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ACCESSIBILITY AND HYPERBOLICITY

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ABSTRACT. We examine conditions under which a point in the stable set of a hyperbolic invariant set for a C^1 surface diffeomorphism is accessible via a path from the complement of the stable set. Let M be a surface, and let Λ be a compact saturated hyperbolic locally stably closed invariant set possessing a local product structure. Denote the stable set of Λ by $W^s(\Lambda)$. Our main result states that $z \in W^s(\Lambda)$ is accessible from $M \setminus W^s(\Lambda)$ if and only if z lies on the stable manifold of a periodic point p, and there is a branch of a local unstable manifold of p disjoint from $W^s(\Lambda)$.

Let Λ be a compact hyperbolic invariant set for a C^1 diffeomorphism of a surface M. The objective of this article is to establish necessary and sufficient hyperbolic conditions for a point in the stable set of Λ, denoted by $W^s(\Lambda)$, to be accessible from $M \setminus W^s(\Lambda)$. Intuitively, a point $z \in W^s(\Lambda)$ is accessible from $M \setminus W^s(\Lambda)$ if the stable manifolds of points in Λ accumulate on at most one side of the stable manifold through z. Precisely, identifying accumulation on one side of a stable manifold involves analyzing branches of local unstable manifolds and their intersections with $W^s(\Lambda)$. The main result of this paper asserts that if Λ is a compact saturated hyperbolic locally stably closed invariant set possessing a local product structure, then $z \in W^s(\Lambda)$ is accessible from $M \setminus W^s(\Lambda)$ if and only if z lies on the stable manifold of a periodic point $p \in \Lambda$, and there is a branch of a local unstable manifold of p disjoint from $W^s(\Lambda)$. In this manner we relate accessibility with periodicity and hyperbolicity.

1. Definitions

Recall that a *path* γ in a topological space X is a continuous function $\gamma\colon [0,1] \to X$.

DEFINITION 1.1 (accessible point). Let X be a topological space and $Y \subseteq X$. A point $y \in Y$ is *accessible* from $X \setminus Y$ if there exists a path γ in X

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satisfying $\gamma(0) = y$, and $\gamma((0,1]) \subseteq X \setminus Y$. A point $y \in Y$ is *inaccessible* from $X \setminus Y$ if y is not accessible from $X \setminus Y$.

We will assume that manifolds are smooth, Hausdorff, and admit countable bases for their topologies. Consequently, they support metrics. A *surface* is a two-dimensional manifold.

Denote the stable and unstable manifolds of a point x by $W^s(x)$ and $W^u(x)$, respectively. If (X,d) is a metric space, then

$$W^s(x) = \{y \in X \mid d(f^n(x), f^n(y)) \to 0 \text{ as } n \to \infty\},$$

and

$$W^u(x) = \{y \in X \mid d(f^{-n}(x), f^{-n}(y)) \to 0 \text{ as } n \to \infty\}.$$

Definition 1.2 (branches of stable and unstable manifolds). Let Λ be a compact hyperbolic invariant set for a C^1 surface diffeomorphism with $x \in \Lambda$. Suppose that $\dim W^s(x) = \dim W^u(x) = 1$. As a consequence of the Stable Manifold Theorem [2], $W^s(x)$ and $W^u(x)$ are injectively immersed images of the real numbers.

The *branches of the global stable manifold of* x are the two connected components of $W^s(x) \setminus \{x\}$ in the intrinsic topology of $W^s(x)$. When distinguishing between the two branches at x, we denote the individual branches of the global stable manifold of x by $W^{s+}(x)$ and $W^{s-}(x)$.

Similarly, the *branches of the global unstable manifold of* x are the two connected components of $W^u(x) \setminus \{x\}$ in the intrinsic topology of $W^u(x)$. When distinguishing between the two branches at x, we denote the individual branches of the global unstable manifold of x by $W^{u+}(x)$ and $W^{u-}(x)$.

Given a local stable manifold $W^s_\epsilon(x)$ for some $\epsilon > 0$, the *branches of the local stable manifold of* x are the two connected components of $W^s_\epsilon(x) \setminus \{x\}$. When distinguishing between the local branches at x, our labeling agrees with the labeling of the global branches,

$$W^{s\pm}_\epsilon(x) = W^{s\pm}(x) \cap W^s_\epsilon(x).$$

Similarly, given a local unstable manifold $W^u_\epsilon(x)$ for some $\epsilon > 0$, the *branches of the local unstable manifold of* x are the two connected components of $W^u_\epsilon(x) \setminus \{x\}$. When distinguishing between the local branches at x, our labeling agrees with the labeling of the global branches,

$$W^{u\pm}_\epsilon(x) = W^{u\pm}(x) \cap W^u_\epsilon(x).$$

Remark 1.3. If $\dim W^s(x) = 0$, then $W^u(x)$ is an open disk, and the branches of $W^u(x)$ are undefined. Similarly, if $\dim W^s(x) = 2$, then $W^u(x) = \{x\}$, and the branches of $W^u(x)$ are undefined.

It is useful to distinguish between the collection of stable manifolds of points in a compact hyperbolic invariant set Λ and the stable set of Λ.

DEFINITION 1.4 (stable set of an invariant set). Let X be a metric space and $f\colon X \to X$ with an invariant set $\Lambda \subset X$. Define the *stable set of* Λ to be the set

$$W^s(\Lambda) = \{x \in X \mid \omega(x) \subset \Lambda\},$$

where $z \in \omega(x)$ if there exists a sequence of integers $n_i \to \infty$ such that $f^{n_i}(x) \to z$.

For a compact hyperbolic invariant set Λ, the stable set of Λ consists of all points which are forward asymptotic to the set Λ, though a point in $W^s(\Lambda)$ need not actually lie in the stable manifold of a point in Λ. However, if Λ possesses a local product structure, or equivalently, if Λ is isolated, then a point forward asymptotic to Λ must lie on the stable manifold of a point in Λ. In this instance the stable set and the collection of stable manifolds coincide. This is called the In Phase Property. For a proof of the In Phase Property see Robinson [2].

DEFINITION 1.5 (local product structure). Let Λ be a hyperbolic invariant set for a diffeomorphism of a manifold equipped with a distance function d. The set Λ possesses a *local product structure* if there exist ϵ, $\delta > 0$ such that whenever $x, y \in \Lambda$, and $d(x, y) < \delta$, there exists $z \in \Lambda$ with

$$W^u_\epsilon(x) \cap W^s_\epsilon(y) = \{z\}.$$

THEOREM 1.6 (In Phase Property). *Let Λ be a compact hyperbolic invariant set possessing a local product structure. Then*

$$W^s(\Lambda) = \bigcup_{x\in\Lambda} W^s(x).$$

For the sake of notational economy, we will identify the collection of stable manifolds of points in Λ with $W^s(\Lambda)$ and employ the notation $\bigcup_{x\in\Lambda} W^s(x)$ only when not requiring the existence of a local product structure.

A saturated hyperbolic set is maximal in the sense that intersections of stable and unstable manifolds of points in the hyperbolic set must also lie in the hyperbolic set. In a certain sense, a saturated hyperbolic set has a global version of local product structure.

DEFINITION 1.7 (saturated). A hyperbolic set Λ of a manifold diffeomorphism is called *saturated* if for all $x, y \in \Lambda$, $W^s(x) \cap W^u(y) \subset \Lambda$.

An invariant set Λ is locally stably closed if accumulation points of $W^s(\Lambda)$ which are close to Λ also lie in $W^s(\Lambda)$.

DEFINITION 1.8 (locally stably closed). An invariant set Λ is called *locally stably closed* if there exists a neighborhood U of Λ such that

$$\overline{W^s(\Lambda)} \cap U = W^s(\Lambda) \cap U.$$

The Smale horseshoe extended to a diffeomorphism of S^2 by adding a source and a sink (see Robinson [2]) provides an example of a compact hyperbolic invariant set which is saturated, locally stably closed, and possesses a local product structure. More generally, if a diffeomorphism admits some global hyperbolic structure (e.g., it is Axiom A or has a hyperbolic chain recurrent set), then each basic set (see Robinson [2]) for the diffeomorphism has a local product structure and is saturated and locally stably closed.

2. Accessibility and periodicity

Let Λ be a compact saturated hyperbolic invariant set for a C^1 diffeomorphism of a surface M. This section provides a necessary condition for accessibility involving hyperbolicity and periodicity. We show that points in $W^s(\Lambda)$ accessible from $M \setminus W^s(\Lambda)$ lie on stable manifolds of periodic points in Λ which have branches of their local unstable manifolds disjoint from $W^s(\Lambda)$. This theorem extends a result due to Newhouse and Palis [1].

Let Λ be a compact hyperbolic invariant set possessing a local product structure. We begin by demonstrating that a point in Λ which has a branch of its local unstable manifold disjoint from Λ must lie in the stable manifold of a periodic point in Λ with the same property. This essentially recapitulates the result of Newhouse and Palis [1]. We offer a proof in the interest of completeness.

PROPOSITION 2.1. *Let M be a surface and $f\colon M \to M$ a C^1 diffeomorphism with a compact hyperbolic invariant set Λ possessing a local product structure. If $z \in \Lambda$, and there exists a branch of a local unstable manifold of z disjoint from Λ, then z lies in the stable manifold of a periodic point p, and there is a branch of a local unstable manifold of p disjoint from Λ.*

NOTATION 2.2. $B(x, r)$ denotes the open ball of radius r centered at x. $\mathcal{O}(x)$ denotes the full orbit of x.

Proof. Let $z \in \Lambda$ such that there is a branch of a local unstable manifold of z disjoint from Λ. If z is periodic, then the proof is complete, so assume that z is not periodic. Since Λ is invariant and compact, the orbit of z accumulates on a point $q \in \Lambda$. We will prove that z lies in the stable set of the orbit of q. Then, we will prove that q lies on the stable manifold of a periodic point, so that z must lie on the stable manifold of a point p in the orbit of this periodic point. Finally, we show that there is a branch of a local unstable manifold of p which is disjoint from Λ.

By hypothesis the unstable manifold of z has well-defined branches, so $\dim W^u(z) = 1$. The orbit of z accumulates on q, and the dimensions of unstable manifolds are locally constant. Thus, $\dim W^u(q) = 1$.

Choose $\epsilon > 0$ and local coordinates (ϕ_1, ϕ_2) in $B(q, \epsilon)$ so that

$$\phi_1(W^u_\epsilon(q)) = \{0\}, \quad \text{and} \quad \phi_2(W^s_\epsilon(q)) = \{0\}.$$

By hypothesis, there is a branch of a local unstable manifold of z disjoint from Λ. In particular, there exists $\eta \in (0, \epsilon/2)$ and a branch of the unstable manifold of z disjoint from Λ. The local product structure of Λ guarantees the existence of $\delta > 0$ such that if $x, y \in \Lambda$ with $d(x, y) < \delta$, then $W^s_\eta(x) \cap W^u_\eta(y)$ is a single point in Λ. Assume that $\delta < \epsilon/2$.

To prove that z lies in the stable set of the orbit of q we need only prove that $\mathcal{O}(z) \cap W^s(\mathcal{O}(q)) \neq \emptyset$ since the stable set of the orbit of q is invariant. Seeking a contradiction, assume that

$$\mathcal{O}(z) \cap W^s(\mathcal{O}(q)) = \emptyset.$$

The orbit of z accumulates on q, so there exists a sequence of integers $n_i \to \infty$ as $i \to \infty$ such that $f^{n_i}(z) \in B(q, \delta/2)$ for every $i > 0$. The assumption that $\mathcal{O}(z) \cap W^s(q) = \emptyset$ implies that $\phi_2(f^{n_i}(z)) \neq 0$ for every $i > 0$. Thus, for infinitely many $i > 0$ either

$$\phi_2(f^{n_i}(z)) > 0, \quad \text{or} \quad \phi_2(f^{n_i}(z)) < 0.$$

Assume, without loss of generality, that $\phi_2(f^{n_i}(z)) > 0$ for infinitely many $i > 0$. Then the sequence $\{\phi_2(f^{n_i}(z))\}_{i=0}^\infty$ has a strictly positive subsequence accumulating on zero.

Given our previous choice of $\delta > 0$, there exist indices $I, J, K > 0$ such that

$$\phi_2(f^{n_I}(z)) > \phi_2(f^{n_J}(z)) > \phi_2(f^{n_K}(z)),$$

and

$$\{f^{n_I}(z), f^{n_J}(z), f^{n_K}(z)\} \subset B(q, \delta/2).$$

From the local product structure on Λ there exist $s_I, s_K \in \Lambda$ such that

$$W^s_\eta(f^{n_I}(z)) \cap W^u_\eta(f^{n_J}(z)) = \{s_I\},$$

and

$$W^s_\eta(f^{n_K}(z)) \cap W^u_\eta(f^{n_J}(z)) = \{s_K\}.$$

Thus, both branches of $W^u_\eta(f^{n_J}(z))$ contain a point of Λ. Consequently, both branches of $W^u_\eta(z)$ contain a point of Λ, contradicting the hypothesis that a branch of $W^u_\eta(z)$ is disjoint from Λ. So, $\phi_2(f^{n_i}(z)) = 0$ for all but finitely many $i > 0$. Therefore, z lies in the stable set of the orbit of q. In fact, we have shown that

$$f^{n_i}(z) \in W^s_{\delta/2}(q)$$

for infinitely many $i > 0$.

Next, we show that q lies on the stable manifold of a periodic point p. Since $f^{n_i}(z) \in W^s_{\delta/2}(q)$ for infinitely many $i > 0$, $W^s_{\delta/2}(q)$ contains $f^r(z)$ and $f^s(z)$ for some $s > r$. This implies that $f^{s-r}(W^s_{\delta/2}(q)) \subset W^s_{\delta/2}(q)$. It is a consequence of the Banach Contraction Principle that a stable manifold which maps into itself under f^{s-r} contains a fixed point of f^{s-r}. Therefore, q lies on the stable manifold of a fixed point of f^{s-r}, and z lies in the stable set of the orbit of a periodic point of f. Thus, z lies on the stable manifold of a point p in the orbit of this periodic point.

Finally, we demonstrate that there is a local unstable manifold of p with a branch disjoint from Λ. Seeking a contradiction, assume that each branch of each local unstable manifold of p contains a point in Λ. Then there exist

$$q^+ \in W^{u+}_{\delta/2}(p) \cap \Lambda, \qquad \text{and} \qquad q^- \in W^{u-}_{\delta/2}(p) \cap \Lambda.$$

Since $z \in W^s(p)$ there exists a positive integer m such that $f^m(z) \in W^s_{\delta/2}(p)$. The hyperbolic invariant set Λ has a local product structure, so both branches of $W^u_\eta(f^m(z))$ contain a point of Λ. Since Λ is invariant, and $W^u_\eta(f^m(z)) \subset f^m(W^u_\eta(z))$, both branches of $W^u_\eta(z)$ contain a point of Λ, contradicting the hypothesis that a branch of $W^u_\eta(z)$ is disjoint from Λ. Thus, there is a branch of a local unstable manifold of p which is disjoint from Λ. □

The next proposition provides a necessary hyperbolic condition for accessibility. Specifically, a point in $\bigcup_{x\in\Lambda} W^s(x)$ which is accessible from $M \setminus \bigcup_{x\in\Lambda} W^s(x)$ lies in the stable manifold of a point in Λ with a branch of its local unstable manifold disjoint from Λ.

Proposition 2.3. *Let Λ be a compact hyperbolic invariant set for a C^1 diffeomorphism of a surface M. If $z \in \bigcup_{x\in\Lambda} W^s(x)$ is accessible from $M \setminus \bigcup_{x\in\Lambda} W^s(x)$, then for every $q \in \Lambda$ such that $z \in W^s(q)$, there is a branch of a local unstable manifold of q disjoint from Λ.*

Proof. We prove the contrapositive: if there exists $q \in \Lambda$ such that $z \in W^s(q)$, and each branch of each local unstable manifold of q intersects Λ, then z is inaccessible from $M \setminus \bigcup_{x\in\Lambda} W^s(x)$.

It suffices to prove that for every path γ in M with $\gamma(0) = z$,

$$\gamma((0,1]) \cap \bigcup_{x\in\Lambda} W^s(x) \neq \emptyset.$$

Choose a chart (U, ϕ) around z with coordinate functions (ϕ_1, ϕ_2) so that

$$\phi(z) = (0,0), \qquad \text{and} \qquad \phi_2(\mathrm{comp}_z(W^s(q) \cap U)) = 0,$$

where $\mathrm{comp}_z(W^s(q) \cap U)$ denotes the connected component of $W^s(q) \cap U$ containing z. Let γ be any path in M with $\gamma(0) = z$. If $\gamma((0,1])$ intersects $W^s(q)$ then the proof is complete, so assume that

$$\gamma((0,1]) \cap W^s(q) = \emptyset.$$

Express γ in the local coordinates by $\gamma_1 = \phi_1 \circ \gamma$ and $\gamma_2 = \phi_2 \circ \gamma$. Let $V \subset U$ be a compact rectangular neighborhood of z. Continuity of γ assures that there exists $t_1 \in (0,1]$ so that $\gamma([0,t_1]) \subset V$. Without loss of generality assume that $\gamma_2(t_1) > 0$.

Stable manifolds of points in Λ vary continuously. Therefore, there exists a neighborhood B around q such that whenever $x \in B \cap \Lambda$, a connected component $\mathcal{C}$ of the submanifold $W^s(x) \cap V$ will be within $\gamma_2(t_1)$ of $\mathrm{comp}_z(W^s(q) \cap V)$ in the C^1 topology. The assumption that every branch of every local unstable manifold of q intersects Λ provides the existence of a point $x \in B \cap \Lambda \cap W^{u\pm}_\eta(q)$ for all sufficiently small $\eta > 0$ and both choices of branches of $W^u(q)$. Moreover, by the Stable Manifold Theorem, $\phi(\mathcal{C})$ is the graph of a C^1 function $g\colon \phi_1(\mathcal{C}) \to \mathbb{R}$.

Define a function $\psi\colon [0, t_1] \to \mathbb{R}$ by

$$\psi(t) = \gamma_2(t) - g(\gamma_1(t))$$

measuring the signed distance in the ϕ_2 direction from $\phi(\gamma([0, t_1]))$ to $\phi(\mathcal{C})$. Then,

$$\psi(0) < 0, \qquad \text{and} \qquad \psi(t_1) > 0.$$

The function ψ is continuous on a compact space. By the Intermediate Value Theorem there exists $t_2 \in (0, t_1)$ with $\psi(t_2) = 0$. Therefore,

$$\gamma((0,1]) \cap \bigcup_{x \in \Lambda} W^s(x) \neq \emptyset. \qquad \square$$

Finally, we obtain a necessary condition for accessibility involving both hyperbolicity and periodicity. Accessible points lie on stable manifolds of periodic points which have branches of their local unstable manifolds disjoint from the invariant set.

Theorem 2.4. *Let Λ be a compact hyperbolic invariant set possessing a local product structure for a C^1 diffeomorphism of a surface M. A point in $W^s(\Lambda)$ accessible from $M \setminus W^s(\Lambda)$ lies in the stable manifold of a periodic point $p \in \Lambda$, and there is a branch of a local unstable manifold of p disjoint from Λ.*

Proof. Let $z \in W^s(\Lambda)$ be accessible from $M \setminus W^s(\Lambda)$. By Proposition 2.3, for every $q \in \Lambda$ such that $z \in W^s(q)$ there is a branch of a local unstable manifold of q which is disjoint from Λ. Choose one such point q. According to Proposition 2.1, the point q, and hence the point z, lies on the stable manifold of a periodic point p, and there is a branch of a local unstable manifold of p disjoint from Λ. $\square$

Let Λ be a compact saturated hyperbolic invariant set for a C^1 surface diffeomorphism. We can sharpen our previous results by demonstrating that a branch of a local unstable manifold disjoint from Λ is also disjoint from $W^s(\Lambda)$.

Proposition 2.5. *Let Λ be a compact saturated hyperbolic invariant set for a C^1 surface diffeomorphism. A branch of a local unstable manifold of a point in Λ is disjoint from Λ if and only if it is disjoint from $W^s(\Lambda)$.*

Proof. Suppose that $z \in \Lambda$, and $W^{u+}_\eta(z)$ is disjoint from Λ for some $\eta > 0$. Seeking a contradiction, assume that $W^{u+}_\eta(z)$ intersects $W^s(\Lambda)$ in a point x. Then there exists a point $q \in \Lambda$ such that

$$x \in W^s(q) \cap W^u(z).$$

Since Λ is saturated and contains z and q, the point x lies in Λ, contradicting the hypothesis that $W^{u+}_\eta(z)$ is disjoint from Λ. Therefore, $W^{u+}_\eta(z)$ is disjoint from $W^s(\Lambda)$. Since Λ is compact and invariant, $\Lambda \subset W^s(\Lambda)$, so the reverse implication is trivial. □

Proposition 2.5 delivers analogues of Proposition 2.1 and Theorem 2.4 for a compact saturated hyperbolic invariant set with a local product structure.

PROPOSITION 2.6. *Let Λ be a compact saturated hyperbolic invariant set possessing a local product structure for a C^1 surface diffeomorphism. If $z \in \Lambda$, and there exists a branch of a local unstable manifold of z disjoint from Λ, then z lies in the stable manifold of a periodic point p, and there is a branch of a local unstable manifold of p disjoint from $W^s(\Lambda)$.*

Proof. By Proposition 2.1, the point z lies on the stable manifold of a periodic point p, and there is a branch of a local unstable manifold of p disjoint from Λ. Moreover, Λ is saturated, so Proposition 2.5 guarantees that this branch is disjoint from $W^s(\Lambda)$. □

THEOREM 2.7. *Let Λ be a compact saturated hyperbolic invariant set possessing a local product structure for a C^1 diffeomorphism of a surface M. A point in $W^s(\Lambda)$ accessible from $M \setminus W^s(\Lambda)$ lies in the stable manifold of a periodic point p, and there is a branch of a local unstable manifold of p disjoint from $W^s(\Lambda)$.*

Proof. Let $z \in W^s(\Lambda)$ be accessible from $M \setminus W^s(\Lambda)$. According to Theorem 2.4, the point z lies on the stable manifold of a periodic point p, and there is a branch of a local unstable manifold of p disjoint from Λ. By Proposition 2.5 this branch of the local unstable manifold of p is disjoint from $W^s(\Lambda)$. □

The hypotheses of Theorem 2.7 are not sufficient to guarantee the accessibility of a point in the stable set. Example 3.1 in the next section motivates the need for a further assumption on the nature of the invariant sets under consideration.

3. Accessibility and locally stably closed sets

Let Λ be a compact hyperbolic invariant set possessing a local product structure for a C^1 diffeomorphism of a surface M. Suppose that $q \in \Lambda$ has a branch of one of its local unstable manifolds disjoint from Λ. The following example shows that points on $W^s(q)$ are not necessarily accessible from $M \setminus W^s(\Lambda)$.

Example 3.1. Let p_1 and p_2 be hyperbolic fixed points for a C^1 diffeomorphism of a surface M. Then $\Lambda = \{p_1\}$ is a compact hyperbolic invariant set with a trivial local product structure.

Choose branches of the stable and unstable manifolds of p_1 and p_2. Assume that $W^{u+}(p_1) = W^{s+}(p_2)$. Suppose that $W^{s+}(p_1)$ intersects $W^{u+}(p_2)$ transversely so that $W^{s+}(p_1)$ accumulates on $W^s(p_2) \cup W^{s+}(p_1)$. Let $W^{s-}(p_1)$ intersect $W^{u-}(p_1)$ transversely so that $W^{s-}(p_1)$ accumulates on $W^s(p_1)$. Every $z \in W^{s+}(p_1)$ is inaccessible from $M \setminus W^s(\Lambda)$ even though $W^{u+}(p_1)$ is disjoint from $\Lambda = \{p_1\}$. Moreover, $W^{u+}(p_1)$ is disjoint from $W^s(\Lambda) = W^s(p_1)$.

Example 3.1 shows that if $z \in W^s(q)$, and a branch of the unstable manifold of q intersects the closure of the stable set, then z may be inaccessible. However, this is the only barrier to obtaining a sufficient hyperbolic condition for accessibility.

Theorem 3.2. *Let f be a C^1 diffeomorphism of a surface M. Let Λ be a compact hyperbolic invariant set for f possessing a local product structure. If $z \in W^s(p)$ for some $p \in \Lambda$, and there is a branch of a local unstable manifold of p disjoint from $\overline{W^s(\Lambda)}$, then z is accessible from $M \setminus \overline{W^s(\Lambda)}$.*

Proof. Assume that there is a branch $W^{u+}_\epsilon(p)$ of the local unstable manifold of p disjoint from $\overline{W^s(\Lambda)}$. By Proposition 2.1, the point p lies in the stable manifold of a periodic point. Since $z \in W^s(p)$, the point z also lies in the stable manifold of this periodic point. Thus, it suffices to consider the case in which p is itself periodic.

Let m be the period of p, and define $g = f^{2m}$. Then p is a hyperbolic fixed point of g. By the Hartman-Grobman Theorem [2], there are neighborhoods U of p and V of $0 \in T_pM$ such that $g|_U$ and $Dg_p|_V$ are topologically conjugate via a homeomorphism $h \colon V \to U$.

Choose coordinates (ϕ_1, ϕ_2) in T_pM so that

$$\phi_1(T_pW^u(p)) = \{0\}, \qquad \text{and} \qquad \phi_2(T_pW^s(p)) = \{0\}.$$

Assume that $\phi_2(h^{-1}(W^{u+}_\epsilon(p))) \subset (0, \infty)$. Let $x \in W^{u+}_\epsilon(p) \cap U$. For each $\delta > 0$ define a subset B_δ of V by

$$B_\delta = \{v \in V \mid -\delta \leq \phi_1(v) \leq \delta, \text{ and } \phi_2(h^{-1}(g^{-1}(x))) \leq \phi_2(v) \leq \phi_2(h^{-1}(x))\}.$$

Since $\overline{W^s(\Lambda)}$ is disjoint from $W^{u+}_\epsilon(p)$, there exists $\delta > 0$ such that $h^{-1}(\overline{W^s(\Lambda)} \cap U)$ and B_δ are disjoint. Define

$$\mathcal{S} = h\Big(\bigcup_{k=1}^{\infty} [Dg_p^{-k}(B_\delta) \cap V]\Big).$$

Then $W^s_\eta(p) \cap U \subset \partial\mathcal{S}$ for some $\eta > 0$, and $\mathcal{S}$ is disjoint from $\overline{W^s(\Lambda)}$ because $\overline{W^s(\Lambda)}$ is invariant. Since $z \in W^s(p)$, there exists $n > 0$ such that $g^n(z) \in W^s_\eta \cap U$. Let $\gamma\colon [0,1] \to M$ be a path such that $\gamma(0) = g^n(z)$, and $\gamma((0,1]) \subset \mathcal{S}$. Then $g^{-n} \circ \gamma$ is a path such that $g^{-n}(\gamma(0)) = z$, and $g^{-n}(\gamma((0,1])) \cap \overline{W^s(\Lambda)} = \emptyset$. Therefore, z is accessible from $M \setminus \overline{W^s(\Lambda)}$. □

Example 3.1 reveals that the impediment to a sufficient condition for accessibility involves the intersection of an unstable manifold with the closure of the stable set. Restricting our attention to locally stably closed sets avoids this problem.

Theorem 3.2 produces a sufficient hyperbolic condition for accessibility in the context of a compact saturated hyperbolic locally stably closed invariant set possessing a local product structure. The stable manifold of a point which has a branch of its local unstable manifold disjoint from the stable set contains only accessible points.

Theorem 3.3. *Let Λ be a compact hyperbolic locally stably closed invariant set possessing a local product structure for a C^1 diffeomorphism of a surface M. If $z \in W^s(q)$ for some $q \in \Lambda$, and there is a branch of a local unstable manifold of q disjoint from $W^s(\Lambda)$, then z is accessible from $M \setminus W^s(\Lambda)$.*

Proof. Suppose that $z \in W^s(q)$ for some $q \in \Lambda$, and that there is a branch of a local unstable manifold of q disjoint from $W^s(\Lambda)$. Since Λ is locally stably closed, there exists a neighborhood U of Λ such that

$$\overline{W^s(\Lambda)} \cap U = W^s(\Lambda) \cap U.$$

For some $\eta > 0$ there is a branch $W^{u+}_\eta(q)$ disjoint from $W^s(\Lambda)$ and contained in U. Then $W^{u+}_\eta(q)$ is disjoint from $\overline{W^s(\Lambda)}$. By Theorem 3.2, the point z is accessible from $M \setminus \overline{W^s(\Lambda)}$. Since $W^s(\Lambda) \subset \overline{W^s(\Lambda)}$, the point z is accessible from $M \setminus W^s(\Lambda)$. □

Our main theorem asserts that a point in the stable set of a compact saturated hyperbolic locally stably closed invariant set possessing a local product structure is accessible exactly when it lies on the stable manifold of a periodic point which has a branch of one of its local unstable manifolds disjoint from the stable set.

Theorem 3.4. *Let Λ be a compact saturated hyperbolic locally stably closed invariant set possessing a local product structure for a C^1 diffeomorphism of a surface M. A point $z \in W^s(\Lambda)$ is accessible from $M \setminus W^s(\Lambda)$ if and only if $z \in W^s(p)$ for some periodic point $p \in \Lambda$, and there is a branch of a local unstable manifold of p disjoint from $W^s(\Lambda)$.*

Proof. Theorems 2.7 and 3.3 prove both implications. □

References

[1] S. Newhouse and J. Palis, *Hyperbolic nonwandering sets on two-dimensional manifolds*, Dynamical systems (M. M. Peixoto, ed.), Academic Press, New York, 1973.

[2] C. Robinson, *Dynamical systems. Stability, symbolic dynamics, and chaos*, CRC Press, Boca Raton, FL, 1995.

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