \{t \in T : s \leq t\}.$$

For $s \neq e$, let s^- denote the unique vertex less or equal to s of length $|s| - 1$. A vertex having exactly one neighbour is called a *terminal* vertex.

Denote by Ω the set of infinite geodesics starting at e together with all finite geodesic paths from e to a terminal vertex. We denote a typical element of Ω by $\omega = \{\omega_0, \omega_1, \omega_2, \dots\}$. For $s \in T$ we say that s is *less or equal to* ω , and write $s \leq \omega$, if $s = \omega_n$ for some n . For $s \in T$, define the *interval* $I_s = \{\omega \in \Omega : s \leq \omega\}$. The set of all intervals $\{I_{\omega_n} : n \geq 0\}$ forms the base of neighbourhoods at ω of a compact topology on Ω . If T is given the discrete topology, Ω is the boundary of T in a compactification of T . A sequence

$\{t_n\}_{n \geq 0}$ in T converges to a boundary point ω in this topology if for every $M \in \mathbb{Z}^+$ there exists $N \in \mathbb{Z}^+$ such that $t_n \in T_{\omega_M}$ for all $n \geq N$.

For $s, t \in T$, we denote by $s \wedge t$ the unique vertex of largest length that is less or equal to both s and t . For $\omega \in \Omega$, we let $\omega \wedge s$ denote the unique vertex of maximal length that is less or equal to both s and ω .

Let p denote a *nearest neighbour transition probability* on $T \times T$. This means that $p : T \times T \rightarrow [0, 1]$, $p(s, t) = 0$ if s and t are not neighbours, $p(s, t) > 0$ if $s \sim t$, and

$$\sum_{t \in T} p(s, t) = 1$$

for every $s \in T$. If $c = \{s_0, s_1, \dots, s_n\}$ is a finite path in T , define

$$p(c) = \prod_{i=0}^{n-1} p(s_i, s_{i+1})$$

if $n \neq 0$, and $p(c) = 1$ if $n = 0$.

A *kernel* is any function that maps $T \times T$ into $[0, \infty]$. Let C be any set of paths, and let $C_{s,t}$ be the subset of paths in C from s to t . Then U_C , defined by

$$U_C(s, t) = \sum_{c \in C_{s,t}} p(c)$$

(where the empty sum is defined to be 0), is a kernel. In particular if we take C to be the set of paths of length 0 (respectively 1) we denote the resulting kernel by I (respectively P).

Let $f : T \rightarrow \mathbb{R}$. If U is a kernel, then $Uf : T \rightarrow \mathbb{R}$ is defined by

$$Uf(s) = \sum_{t \in T} U(s, t)f(t).$$

The *Laplacian* kernel is $\Delta = P - I$. Thus

$$\Delta f(s) = \sum_{t \in T} P(s, t)f(t) - f(s) = \sum_{t \sim s} p(s, t)f(t) - f(s).$$

If U', U'' are two kernels, then the product kernel $U'U''$ is defined to be

$$(U'U'')(s, t) = \sum_{r \in T} U'(s, r)U''(r, t).$$

Inductively we can define any finite power of a kernel.

To get further examples of kernels, let

- $\Gamma =$ the set of all finite paths in T ,
- $\Gamma' = \{(s_0, \dots, s_n) \in \Gamma : n > 0, s_j \neq s_n \text{ for all } j \in (0, n)\}$,
- $\Gamma'_k = \{(s_0, \dots, s_n) \in \Gamma' : d(s_j, s_n) \leq k \text{ for every } j\}$,

for k any positive integer. For these sets of paths, define the kernels

$$(1) \quad G(s, t) = U_{\Gamma}(s, t), \quad F(s, t) = U_{\Gamma'}(s, t), \quad F_k(s, t) = U_{\Gamma_k}(s, t).$$

G is referred to as the *Green function* of T .

The following result gives a useful way to calculate certain kernels.

THEOREM A ([C, p. 223]). *Let C, C', C'' be sets of paths. Suppose each path of the form $c'c''$ with $c' \in C'$ and $c'' \in C''$ is in C and conversely suppose that every path in C has a unique decomposition of this form. Then $U_C = U_{C'}U_{C''}$.*

By Theorem A, it follows that for any positive integer k , $P^k(s, t)$ is $\sum p(c)$, where the sum is taken over all paths from s to t of length k . Thus

$$G(s, t) = I(s, t) + \sum_{k=1}^{\infty} P^k(s, t).$$

It also follows using Theorem A that

$$(2) \quad G(s, t) = I(s, t) + (PG)(s, t) = I(s, t) + (GP)(s, t).$$

We assume throughout that for some $s, t \in T$, $G(s, t) < \infty$. This implies it is true for all $s, t \in T$. This condition on G is equivalent to the condition that the associated random walk is transient.

Let $s, t \in T$ with $s \neq t$ and let $\{s_0, s_1, \dots, s_n\}$ be the geodesic path from s to t . The following are shown using Theorem A and (2):

$$(3) \quad G(s, t) = F(s, t) G(t, t),$$

$$(4) \quad G(t, t) = \frac{1}{1 - F(t, t)},$$

$$(5) \quad F(s, t) = \prod_{i=0}^{n-1} F(s_i, s_{i+1}),$$

$$(6) \quad F(s, s) = \sum_{r \sim s} p(s, r) F(r, s).$$

It follows that

$$F(s, t) \leq 1 \text{ for all } s, t \in T.$$

Let $\omega \in \Omega$. The kernel $K_{\omega} : T \rightarrow \mathbb{R}$ is defined to be

$$(7) \quad K_{\omega}(s) = \frac{G(s, \omega \wedge s)}{G(e, \omega \wedge s)}.$$

It follows from (3) and (5) that for each $s \in T$, $\omega \mapsto K_{\omega}(s)$ is locally constant:

$$K_{\omega}(s) = \frac{G(s, t)}{G(e, t)}, \quad \text{for } \omega \wedge s = t \wedge s.$$

By a *distribution* on Ω we mean a finitely additive set function defined on finite unions of intervals. Any nonnegative distribution extends to a Borel measure on Ω . If μ is a distribution, the function $K\mu$ is defined on T by $K\mu(t) = \int_{\Omega} K_{\omega}(t) d\mu(\omega)$. This definition makes sense for any distribution because for each $t \in T$, $\omega \mapsto K_{\omega}(t)$ is constant on each member of a finite partition of Ω consisting of differences of intervals. Specifically, if $t \neq e$, we have by (3)

$$\begin{aligned}
 (8) \quad K\mu(t) &= \sum_{j=0}^{n-1} \int_{I_{t_j} - I_{t_{j+1}}} K_{\omega}(t) d\mu(\omega) + \int_{I_{t_n}} K_{\omega}(t) d\mu(\omega) \\
 &= \sum_{j=0}^{n-1} \frac{G(t, t_j)}{G(e, t_j)} \mu(I_{t_j} - I_{t_{j+1}}) + \frac{G(t, t_n)}{G(e, t_n)} \mu(I_{t_n}) \\
 &= \sum_{j=0}^{n-1} \frac{F(t, t_j)}{F(e, t_j)} \mu(I_{t_j} - I_{t_{j+1}}) + \frac{1}{F(e, t_n)} \mu(I_{t_n}),
 \end{aligned}$$

where $[e, t] = \{t_0, t_1, \dots, t_n\}$, and $K\mu(e) = \mu(\Omega)$. The integral $K\mu$ of a distribution is harmonic and conversely every harmonic function u is $K\mu$ of a unique distribution, which we shall denote by μ_u . If u is positive, the distribution μ_u extends to a measure on the Borel subsets of Ω .

The definition of K in terms of the Green function given in (7) and the integral representation of all positive harmonic functions described above imply that the geometric boundary Ω of T is the Martin boundary of T viewed as a harmonic space. Hence we shall refer to the topology we have defined on $T \cup \Omega$ as the *Martin topology* and we shall call K the *Martin kernel*.

A function $v : T \rightarrow \mathbb{R}$ is called *superharmonic* at a nonterminal vertex $s \in T$ provided $\Delta v(s) \leq 0$. For $E \subset T$, we let $\mathbf{S}(E)$ denote the set of functions superharmonic at each nonterminal vertex of E . We refer to an element of $\mathbf{S}(T)$ as a *superharmonic function*. The sets $\mathbf{s}(E) = \{v : -v \in \mathbf{S}(E)\}$ and $\mathbf{H}(E) = \mathbf{S}(E) \cap \mathbf{s}(E)$ denote the set of functions that are *subharmonic* and *harmonic* at each nonterminal vertex of E respectively. We define $S^+(E)$, $s^+(E)$ and $H^+(E)$ to be the elements of these sets that are positive.

Let $f : T \rightarrow [0, \infty)$. The *potential* of f is the function Gf , i.e.,

$$Gf(s) = \sum_{t \in T} G(s, t)f(t).$$

We refer to f as the *density* of the potential. Gf is finite at each vertex of T if and only if $\sum_{t \in T} G(e, t)f(t) < \infty$, in which case it defines a positive superharmonic function on T . Gf satisfies the Poisson equation $\Delta(Gf) = -f$ and so Gf is harmonic outside the support of f . In particular, the function $s \mapsto G(s, t)$ is positive, superharmonic on T , and harmonic outside of t . Every positive superharmonic function v on T can be uniquely written as the sum of a potential and a nonnegative harmonic function. The density of this potential

is the negative of the Laplacian of v . Thus potentials are precisely the positive superharmonic functions which majorize no positive harmonic function. The *harmonic support* of a potential Gf is the complement of the set of vertices where Gf is harmonic. The harmonic support of Gf equals the support of f .

If f is any real-valued function on T , $\liminf_{\infty} f$, $\limsup_{\infty} f$ and $\lim_{\infty} f$ refer to the upper limit, lower limit and limit of the function f at the point at ∞ respectively, where the neighbourhoods of ∞ are the complements of finite subsets of T .

We shall always assume that the Green function satisfies:

$$(9) \quad 0 < G(s, t) < \infty \quad \text{for all } s, t \in T.$$

This occurs if and only if there is a positive, superharmonic, nonharmonic function on T .

We shall also consider two conditions on the transition probabilities which imply that the Green function satisfies more stringent behaviour:

$$(10) \quad \lim_{s \rightarrow \infty} G(s, t) = 0 \quad \text{for some (equivalently every) } t \in T;$$

$$(11) \quad \text{there exists } 0 < \delta < 1/4 \text{ such that } \delta \leq p(s, t) \leq \frac{1}{2} - \delta.$$

A theorem of Picardello and Woess, which is proved in the appendix of [KPT], says that (11) implies that for all $s, t \in T$,

$$(12) \quad F(s, t) \leq \left(\frac{\frac{1}{2} - \delta}{\frac{1}{2} + \delta} \right)^{d(s,t)}$$

and

$$(13) \quad G(s, t) \leq M \left(\frac{\frac{1}{2} - \delta}{\frac{1}{2} + \delta} \right)^{d(s,t)},$$

where

$$M = \frac{1 + 2\delta}{4\delta}.$$

Examples of trees satisfying (11) are the homogeneous trees of degree $q \geq 3$ with isotropic transition probabilities: every vertex has exactly q neighbours and the transition probabilities are all given by $1/q$. Notice that for any tree satisfying (11), every vertex has at least three neighbours.

In general it is clear that (13) implies (10). Example 2.1 which follows shows that (10) need not hold, and Example 2.2 shows that (10) can hold without (13). We shall make it clear in each section which of (9), (10), or (11) we assume.

EXAMPLE 2.1. Let T be the set of rational integers. Let $1/2 < p < 1$ and let $q = 1 - p$. Define the transition probabilities $p(n, n+1) = p$, $p(n, n-1) = q$.

Let

$$v(n) = \begin{cases} \frac{1}{2p-1}, & n \leq 0, \\ \frac{1}{2p-1} \left(\frac{q}{p}\right)^n, & n > 0. \end{cases}$$

Then $\Delta v = -\delta_0$, so v is a positive, superharmonic, nonharmonic function on T . It follows that T has a finite Green function. Thus $F(n, n) < 1$ for all n . Clearly $F(n, n + 1) = F(n + 1, n + 2)$ and $F(n, n - 1) = F(n - 1, n - 2)$. Thus $F(n, n + 1) = p + q F(n - 1, n + 1) = p + q F^2(n, n + 1)$ and so $[qF(n, n + 1) - 1][F(n, n + 1) - 1] = 0$. Since $p/q > 1$, it follows that $F(n, n + 1) = 1$. We also have $F(n, n) = pF(n + 1, n) + qF(n - 1, n) = pF(n + 1, n) + q$. Since $F(n, n) < 1$, we deduce that $F(n + 1, n) < 1$. Using a similar argument as above, we get $[pF(n + 1, n) - q][F(n + 1, n) - 1] = 0$, and so $F(n + 1, n) = q/p$. Thus $F(n, n) = 2q = 2(1 - p)$, and so $G(n, n) = 1/(1 - F(n, n)) = 1/(2p - 1)$. It follows that

$$G(m, n) = \begin{cases} \frac{1}{2p-1}, & m \leq n, \\ \frac{1}{2p-1} \left(\frac{1-p}{p}\right)^{m-n}, & m > n. \end{cases}$$

In particular this gives a tree for which $\lim_{m \rightarrow -\infty} G(m, n) \neq 0$. □

EXAMPLE 2.2. Again we take T to be the set of integers. Let $\{x_n\}_{-\infty}^{\infty}$ be any positive sequence of real numbers such that $x_n \rightarrow 0$ as $n \rightarrow \pm\infty$, $x_0 = 2$, $x_{-1} = x_1 = 1$, $x_0 > x_1 > x_2 > \dots$ and $x_0 > x_{-1} > x_{-2} > \dots$. Define $p_n = \frac{x_n - x_{n-1}}{x_{n+1} - x_{n-1}}$ if $n \neq 0$ and let p_0 be any number strictly between 0 and 1. Then for all n , $0 < p_n < 1$. We define transition probabilities as follows: for $n \geq 1$ define $p(n, n + 1) = p_n$ and $p(n, n - 1) = 1 - p_n$; for $n \leq -1$ define $p(n, n - 1) = p_n$ and $p(n, n + 1) = 1 - p_n$; $p(0, 1) = p_0$ and $p(0, -1) = 1 - p_0$. Let $v(n) = x_n$. It is easy to see that $\Delta v = -\delta_0$, so as in Example 2.1 the Green function is finite. Since $v \rightarrow 0$ as $n \rightarrow \pm\infty$, it follows from the maximum principle that v cannot majorize a positive harmonic function, so v is a potential. Since v is harmonic outside of $\{0\}$, it follows that v is a multiple of the Green function: $x_n = v(n) = c G(n, 0)$. Thus by choosing $\{x_n\}$ to tend to 0 at an arbitrarily slow rate, we get examples of trees satisfying (10) but not (13). □

3. Growth of positive superharmonic functions

ASSUMPTION IN THIS SECTION. *The Green function satisfies condition (9).*

We show now that positive superharmonic functions grow at maximum and minimum rates determined by the kernels F and F_k (recall (1)).

THEOREM 3.1. *Let $s, t \in T$, with $s \sim t$. For $k \geq 1$, let $T_k(s, t) = \{w \in T : t \in [s, w], 1 \leq d(s, w) \leq k\}$. Let $T(s, t) = T_\infty(s, t) = \{w \in T : t \in [s, w]\}$.*

If $v \in \mathbf{S}^+(T_k(s, t))$ then

$$(14) \quad v(t) > F_k(t, s) v(s),$$

if $v \in \mathbf{S}^+(T(s, t))$ then

$$(15) \quad v(t) \geq F(t, s) v(s)$$

and if $v \in \mathbf{S}^+(T)$ then

$$(16) \quad F(t, s) v(s) \leq v(t) \leq \frac{v(s)}{F(s, t)}.$$

Proof. We first note some facts about $F(t, s)$ and $F_k(t, s)$. It is immediate from the definition of F_1 that $F_1(t, s) = p(t, s)$. Let t_1, \dots, t_n be the neighbours of t other than s . Let $k \geq 2$. Applying Theorem A twice we get

$$\begin{aligned} F_k(t, s) &= p(t, s) + \sum_{j=1}^n p(t, t_j) F_k(t_j, s) \\ &= p(t, s) + \sum_{j=1}^n p(t, t_j) F_{k-1}(t_j, t) F_k(t, s). \end{aligned}$$

We deduce from this that

$$(17) \quad 1 - \sum_{j=1}^n F_{k-1}(t_j, t) p(t, t_j) > 0$$

and

$$(18) \quad F_k(t, s) = \frac{p(t, s)}{1 - \sum_{j=1}^n F_{k-1}(t_j, t) p(t, t_j)}.$$

We proceed to prove formula (14) by induction on k . If $v \in \mathbf{S}^+(T_1(s, t)) (= S^+(\{t\}))$ then

$$v(t) \geq \sum_{j=1}^n p(t, t_j) v(t_j) + p(t, s) v(s) > p(t, s) v(s) = F_1(t, s) v(s),$$

proving the formula for $k = 1$. Suppose $v \in \mathbf{S}^+(T_k(s, t)), k \geq 2$. Then $v \in \mathbf{S}^+(T_{k-1}(t, t_j))$, for each $j = 1, \dots, n$, so by the inductive hypothesis we have $v(t_j) > F_{k-1}(t_j, t) v(t), j = 1, \dots, n$. Thus

$$\begin{aligned} v(t) &\geq \sum_{j=1}^n p(t, t_j) v(t_j) + p(t, s) v(s) \\ &> \sum_{j=1}^n p(t, t_j) F_{k-1}(t_j, t) v(t) + p(t, s) v(s). \end{aligned}$$

Applying this together with (17) and (18), we get

$$v(t) > \frac{p(t, s) v(s)}{1 - \sum_{j=1}^n p(t, t_j) F_{k-1}(t_j, t)} = F_k(t, s) v(s)$$

which proves (14). Formula (15) follows by letting k go to ∞ , and formula (16) follows by interchanging s and t in (15). □

In the following theorem we deduce as a consequence of Theorem 3.1 that among all positive superharmonic functions, the Martin kernel K_ω grows as quickly as possible along the vertices of ω and decreases as rapidly as possible outside the vertices of ω as we move away from the root.

THEOREM 3.2. *Let $\omega \in \Omega$. Let c_1 be the geodesic path ω and, for $n \geq 0$, let c_2 be a geodesic path starting at ω_n consisting of vertices in $T_{\omega_n} - T_{\omega_{n+1}}$. Let s, t be any pair of vertices in c_1 (respectively c_2) with $s \sim t$ and $s \leq t$. Among all positive superharmonic functions v , the ratio $v(t)/v(s)$ is maximized (respectively minimized) if $v = K_\omega$ is the Martin kernel.*

Proof. Suppose first $s = \omega_j$ and $t = \omega_{j+1}$. We have

$$K_\omega(\omega_j) = \frac{G(\omega_j, \omega_j)}{G(e, \omega_j)} = \frac{G(\omega_j, \omega_j)}{F(e, \omega_j)G(\omega_j, \omega_j)} = \frac{1}{F(e, \omega_j)},$$

so

$$\begin{aligned} K_\omega(\omega_{j+1}) &= \frac{1}{F(e, \omega_{j+1})} = \frac{1}{F(e, \omega_j)F(\omega_j, \omega_{j+1})} \\ &= \frac{K_\omega(\omega_j)}{F(\omega_j, \omega_{j+1})}, \end{aligned}$$

which by Theorem 3.1 proves the first assertion.

Suppose now $s, t \in c_2$. We have $s \wedge \omega = t \wedge \omega = \omega_n$, so by (3) and (5)

$$\begin{aligned} K_\omega(t) &= \frac{G(t, \omega_n)}{G(e, \omega_n)} = \frac{F(t, \omega_n)}{F(e, \omega_n)} = \frac{F(t, s)F(s, \omega_n)}{F(e, \omega_n)} \\ &= F(t, s)K_\omega(s). \end{aligned}$$

The result follows by Theorem 3.1. □

4. Dirichlet problem and the reduced function

ASSUMPTION IN THIS SECTION. *The Green function satisfies conditions (9) and (10).*

If u, v are two functions defined on a subset S of T , $u \leq v$ (respectively $u = v$) means that $u(s) \leq v(s)$ (respectively $u(s) = v(s)$) for every $s \in S$.

PROPOSITION 4.1 (Minimum Principle). *Let E be a finite subset of T . Let v, f be functions on T such that $v \geq f$ on E , v is superharmonic at every vertex of $T - E$ and $\liminf_\infty v \geq 0$. Then $v \geq \min(0, \min(f))$.*

Proof. Suppose there exists a vertex t such that $v(t) < \min(0, \min(f))$. Since $\liminf_{\infty} v \geq 0$, it follows that v assumes its absolute minimum at some point t_0 of T . If t_0 were an element of $T - E$, the superharmonicity of v at t_0 would imply $v(t) = v(t_0)$ for each neighbour t of t_0 . By considering the path from t_0 to the nearest point of E , we deduce that there exists a point t of E such that $v(t) = v(t_0) < \min(0, \min(f))$. Since this is impossible, the result follows. \square

Let E be a finite subset of T . For $s, t \in T$, define

$$F^E(s, t) = \sum p(c),$$

where the sum extends over all paths from s to t such that t is the first vertex of c in E . Then

$$(19) \quad \begin{aligned} F^E(s, t) &\leq F(s, t), \\ F^E(s, t) &= 0 \text{ if } t \notin E, \\ F^E(s, t) &= \delta_s(t) \text{ if } s, t \in E, \\ s \mapsto F^E(s, t) &\text{ is harmonic outside } E. \end{aligned}$$

For the last property, let $[s, t]$ denote the geodesic from $s \in T - E$ to $t \in T$. If $[s, t]$ contains a vertex of E other than t , then the same is true for $[s', t]$ where $s \sim s'$, and so $F^E(s, t) = F^E(s', t) = 0$; if $[s, t]$ does not contain a vertex of E other than t , the same is true for $[s', t]$ and it is easy to see in this case that the mean value property holds. Thus the last property in (19) holds.

For E a finite subset of T , we define

$$F^E f(s) = \sum_{t \in E} F^E(s, t) f(t).$$

From (19), the fact that E is finite, the minimum principle and (10) we deduce that $F^E f$ is the solution of the Dirichlet problem on $T - E$ with boundary function f :

PROPOSITION 4.2. *$F^E f$ is the unique function on T which equals f on E , is harmonic on $T - E$ and has a limit of 0 at ∞ .*

In the special case that f is superharmonic on T , we deduce the following.

PROPOSITION 4.3. *Let v be positive superharmonic on T . Then $F^E v \leq v$ and $F^E v$ is superharmonic on T .*

Proof. We have seen that $v = F^E v$ on E . The function $w = v - F^E v$ is superharmonic on $T - E$, is 0 on E and $\liminf_{\infty} w \geq 0$ since $F^E \leq G$. Thus by the minimum principle $w \geq 0$, proving that $F^E v \leq v$ on T . We already

know that $F^E v$ is harmonic on $T - E$. If $s \in E$,

$$\sum_{s' \sim s} p(s, s') F^E v(s') \leq \sum_{s' \sim s} p(s, s') v(s') \leq v(s) = F^E v(s),$$

proving that $F^E v$ is superharmonic on T . \square

We deduce the following explicit formula for the solution of the Dirichlet problem in case E is finite.

THEOREM 4.1. *If $E = \{s_1, \dots, s_n\}$, then the matrix M given by $M_{ij} = G(s_i, s_j)$ is invertible, and*

$$(20) \quad F^E f(s) = \sum_{i,j} G(s, s_i) M_{ij}^{-1} f(s_j).$$

Proof. Set $C_{ij} = -\Delta(s \mapsto F^E(\cdot, s_j))|_{s=s_i}$. Then the function

$$(21) \quad h_j(s) = \sum_i G(s, s_i) C_{ij} - F^E(s, s_j)$$

is harmonic on T and tends to 0 at ∞ . By the minimum principle, h_j is identically 0. By taking s to be in E and applying (19) we deduce that MC is the identity matrix. Replacing C_{ij} by M_{ij}^{-1} in (21) completes the proof. \square

The following is a consequence of the theorem. In the classical setting the result is one of approximation (see Lemma 6.1 in [H]) whereas on trees it is exact.

COROLLARY 4.1. *Every function defined on a finite set E of T extends uniquely to T as the difference of two positive potentials with harmonic support in E .*

Proof. The uniqueness follows from the fact that any such difference of positive potentials necessarily equals $F^E f$ by the minimum principle. The existence follows from (20). \square

REMARK 4.1. We recall that in \mathbb{R}^n a compact polar set K (with more than one point) is characterized by the fact that every positive continuous function on K is the restriction to K of a positive continuous potential [W]. See [J] for a generalization to Brelot spaces. Such a result is not meaningful for trees with positive potentials since there are no polar sets in such a setting. It also follows immediately from Theorem 3.1 that we cannot expect to write every positive function on a finite set of vertices as the restriction of a potential.

We now define the *reduced function*, which plays a key role in the development of the minimal fine topology.

DEFINITION 4.1. Let $E \subset T$ (not necessarily finite) and let $v \in \mathbf{S}^+(T)$. For each $t \in T$, define the reduced function by

$$R_v^E(t) = \inf\{w(t) : w \in \mathbf{S}^+(T), w(s) \geq v(s) \text{ for all } s \in E\}.$$

We collect the elementary properties of the reduced function in the following proposition. We omit the proofs as they are either similar to those in the abstract setting (see for example [B]) or follow easily.

PROPOSITION 4.4. Let $v \in \mathbf{S}^+(T)$ and let $E \subset T$.

- (a) R_v^E is positive and superharmonic at each vertex of T , harmonic at each vertex of $T - E$, agrees with v at each vertex of E and is less than or equal to v on T .
- (b) Let $E, F, \{G_n\}_{n \geq 1}$ be subsets of T with $E \subset F$, let $c \in \mathbb{R}$, and let $w \in \mathbf{S}^+(T)$ with $v \leq w$. Then $R_v^E \leq R_v^F$, $R_v^{\bigcup_n G_n} \leq \sum_n R_v^{G_n}$, $R_v^E \leq R_w^E$, and $R_{cv}^E = cR_v^E$.
- (c) If $\{E_j\}_j$ is an increasing sequence of subsets of T with union E , then for each $t \in T$,

$$\lim_{j \rightarrow \infty} R_v^{E_j}(t) = R_v^E(t).$$

- (d) If $\{w_j\}_j$ is an increasing sequence of positive superharmonic functions on T whose limit w is finite (and hence superharmonic), then, for every subset E of T ,

$$\lim_{j \rightarrow \infty} R_{w_j}^E(t) = R_w^E(t).$$

In case E is finite, the following Corollary shows that the reduced function is the solution of the Dirichlet problem and so can be represented by an explicit formula.

COROLLARY 4.2. In case $E = \{s_1, \dots, s_n\}$ is a finite subset of T and v is positive superharmonic on T , then $R_v^E = F^E v$ on T , and so is given by formula (20):

$$(22) \quad R_v^E(s) = \sum_{i,j} G(s, s_i) M_{i,j}^{-1} v(s_j).$$

Proof. Since $F^E v$ is a positive superharmonic function on T which equals v on E , by definition $F^E v \geq R_v^E$. On the other hand, if w is a positive superharmonic function on T such that $w \geq v$ on E , then by the minimum principle $w \geq F^E v$ on T , so taking the inf over all such w gives $R_v^E \geq F^E v$. \square

By combining formula (22) with the limit results in Proposition 4.4, we can deduce many properties of the reduced function of a superharmonic function.

In the next theorem we use this technique to show that the reduced function satisfies certain additivity properties. We also show that the processes of calculating the reduced function and writing the integral representation of a positive harmonic function commute.

THEOREM 4.2.

- (a) Let $\{v_k\}_{k \geq 1}$ be a sequence of positive superharmonic functions on T such that the sum $v(s) = \sum_{k=1}^{\infty} v_k(s)$ is finite and so defines a superharmonic function. Let E be any subset of T . Then

$$(23) \quad R_v^E = \sum_{k=1}^{\infty} R_{v_k}^E.$$

- (b) If Gf is a potential, then for each $t \in T$,

$$(24) \quad R_{Gf}^E(t) = \sum_{s \in T} f(s) R_{G(\cdot, s)}^E(t).$$

- (c) Let u be positive harmonic, μ_u its representing measure and $E \subset T$. Then for every $t \in T$,

$$(25) \quad R_u^E(t) = \int R_{K_\omega}^E(t) d\mu_u(\omega).$$

Proof. (a) Suppose first that $E = \{s_1, \dots, s_n\}$ is finite. By (22), we have

$$\begin{aligned} R_v^E(t) &= \sum_{i=1}^n \sum_{j=1}^n M_{ij}^{-1} v(s_j) G(t, s_i) \\ &= \sum_{i=1}^n \sum_{j=1}^n M_{ij}^{-1} \sum_{k=1}^{\infty} v_k(s_j) G(t, s_i) \\ &= \sum_{k=1}^{\infty} \sum_{i=1}^n \sum_{j=1}^n M_{ij}^{-1} v_k(s_j) G(t, s_i) \\ &= \sum_{k=1}^{\infty} R_{v_k}^E(t), \end{aligned}$$

so (23) holds in case E is finite. In the general case, there exists an increasing sequence of finite sets E_m whose union is E . Then, by Proposition 4.4,

$$\begin{aligned}
 R_v^E(t) &= \lim_{m \rightarrow \infty} R_v^{E_m}(t) \\
 &= \lim_{m \rightarrow \infty} \sum_{j=1}^{\infty} R_{v_j}^{E_m}(t) \\
 &= \lim_{m \rightarrow \infty} \lim_{J \rightarrow \infty} \sum_{j=1}^J R_{v_j}^{E_m}(t) \\
 &= \lim_{J \rightarrow \infty} \lim_{m \rightarrow \infty} \sum_{j=1}^J R_{v_j}^{E_m}(t) \\
 &= \lim_{J \rightarrow \infty} \sum_{j=1}^J R_{v_j}^E(t) \\
 &= \sum_{j=1}^{\infty} R_{v_j}^E(t),
 \end{aligned}$$

which is (23).

(b) Since $Gf(t) = \sum_{s \in T} G(t, s)f(s)$, (24) holds by (23) and (b) of Proposition 4.4.

(c) If E is finite, then (25) holds by replacing v by K_ω in formula (22) and then integrating with respect to μ_u . It holds in general by applying the monotone convergence theorem and (c) of Proposition 4.4. \square

As another application of formula (22) we give a simple proof of the domination principle.

THEOREM 4.3 (Domination Principle). *Let Gf be a potential having harmonic support E . If v is any positive superharmonic function that majorizes Gf on E , then $v \geq Gf$ on T .*

Proof. We claim that if $s \in E$, then for every $t \in T$,

$$R_{G(\cdot, s)}^E(t) = G(t, s).$$

To see this, suppose first that $E = \{s_1, \dots, s_n\}$ is finite. Let $s = s_{i_0} \in E$. Since

$$\sum_{j=1}^n M_{ij}^{-1} \cdot G(s_j, s_{i_0}) = \delta_{i i_0},$$

it follows that

$$\begin{aligned} R_{G(\cdot, s_{i_0})}^E(t) &= \sum_{i=1}^n \sum_{j=1}^n M_{ij}^{-1} G(s_j, s_{i_0}) G(t, s_i) \\ &= \sum_{i=1}^n \delta_{ii_0} G(t, s_i) \\ &= G(t, s_{i_0}). \end{aligned}$$

If E is any subset of T and $s_{i_0} \in E$, choose an increasing sequence $\{E_m\}$ of finite subsets of T with union E such that each set in the sequence contains s_{i_0} . Then

$$R_{G(\cdot, s_{i_0})}^{E_m}(t) = G(t, s_{i_0}).$$

Letting $m \rightarrow \infty$ and applying (c) of Proposition 4.4 completes the proof of the claim.

To finish the proof, we need to show that $R_{Gf}^E = Gf$. This follows from Theorem 4.2:

$$R_{Gf}^E(t) = \sum_{s \in E} f(s) R_{G(\cdot, s)}^E(t) = \sum_{s \in E} f(s) G(t, s) = Gf(t). \quad \square$$

5. Minimally thin sets and the minimal fine filter

ASSUMPTION IN THIS SECTION. *The Green function satisfies conditions (9) and (10). We shall at times assume that the transition probabilities satisfy (11), and hence G satisfies (13).*

Let $\omega \in \Omega$. By the uniqueness of integral representation of positive harmonic functions, K_ω is *minimal* in the sense that any positive harmonic minorant of it is necessarily a multiple of K_ω . From this, it follows that if $E \subset T$, then the reduced function $R_{K_\omega}^E$ is either identically K_ω or a potential.

DEFINITION 5.1. Let $\omega \in \Omega$ and let $E \subset T$. We say that E is *minimally thin* at ω if $R_{K_\omega}^E$ is a potential. Equivalently, this occurs if $R_{K_\omega}^E$ is not identically equal to K_ω . Thus E is minimally thin at ω if and only if there exists a positive superharmonic function which majorizes K_ω on E but not on all of T .

We collect some properties of minimally thin sets in the next result.

PROPOSITION 5.1. *Let $\omega \in \Omega$. For brevity, in what follows, the term “minimally thin” means minimally thin at ω .*

- (a) *The class of minimally thin sets is closed under the operations of taking subsets and finite unions.*

- (b) For each $n \geq 0$, $T - T_{\omega_n}$ is minimally thin and T_{ω_n} is not minimally thin.
- (c) E is minimally thin if and only if

$$\lim_{n \rightarrow \infty} R_{K_\omega}^{E \cap T_{\omega_n}}(e) = 0.$$

- (d) Let $\{E_n\}$ be a sequence of sets, each minimally thin. If $\sum_n R_{K_\omega}^{E_n}(e) < \infty$, then $E = \bigcup_n E_n$ is minimally thin.

Proof. (a) The proof is obvious.

(b) Fix any $n \geq 0$. The reduced function of any finite set is a potential. It follows by Theorem 3.2 that the potential $R_{K_\omega}^{\{\omega_0, \dots, \omega_n\}}$ majorizes K_ω on $\bigcup_{j=0}^n (T_{\omega_j} - T_{\omega_{j+1}}) = T - T_{\omega_{n+1}}$. Thus $T - T_{\omega_{n+1}}$ is minimally thin. It is now obvious that T_{ω_n} is not minimally thin for any n .

(c) Suppose first that E is minimally thin. The function $\lim_{n \rightarrow \infty} R_{K_\omega}^{E \cap T_{\omega_n}}$ is a harmonic minorant of the potential $R_{K_\omega}^E$, and so is 0 on T . Conversely, suppose E is not minimally thin. Then by (a) and (b), $E \cap T_{\omega_n}$ is not minimally thin for every n . Thus $R_{K_\omega}^{E \cap T_{\omega_n}} = K_\omega$, and so the limit in the statement is not 0.

(d) Choose N so that $\sum_{n \geq N} R_{K_\omega}^{E_n}(e) < K_\omega(e)$. Then the countable subadditivity of the reduced function implies $\bigcup_{n \geq N} E_n$ is minimally thin. It follows that E is minimally thin since it is the finite union of minimally thin sets: $E = \bigcup_{n \leq N-1} E_n \cup \bigcup_{n \geq N} E_n$. \square

The following result gives further examples of sets which are not minimally thin.

THEOREM 5.1. Fix $\omega = \{\omega_0, \omega_1, \dots\} \in \Omega$.

- (a) Any subsequence $\{\omega_{j_k}\}_k$ is not minimally thin at ω .
- (b) Suppose that the transition probabilities satisfy (11). Let $\{n_j\}_{j \geq 1}$ be any increasing sequence of positive integers and let n be a positive integer. For each j , let t_j be any vertex in $T_{\omega_{n_j}} - T_{\omega_{n_j+1}}$ such that $d(t_j, \omega_{n_j}) \leq n$. Then $E = \bigcup_{j=1}^\infty \{t_j\}$ is not minimally thin at ω .

Proof. Choose v , a positive superharmonic function that majorizes K_ω on $\{\omega_{j_k} : k \geq 1\}$. According to Theorem 3.2, K_ω increases at least as fast along $\{\omega_0, \omega_1, \omega_2, \dots\}$ as v , thus v must majorize K_ω on all of $\{\omega_0, \omega_1, \omega_2, \dots\}$. For any $n \geq 0$, let $\{s_j\}$ be any infinite geodesic in $T_{\omega_n} - T_{\omega_{n+1}}$ starting at ω_n . Again by Theorem 3.2, K_ω decreases at least as fast along $\{s_j\}$ as v , so v must majorize K_ω on $\{s_j\}$. We deduce that $v \geq K_\omega$ on T , and so $R_{K_\omega}^E = K_\omega$, proving that $\{\omega_{j_k} : k \geq 1\}$ is not minimally thin at ω .

Assume now that the transition probabilities satisfy (11). We may assume without loss of generality that $d(t_j, \omega_{n_j}) = n$ for every j , since there exists a

subsequence for which these distances are all the same. We show that $R_{K_\omega}^E$ is not a potential. Suppose on the contrary that

$$R_{K_\omega}^E = \sum_{j=1}^{\infty} G(\cdot, t_j) f(t_j),$$

where $\sum_{j=1}^{\infty} G(e, t_j) f(t_j) < \infty$. Fix a positive integer k . Recalling (11)–(13), we have

$$\frac{G(\omega_{n_j}, t_j)}{G(t_j, t_j)} = F(\omega_{n_j}, t_j) \geq \delta^n$$

and for $j \neq k$

$$\frac{G(\omega_{n_k}, t_j)}{G(t_k, t_j)} = \frac{F(\omega_{n_k}, t_j)}{F(t_k, t_j)} = \frac{1}{F(t_k, \omega_{n_k})} \geq \left(\frac{\frac{1}{2} + \delta}{\frac{1}{2} - \delta}\right)^n > \delta^n.$$

Also

$$\frac{K_\omega(t_k)}{K_\omega(\omega_{n_k})} = \frac{G(t_k, \omega_{n_k})}{G(\omega_{n_k}, \omega_{n_k})} = F(t_k, \omega_{n_k}) \geq \delta^n.$$

Thus

$$\begin{aligned} R_{K_\omega}^E(\omega_{n_k}) &= \sum_{j=1}^{\infty} G(\omega_{n_k}, t_j) f(t_j) \\ &= \sum_{j=1}^{\infty} G(t_k, t_j) \frac{G(\omega_{n_k}, t_j)}{G(t_k, t_j)} f(t_j) \\ &\geq \sum_{j=1}^{\infty} \delta^n G(t_k, t_j) f(t_j) \\ &= \delta^n R_{K_\omega}^E(t_k) \\ &= \delta^n K_\omega(t_k) \\ &\geq \delta^{2n} K_\omega(\omega_{n_k}) \end{aligned}$$

and so $\delta^{-2n} R_{K_\omega}^E$ is a potential that majorizes K_ω on $\bigcup_{j=1}^{\infty} \{\omega_{n_j}\}$. This contradicts the result of the previous paragraph. Thus E is not minimally thin at ω . \square

We now show how to generate examples of sets that are minimally thin.

THEOREM 5.2. *Let $\omega = \{\omega_0, \omega_1, \omega_2, \dots\} \in \Omega$.*

- (a) *For each n choose $x_n \in T_{\omega_n} - T_{\omega_{n+1}}$ such that $\sum_n G(x_n, \omega_n) < \infty$. Then $E = \bigcup_n \{x_n\}$ is minimally thin at ω .*
- (b) *Assume in addition that the transition probabilities satisfy (11). Let $\{d_n\}_{n \geq 1}, \{M_n\}_{n \geq 1}$ be two sequences of positive integers such that $M_n \leq \left(\frac{1}{\delta} - 1\right)^{d_n}$ and $\sum_{n=1}^{\infty} M_n \left(\frac{\frac{1}{2} - \delta}{\frac{1}{2} + \delta}\right)^{2d_n} < \infty$. For each n , choose*

at most M_n vertices of $T_{\omega_n} - T_{\omega_{n+1}}$ a distance d_n from ω_n . Denote each of them by $t_{nj}, j = 1, \dots, M_n$. Let $E = \bigcup_{n=1}^{\infty} \bigcup_{j=1}^{M_n} T_{t_{nj}}$. Then E is minimally thin at ω .

Proof. (a) Define

$$f(x) = \sum_{n=1}^{\infty} \frac{G(x, x_n)}{G(x_n, x_n)} \frac{G(x_n, \omega_n)}{G(e, \omega_n)}.$$

Since $F(e, x_n) \leq 1 \leq G(e, \omega_n)$,

$$\frac{G(e, x_n)}{G(x_n, x_n)} \frac{G(x_n, \omega_n)}{G(e, \omega_n)} = F(e, x_n) \frac{G(x_n, \omega_n)}{G(e, \omega_n)} \leq G(x_n, \omega_n).$$

It follows that $f(e) < \infty$, so f is a potential. Also

$$f(x_n) \geq \frac{G(x_n, \omega_n)}{G(e, \omega_n)} = K_{\omega}(x_n).$$

Thus E is minimally thin at ω .

(b) To prove this, define $f : T \rightarrow \infty$ with support on $\bigcup_{n=1}^{\infty} \bigcup_{j=1}^{M_n} \{t_{nj}\}$ by

$$f(t_{nj}) = \frac{K_{\omega}(t_{nj})}{G(t_{nj}, t_{nj})}.$$

Then

$$\begin{aligned} \sum_{t \in T} G(e, t) f(t) &= \sum_{n=1}^{\infty} \sum_{j=1}^{M_n} G(e, t_{nj}) \frac{K_{\omega}(t_{nj})}{G(t_{nj}, t_{nj})} \\ &= \sum_{n=1}^{\infty} \sum_{j=1}^{M_n} F(t_{nj}, \omega_n) F(\omega_n, t_{nj}) \\ &\leq \sum_{n=1}^{\infty} M_n \left(\frac{\frac{1}{2} - \delta}{\frac{1}{2} + \delta} \right)^{2d_n} \\ &< \infty, \end{aligned}$$

so Gf is a potential. Also for each n, j

$$Gf(t_{nj}) \geq G(t_{nj}, t_{nj}) \frac{K_{\omega}(t_{nj})}{G(t_{nj}, t_{nj})} = K_{\omega}(t_{nj}),$$

so $Gf \geq K_{\omega}$ on $\bigcup_{n,j} \{t_{nj}\}$. By Theorem 3.2, Gf majorizes K_{ω} on all of E . This completes the proof. \square

We next define the *minimal fine filter* corresponding to a boundary point $\omega \in \Omega$.

DEFINITION 5.2. For each $\omega \in \Omega$, define

$$\mathcal{F}_\omega = \{T - E : E \text{ minimally thin at } \omega\}.$$

By Proposition 5.1, \mathcal{F}_ω is a filter which we call the *minimal fine filter* corresponding to ω . We refer to limits following this filter as *minimal fine limits*. Thus if $f : T \rightarrow \mathbb{R}$ and $L \in \mathbb{R}$, we write

$$\text{mfine } \lim_{t \rightarrow \omega} f(t) = L$$

if for every $\epsilon > 0$ there exists $F \in \mathcal{F}_\omega$ such that $|f(t) - L| < \epsilon$ for all $t \in F$. We define minimal fine limits of $\pm\infty$ as well as minimal fine limsup and liminf in the obvious way.

It follows from Proposition 5.1 that the minimal fine filter at ω is at least as fine as the neighbourhood filter defined by the Martin topology. On the other hand, let E be the set in Theorem 5.2(a). Then $T - E$ is in the minimal fine filter \mathcal{F}_ω , but it is not a Martin neighbourhood of ω since it does not contain T_{ω_n} for any $n \geq 1$. Thus we have shown:

THEOREM 5.3. *The minimal fine filter together with the discrete topology on T defines a topology that is strictly finer than the Martin topology on $T \cup \Omega$.*

The following result gives a more intuitive sense of what it means for a function to have a minimal fine limit at a boundary point.

PROPOSITION 5.2. *Let $f : T \rightarrow \mathbb{R}$, $\omega \in \Omega$. The minimal fine limit of f at ω is L if and only if there exists a set E minimally thin at ω such that $f(t)$ converges to L as t converges in $T - E$ to ω with respect to the Martin topology.*

Proof. The condition is clearly sufficient for the minimal fine limit to equal L . Suppose then that the minimal fine limit of f at ω is L . For each $n \geq 1$ there exists E_n minimally thin at ω and $m(n)$ such that $|f(t) - L| < 1/n$ for all $t \in T_{\omega_{m(n)}} - E_n$. By Proposition 5.1, $R_{K_\omega}^{T_{\omega_j} \cap E_n}(e) \rightarrow 0$ as $j \rightarrow \infty$. Thus we can assume $m(n)$ is chosen so that $R_{K_\omega}^{T_{\omega_{m(n)}} \cap E_n}(e) < 1/2^n$. Let $E = \bigcup_n (E_n \cap T_{\omega_{m(n)}})$. By Proposition 5.1(d), E is minimally thin at ω and it has the property of the set in the statement. \square

We now define approach regions that are more geometric in nature than the approach regions of the minimal fine filter.

DEFINITION 5.3. Let $\omega \in \Omega$. Let $\alpha \geq 0$. The nontangential region at ω of aperture α is

$$S_\alpha(\omega) = \{t \in T : t \geq \omega_n \text{ for some } n \text{ and } d(t, \omega_n) \leq \alpha\}.$$

A function is said to have a nontangential (respectively radial) limit of L at ω if for all $\alpha \geq 0$ (respectively for $\alpha = 0$) and for all $\epsilon > 0$ there exists n such that $|f(t) - L| < \epsilon$ whenever $t \in T_{\omega_n} \cap S_\alpha(\omega)$.

For the rest of this section we consider some of the connections between minimal fine, radial and nontangential limits. In particular we show in the next theorem that if the transition probabilities satisfy (11), then a minimal fine limit at a boundary point implies an equal nontangential limit there. This is in contrast to classical potential theory on a halfspace in \mathbb{R}^n where such a result holds for positive harmonic functions but not for arbitrary functions [BD].

THEOREM 5.4. *Let $f : T \rightarrow \mathbb{R}$ be any function and let $\omega \in \Omega$. Suppose that $\text{mfine lim}_{t \rightarrow \omega} f(t) = L$ exists and is finite.*

- (a) *Then the radial limit of f exists and equals L .*
- (b) *Suppose in addition that the transition probabilities satisfy (11). Then the minimal fine filter at ω is strictly coarser than the filter of nontangential neighbourhoods of ω . Thus the nontangential limit of f exists and equals L , and there exist functions having a finite nontangential limit but not a minimal fine limit at ω .*

Proof. (a) Suppose f does not have a radial limit of L at ω . Then there exist $\epsilon > 0$ and an increasing sequence of integers $\{n_j\}$, such that $|f(\omega_{n_j}) - L| > \epsilon$ for all j . By assumption there exists $F \in \mathcal{F}_\omega$ such that $|f(t) - L| < \epsilon$ for all $t \in F$. By Theorem 5.1(a) $\bigcup_{j \geq 1} \{\omega_{n_j}\}$ is not minimally thin at ω , and so, by Proposition 5.1(a), it intersects F . Thus there exists $j \geq 1$ such that $|f(\omega_{n_j}) - L| < \epsilon$. This contradiction completes the proof of (a).

(b) The proof that f has a nontangential limit of L at ω follows from a similar argument and the use of part (b) rather than part (a) of Theorem 5.1.

We now show that there exists a set E that is not minimally thin at ω and which is not contained in $S_\alpha(\omega)$ for any aperture α . Let $0 = n_0 < n_1 < n_2 < \dots$ be an increasing sequence of integers, to be determined later. Let $\{t_j\}_{j \geq 1}$ be any sequence of vertices such that $t_j \in T_{\omega_j} - T_{\omega_{j+1}}$ and for each $k \geq 1$, if $n_{k-1} + 1 \leq j \leq n_k$, then $d(t_j, \omega_j) = k$. Let $E_k = \{t_j : n_{k-1} + 1 \leq j \leq n_k\}$. By Proposition 4.4(c) and Theorem 5.1(b), we can choose such sequences $\{n_k\}_{k \geq 0}$ and $\{t_j\}_{j \geq 1}$ such that for each $k \geq 1$,

$$R_{K_\omega}^{E_k}(\omega_k) \geq \left(1 - \frac{1}{k}\right) K_\omega(\omega_k).$$

By Theorem 3.2, it follows that

$$(26) \quad R_{K_\omega}^{E_k}(\omega_m) \geq \left(1 - \frac{1}{k}\right) K_\omega(\omega_m) \quad \text{for all } 1 \leq m \leq k.$$

Let $E = \cup_{k \geq 1} E_k$. Then by (26), we have for each $k \geq m \geq 1$ that

$$R_{K_\omega}^E(\omega_m) \geq R_{K_\omega}^{E_k}(\omega_m) \geq \left(1 - \frac{1}{k}\right) K_\omega(\omega_m).$$

Since k is arbitrary, $R_{K_\omega}^E(\omega_m) \geq K_\omega(\omega_m)$ for each m . By Theorem 5.1(a), $R_{K_\omega}^E \equiv K_\omega$, so E is not minimally thin at ω . Since there are vertices in E at an arbitrarily large distance from ω , E cannot be contained in any nontangential region.

Finally, define g on T by $g(t) = d(t, \omega \wedge t)$ if $t \in E$ and $g(t) = 0$ otherwise. Then the minimal fine limsup of g at ω is ∞ and, since E intersects any nontangential region in at most finitely many points, the nontangential limit of g at ω is 0. □

In the following example we show that a tree having a Green function that does not satisfy (11) may not satisfy the conclusion of Theorem 5.4(b): a function on T can have a minimal fine limit at a boundary point ω without having a nontangential limit there. The idea comes from Theorem 5.2(a), which suggests that if the transition probabilities decrease fast enough as we move away from the vertices of ω , then the nontangential region $S_\alpha(\omega)$ with the vertices of ω removed, $S_\alpha(\omega) - \{\omega_n : n \geq 1\}$, can be minimally thin at ω . Example 5.1 demonstrates that this is indeed the case.

EXAMPLE 5.1. In this example we construct a tree T having transition probabilities satisfying (9) and (10) for which any function has a limit at a boundary point following the minimal fine filter if and only if it has a radial limit there and for which there exists a positive superharmonic function on T that has a minimal fine limit at every boundary point but which fails to have a nontangential limit at a particular boundary point.

Let T be the tree consisting of a ray $\omega = \{\omega_0, \omega_1, \omega_2, \dots\}$ together with rays

$$\omega^n = \{\omega_0^n, \omega_1^n, \dots\}$$

for each $n \geq 1$ such that $\omega_0^n = \omega_n$. Define $e = \omega_0^0 = \omega_0$. Note that ω_0 is a terminal vertex and ω is the only boundary point that has sequences converging to it nontangentially but not radially. Choose $\{p_n\}_{n \geq 1}$ such that $\frac{2^n}{1+2^n} < p_n < 1$. Let $q_n = 1 - p_n$. We define transition probabilities as follows: $p(\omega_0, \omega_1) = 1$; $p(\omega_m^n, \omega_{m+1}^n) = p_n$ and $p(\omega_{m+1}^n, \omega_m^n) = q_n$ for $m \geq 0, n \geq 1$; $p(\omega_n, \omega_{n+1}) = p(\omega_n, \omega_{n-1}) = q_n/2$ for $n \geq 1$.

Define $f : T \rightarrow (0, \infty)$ by $f(\omega_m^n) = (q_n/p_n)^m$. Thus f is identically 1 along the vertices of ω and is q_n/p_n on $\omega_1^n, n \geq 0$. It is easy to check that f is harmonic on each ray $\omega^n, n \geq 1$, and

$$\Delta f(\omega_0^n) = p_n \frac{q_n}{p_n} + \frac{1}{2} q_n + \frac{1}{2} q_n - 1 = 2q_n - 1 < 0,$$

so f is positive, superharmonic and nonharmonic. Thus the Green function on T is finite.

For each $n \geq 1$, $F(\omega_m^n, \omega_{m-1}^n)$ is the same for all $m \geq 1$, and by definition of F it must agree with $F(n+1, n)$ in Example 2.1 (with p replaced by p_n), so

$$F(\omega_m^n, \omega_{m-1}^n) = \frac{q_n}{p_n}.$$

Thus

$$F(\omega_m^n, \omega_1) = F(\omega_m^n, \omega_{m-1}^n)F(\omega_{m-1}^n, \omega_1) \leq \frac{q_n}{p_n} \cdot 1 \leq 2^{-n} \rightarrow 0$$

as $n \rightarrow \infty$ and

$$\begin{aligned} F(\omega_n, \omega_{n-1}) &= \frac{1}{2}q_n + p_n F(\omega_n^1, \omega_{n-1}) + \frac{1}{2}q_n F(\omega_{n+1}, \omega_{n-1}) \\ &= \frac{1}{2}q_n + p_n \frac{q_n}{p_n} F(\omega_n, \omega_{n-1}) + \frac{1}{2}q_n F(\omega_{n+1}, \omega_{n-1}) \\ &\leq 2q_n < \frac{1}{2^{(n-1)}}, \end{aligned}$$

so

$$F(\omega_{n+1}, \omega_1) \leq \frac{1}{2^n} \frac{1}{2^{n-1}} \cdots \frac{1}{2^0} = \frac{1}{2^{n(n+1)/2}} \rightarrow 0$$

as $n \rightarrow \infty$. We deduce that G satisfies (10).

Also, for $n \geq 2$,

$$\begin{aligned} F(\omega_n, \omega_n) &= p_n F(\omega_1^n, \omega_n) + \frac{1}{2}q_n F(\omega_{n+1}, \omega_n) + \frac{1}{2}q_n F(\omega_{n-1}, \omega_n) \\ &\leq q_n + \frac{1}{2}q_n + \frac{1}{2}q_n \\ &= 2q_n, \end{aligned}$$

so

$$G(\omega_n, \omega_n) = \frac{1}{1 - F(\omega_n, \omega_n)} \leq \frac{1}{1 - 2q_n} = \frac{1}{2p_n - 1}.$$

Since $p_n > 2^n / (1 + 2^n) \geq 4/5$ for $n \geq 2$, we obtain

$$G(\omega_1^n, \omega_0^n) = F(\omega_1^n, \omega_0^n)G(\omega_n, \omega_n) \leq \frac{q_n}{p_n} \cdot \frac{1}{2p_n - 1} \leq \frac{25}{12}q_n < \frac{25}{12}2^{-n}.$$

By Theorem 5.2(a), $\{\omega_1^n\}_{n \geq 0}$ is minimally thin at ω and so by Theorem 3.2, $T - \{\omega_n : n \geq 0\}$ is minimally thin at ω . Thus a function has a minimal fine limit at ω if and only if it has a radial limit at ω . Clearly this holds at every other boundary point, so radial and minimal fine limits are the same.

Since $q_n/p_n \rightarrow 0$ as $n \rightarrow \infty$, it follows that f is a positive superharmonic function that does not have a nontangential limit at ω but does have a minimal fine limit of 1 at ω . □

6. Fatou-Naim-Doob Theorem

ASSUMPTION IN THIS SECTION. *The Green function satisfies conditions (9) and (10).*

In this section we prove Theorems 6.1 and 6.2 which describe the minimal fine boundary behaviour of quotients of positive harmonic functions. The proof of Theorem 6.1 is an adaptation of the proofs of the main results in [BL1] and [BL2].

We first set some notation. We fix once and for all a positive harmonic function u . For any positive harmonic function h we denote its representing measure by μ_h . The Radon-Nikodym derivative of μ_h with respect to μ_u is denoted by $\frac{d\mu_h}{d\mu_u}$. For each μ_u integrable function ψ on Ω , we define the function u_ψ on T by

$$(27) \quad u_\psi(t) = \int \psi(\omega) K_\omega(t) d\mu_u(\omega).$$

Our main result in this section is as follows.

THEOREM 6.1. *Let $v = Gf + h$ be a positive superharmonic function and u a positive harmonic function on T . Then the minimal fine limit of v/u exists at μ_u -almost every point of Ω and equals $\frac{d\mu_h}{d\mu_u}$.*

Proof. We first show that if Gf is a potential then

$$(28) \quad \text{mfine} \lim_{t \rightarrow \omega} \frac{Gf}{u}(t) = 0 \text{ for } \mu_u\text{-a.e. } \omega \in \Omega.$$

For each $n \in \mathbb{Z}^+$ let $E_n = \{t \in T : Gf(t) > u(t)/n\}$. Let Σ_n be the set of minimal fine limit points of E_n . Thus Σ_n is the set of $\omega \in \Omega$ where E_n is not minimally thin. Since the potential nGf majorizes u on E_n , $R_u^{E_n} \leq nGf$ and so $R_u^{E_n}$ is a potential. On the other hand by Theorem 4.2

$$\begin{aligned} R_u^{E_n}(\cdot) &= \int R_{K_\omega}^{E_n}(\cdot) d\mu_u(\omega) \\ &\geq \int_{\Sigma_n} R_{K_\omega}^{E_n}(\cdot) d\mu_u(\omega) \\ &= \int K_\omega(\cdot) d\mu_u^{\Sigma_n}(\omega), \end{aligned}$$

where $\mu_u^{\Sigma_n}$ is the restriction of μ_u to Σ_n . Since the last integral defines a nonnegative harmonic minorant of a potential, we must have $\mu_u(\Sigma_n) = 0$ for every n . If $\omega \in \Omega - \bigcup_{n=1}^{\infty} \Sigma_n$, then $T - E_n$ is a minimal fine neighbourhood of ω for every n and so $\text{mfine} \lim_{t \rightarrow \omega} \frac{Gf}{u}(t) = 0$. The result follows from the fact that $\mu_u(\bigcup_{n=1}^{\infty} \Sigma_n) = 0$.

We now show that if v is a positive harmonic function such that the representing measures μ_v and μ_u are mutually singular, then

$$(29) \quad \text{mfine} \lim_{t \rightarrow \omega} \frac{v}{u}(t) = 0 \text{ for } \mu_u\text{-a.e. } \omega \in \Omega.$$

The function $\min(u, v)$ is positive superharmonic. By the unique integral representation of positive harmonic functions, there cannot be any positive harmonic function minorizing both u and v . Hence $\min(u, v)$ is a potential. At any ω for which $\text{mfine} \lim_{t \rightarrow \omega} \frac{\min(u, v)}{u}(t) = 0$, it is obvious that $\text{mfine} \lim_{t \rightarrow \omega} \sup \frac{v}{u}(t) = 0$. The result thus follows from (28).

It follows from (28) and (29) that the proof will be complete if we prove that

$$(30) \quad \text{mfine} \lim_{t \rightarrow \omega} \frac{u_\psi}{u}(t) = \psi(\omega) \text{ for } \mu_u\text{-a.e. } \omega \in \Omega,$$

where ψ is a nonnegative, μ_u -integrable function on Ω . Since we are free to change ψ on a set of μ_u -measure zero, we may assume without loss of generality that ψ is finite valued.

For a Borel set $E \subset \Omega$ let I_E denote the characteristic function of E . For each pair of integers m, n with $m \geq 0, n \geq 1$, define

$$E_{mn} = \psi^{-1} \left[\frac{m}{n}, \frac{m+1}{n} \right).$$

We have

$$\frac{u_\psi \cdot I_{\Omega - E_{mn}}}{u} \leq \frac{u_\psi \cdot I_{\Omega - E_{mn}}}{u_{I_{E_{mn}}}} \quad \text{and} \quad \frac{u_{I_{\Omega - E_{mn}}}}{u} \leq \frac{u_{I_{\Omega - E_{mn}}}}{u_{I_{E_{mn}}}}.$$

By (29) we know that the functions on the right in the above two inequalities have minimal fine limits of zero at μ_u -almost every element of E_{mn} . Hence we can find a subset F_{mn} of E_{mn} of μ_u -measure zero such that

$$(31) \quad \text{mfine} \lim_{t \rightarrow \omega} \frac{u_\psi \cdot I_{\Omega - E_{mn}}}{u}(t) = \text{mfine} \lim_{t \rightarrow \omega} \frac{u_{I_{\Omega - E_{mn}}}}{u}(t) = 0$$

for all $\omega \in E_{mn} - F_{mn}$. Let $F = \bigcup_{m,n} F_{mn}$. Then $\mu_u(F) = 0$. Fix $\omega \in \Omega - F$ for the remainder of the proof. Fix any $n \geq 1$. We shall show that

$$(32) \quad \text{mfine} \limsup_{t \rightarrow \omega} \left| \frac{u_\psi}{u}(t) - \psi(\omega) \right| \leq \frac{2}{n}.$$

There exists a unique m such that $\omega \in E_{mn}$ and (31) holds at ω . Let E denote E_{mn} . For any $t \in T$ we have

$$\begin{aligned} |u_\psi(t) - \psi(\omega) \cdot u(t)| &\leq \left| u_\psi(t) - \frac{m}{n} \cdot u(t) \right| + \left| \frac{m}{n} \cdot u(t) - \psi(\omega) \cdot u(t) \right| \\ &\leq \left| u_{\psi \cdot I_E}(t) - \frac{m}{n} \cdot u_{I_E}(t) \right| \\ &\quad + \left| u_{\psi \cdot I_{\Omega-E}}(t) - \frac{m}{n} \cdot u_{I_{\Omega-E}}(t) \right| + \frac{1}{n} \cdot u(t) \\ &\leq \frac{1}{n} \cdot u_{I_E}(t) + u_{\psi \cdot I_{\Omega-E}}(t) + \frac{m}{n} \cdot u_{I_{\Omega-E}}(t) + \frac{1}{n} \cdot u(t), \end{aligned}$$

and so

$$\begin{aligned} \left| \frac{u_\psi}{u}(t) - \psi(\omega) \right| &\leq \frac{1}{n} \cdot \frac{u_{I_E}(t)}{u} + \frac{u_{\psi \cdot I_{\Omega-E}}(t)}{u} + \frac{m}{n} \cdot \frac{u_{I_{\Omega-E}}(t)}{u} + \frac{1}{n} \\ &\leq \frac{2}{n} + \frac{u_{\psi \cdot I_{\Omega-E}}(t)}{u} + \frac{m}{n} \cdot \frac{u_{I_{\Omega-E}}(t)}{u}. \end{aligned}$$

Taking the minimal fine limsup as t goes to ω and applying (31), we deduce (32). Noting that the inequality is true for arbitrary n , we conclude that

$$\text{mfine} \lim_{t \rightarrow \omega} \frac{u_\psi}{u}(t) = \psi(\omega)$$

for all $\omega \in \Omega - F$. □

By applying Theorem 5.4 and Theorem 6.1 we immediately deduce the following result.

THEOREM 6.2. *Let $v = Gf + h$ be a positive superharmonic function and u a positive harmonic function on T . Then the radial limit of v/u exists at μ_u -almost every point of Ω and equals $\frac{d\mu_h}{d\mu_u}$. If in addition the transition probabilities satisfy (11), then we have the same result with nontangential limits.*

7. Harmonic thin sets and the harmonic fine filter

ASSUMPTION IN THIS SECTION. *The Green function satisfies conditions (9) and (10).*

Harmonic thin sets were first introduced for the classical potential theory on NTA domains in \mathbb{R}^n in [A] and later on abstract BreLOT spaces in [Z]. In [Z] it was shown that the set of complements of sets harmonic thin at ω forms a filter, known as the *harmonic fine filter* at ω , and this filter is coarser than the minimal fine filter. It was then shown that, with the assumption of an additional axiom, the quotient of two positive harmonic functions h/u has a limit at $\omega \in \Omega$ following the harmonic fine filter for μ_u -almost every boundary point ω , where μ_u is the representing measure for u . Here we consider the natural analogue of the harmonic fine filter on trees.

DEFINITION 7.1. Let $E \subset T$ and $\omega \in \Omega$. We say that E is *harmonic thin* at $\omega \in \Omega$ if there exists a positive harmonic function that majorizes K_ω on E but not on all of T .

Clearly if a set is harmonic thin at ω then it is also minimally thin at ω . In the settings considered by Aikawa and in all of the examples considered by Zhang the harmonic fine filter is strictly coarser than the minimal fine filter. The following result shows this is not the case on trees.

THEOREM 7.1. *Let $E \subset T$ and $\omega \in \Omega$. Then E is minimally thin at ω if and only if E is harmonic thin at ω .*

Proof. Suppose E is minimally thin at ω . Thus there exists a potential which majorizes K_ω on E . We will be done if we show that there exists a finite measure μ on Ω such that the Martin integral $K\mu$ majorizes K_ω on E but not on all of T .

We first show that any finite set is harmonic thin at ω . Let G be a finite subset of T . Define $n = \max\{|t \wedge \omega| : t \in G\}$. Let u be any positive harmonic function on T which agrees with K_ω at ω_n and whose representing measure does not charge ω . By Theorem 3.2, u majorizes K_ω on $T - T_{\omega_{n+1}}$, hence on all of G . If u majorized K_ω at ω_k for every $k \geq 1$, it again follows by Theorem 3.2 that u would majorize K_ω on T . Since K_ω and u agree at ω_n , this would imply that they are identical. This contradiction proves G is harmonic thin at ω .

It follows from Theorem 5.1 that there can be at most finitely many j such that $\omega_j \in E$. The same proof as in Corollary 2.1.4 in [Z] shows that the finite union of sets harmonic thin at ω is harmonic thin at ω . Thus it is enough to prove the result in case ω_n is not in E for every n . By Theorem 3.2 it follows that if a potential Gf satisfies $Gf(s) \geq K_\omega(s)$ for some $s \in T$ not in $[e, \omega]$, then $Gf(t) \geq K_\omega(t)$ for all $t \in T_s$. Thus we may assume without loss of generality that E is “complete” in the sense that for all $s \in E$, T_s is a subset of E .

For each $s \in E$, there exists a unique vertex s_ω in $E \cap [s, s \wedge \omega]$ of minimal length. Let

$$\tilde{E} = \bigcup \{s_\omega : s \in E\}.$$

Then E can be written as the pairwise disjoint union

$$E = \bigcup_{s \in \tilde{E}} T_s.$$

Let Gf be a potential with harmonic support in \tilde{E} which majorizes K_ω on \tilde{E} but not on all of T . For each $s \in \tilde{E}$ let μ_s be any probability measure whose support is the interval I_s . Thus $\mu(I_t) = 1$ for any $t \leq s$ and $\mu(I_t) = 0$

for any t with $I_t \cap I_s = \emptyset$. We define

$$\mu = \sum_{s \in \tilde{E}} G(e, s) f(s) \mu_s.$$

Since $Gf(e) < \infty$, μ is a finite measure. We shall show that $K\mu$ majorizes K_ω on E but not on all of T .

Fix $s \in \tilde{E}$. Let t be any vertex in $(T - T_s) \cup \{s\}$. Let $\{t_0, t_1, \dots, t_m, \dots, t_n\}$ be the geodesic from e to t and let $t_m = s \wedge t$, where $0 \leq m \leq n$. Then by (8) we have

$$\begin{aligned} K\mu_s(t) &= \sum_{j=0}^{n-1} \frac{G(t, t_j)}{G(e, t_j)} \mu_s(I_{t_j} - I_{t_{j+1}}) + \frac{G(t, t_n)}{G(e, t_n)} \mu_s(I_{t_n}) \\ &= \frac{G(t, s \wedge t)}{G(e, s \wedge t)}, \end{aligned}$$

since all terms on the right side of the first equation are 0 except for one term. But

$$\frac{G(t, s)}{G(e, s)} = \frac{G(t, s \wedge t)}{G(e, s \wedge t)},$$

so

$$G(e, s) f(s) K\mu_s(t) = G(t, s) f(s).$$

It follows that for all $t \in \tilde{E} \cup (T - E)$,

$$K\mu(t) = \sum_{s \in \tilde{E}} G(e, s) f(s) K\mu_s(t) = \sum_{s \in \tilde{E}} G(t, s) f(s) = Gf(t).$$

Thus $K\mu$ does not majorize K_ω on all of T but it does equal Gf on all of \tilde{E} , hence by Theorem 3.2, it majorizes K_ω on all of E . This completes the proof. \square

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