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INSTABILITY AND NONEXISTENCE THEOREMS FOR F-HARMONIC MAPS

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Abstract. In this paper we study the unstability and nonexistence of F-harmonic maps. We introduce the notion of F-strongly unstable and F-unstable manifolds and discuss properties of such manifolds. We classify all compact irreducible F-unstable symmetric spaces.

1. Introduction

The theory of harmonic maps, p-harmonic maps, and exponentially harmonic maps, is a powerful area of differential geometry that has applications to various fields, including topology and physics. Recently, Wei [17] studied the second variational formula for p-harmonic maps and, extending the work of Howard-Wei [6] and Ohnita [10], classified compact irreducible p-superstrongly unstable symmetric spaces.

The author [1] introduced the notions of F-energy and F-harmonic maps which generalize harmonic, p-harmonic and exponentially harmonic maps. In this paper, we first establish the second variational formula of the F-energy and then introduce the notions of F-unstability and F-strong unstability. We consider a nonnegative valued strictly increasing C^2 function F on the interval $[0,\infty)$. We define the F-energy $E_F(\phi)$ for a smooth map ϕ between Riemannian manifolds (M,g) and (N,h) by

$$E_F(\phi) = \int_M F\left(\frac{|d\phi|^2}{2}\right) v_g,$$

where v_g is the volume element of g. Critical mappings of E_F are called *F-harmonic maps* (see [1]). Notice that F-harmonic maps are harmonic, p-harmonic, and exponentially harmonic if $F(t)$ is equal to t, $(2t)^{p/2}/p$ and e^t, respectively. Roughly speaking, our F-harmonic map is *F-stable* if the second variation of the F-energy is nonnegative, and *F-unstable* otherwise. In particular, a compact Riemannian manifold M is *F-unstable* if the identity map is F-unstable, and *F-strongly unstable* if M is neither a domain nor a

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target of any nonconstant stable F-harmonic map. In the case $F(t) = t$, we use the terms *unstable* and *strongly unstable*.

By the definition, F-strong unstability implies F-unstability. In the case when $F(t) = t$ and M is a compact irreducible symmetric space, the converse is true, by a theorem of Howard-Wei [6] and Ohnita [10].

In this paper we first prove a striking F-stability theorem, which says the following: Let F be a strictly increasing C^2 function satisfying $mF''(m/2) + (2-m)F'(m/2) \geq 0$. Then every m-dimensional compact Riemannian manifold M is F-stable (see Theorem 3.1).

We then prove an F-stability theorem, which generalizes results of Urakawa (see [13]) for harmonic maps and of Wei (see [17, Theorem 5.8]) for p-harmonic maps, and which says the following: Assume F is C^2 and strictly increasing and convex. Then every compact Riemannian manifold of constant curvature, except the standard sphere (S^m, can) $(m \geq 3)$, is F-stable (see Theorem 3.7).

The following result clarifies the relation between the notions of F-unstability and F-strong unstability.

Theorem A (see Corollary 4.11). *Let M be a compact irreducible symmetric space. Then there exists a strictly increasing and strictly convex C^2 function $F : [0,\infty) \to [0,\infty)$, such that M is F-strongly unstable if and only if it is F-unstable.*

We remark that in the case $F(t) = t$ this was proved by Howard-Wei and Ohnita. Theorem A says that the same result holds for other function F. Moreover, if $F : [0,\infty) \to [0,\infty)$ is an arbitrary strictly increasing C^2 function satisfying $F'' \geq 0$ everywhere, we can classify all compact irreducible symmetric spaces which are F-unstable (see Theorem 4.12). As a corollary, we can classify all compact irreducible symmetric spaces for which the identity map is unstable as a p-harmonic map (see Corollary 4.13).

Comparing our classifications with that of Wei (see [17]), we find that the notions of superstrong unstability and unstability are different in the case of p-harmonic maps. This is in sharp contrast to the result obtained by Howard-Wei and Ohnita for the case of harmonic maps.

This paper is organized as follow. In Section 2, we recall some facts on F-harmonic maps and the second variational formula of the F-energy. In Section 3, we study unstability as an F-harmonic map for identity maps. In Section 4, we deal with F-strongly unstable and F-unstable manifolds. In Section 5, we give the Bochner formula and prove nonexistence theorems for F-harmonic maps.

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2. F-harmonic maps

Let $F : [0, \infty) \to [0, \infty)$ be a strictly increasing C^2 function. Let $\phi : M \to N$ be a smooth map from an m-dimensional Riemannian manifolds (M, g) to a Riemannian manifold (N, h). We call ϕ an F-harmonic map if it is a critical point of the F-energy functional. That is, ϕ is an F-harmonic map if and only if

$$\left.\frac{d}{dt}\right|_{t=0} E_F(\phi_t) = 0$$

for any compactly supported variation $\phi_t : M \to N$ $(-\epsilon < t < \epsilon)$ with $\phi_0 = \phi$.

Let ∇ and ${}^N\nabla$ denote the Levi-Civita connections of M and N, respectively. Let $\tilde{\nabla}$ be the induced connection on $\phi^{-1}TN$ defined by $\tilde{\nabla}_X W = {}^N\nabla_{\phi_* X} W$, where X is a tangent vector of M and W is a section of $\phi^{-1}TN$. We choose a local orthonormal frame field $\{e_i\}_{i=1}^m$ on M. We define the F-tension field $\tau_F(\phi)$ of ϕ by

$$\begin{aligned}
\tau_F(\phi) &= \sum_{i=1}^{m} \left[\tilde{\nabla}_{e_i} \left\{ F'\left(\frac{|d\phi|^2}{2} \right) \phi_* e_i \right\} - F'\left(\frac{|d\phi|^2}{2} \right) \phi_* \nabla_{e_i} e_i \right] \\
&= F'\left(\frac{|d\phi|^2}{2} \right) \tau(\phi) + \phi_* \operatorname{grad} \left\{ F'\left(\frac{|d\phi|^2}{2} \right) \right\},
\end{aligned}$$

where $\tau(\phi) = \sum_{i=1}^m (\tilde{\nabla}_{e_i} \phi_* e_i - \phi_* \nabla_{e_i} e_i)$ is the tension field of ϕ.

With this notation we have the following result:

THEOREM 2.1 (First variation formula; see [1]).

$$\left.\frac{d}{dt}\right|_{t=0} E_F(\phi_t) = -\int_M h(V, \tau_F(\phi)) v_g,$$

where $V = d\phi_t/dt|_{t=0}$.

Therefore a smooth map $\phi : M \to N$ is an F-harmonic map if and only if the F-tension field $\tau_F(\phi)$ is zero.

Next, we give the second variation formula for F-harmonic maps and describe the F-Jacobi operator J_F.

THEOREM 2.2 (Second variation formula; see [1]). *Let* $\phi : M \to N$ *be an* F*-harmonic map. Let* $\phi_{s,t} : M \to N$ $(-\epsilon < s, t < \epsilon)$ *be a compactly supported two-parameter variation such that* $\phi_{0,0} = \phi$*, and set* $V = \partial \phi_{s,t}/\partial t|_{s,t=0}$ *and*

$W = \partial\phi_{s,t}/\partial s|_{s,t=0}$. *Then*

$$\begin{aligned}\frac{\partial^2}{\partial s\partial t}\bigg|_{s,t=0} E_F(\phi_{s,t}) &= \int_M h(J_{F,\phi}(V), W)v_g \\ &= \int_M F''\left(\frac{|d\phi|^2}{2}\right)\langle\tilde\nabla V, d\phi\rangle\langle\tilde\nabla W, d\phi\rangle v_g \\ &\quad + \int_M F'\left(\frac{|d\phi|^2}{2}\right)\cdot\left\{\langle\tilde\nabla V, \tilde\nabla W\rangle - \sum_{i=1}^m h(R^N(V,\phi_* e_i)\phi_* e_i, W)\right\}v_g,\end{aligned}$$

where $\langle,\rangle$ *is the inner product on* $T^*M\otimes\phi^{-1}TN$, R^N *is the curvature tensor of* N, *and* $J_{F,\phi}(V)$ *is given by*

$$\begin{aligned}J_{F,\phi}(V) &= \tilde\nabla^*\left(F''\left(\frac{|d\phi|^2}{2}\right)\cdot\langle\tilde\nabla V, d\phi\rangle d\phi + F'\left(\frac{|d\phi|^2}{2}\right)\cdot\tilde\nabla V\right) \\ &\quad - F'\left(\frac{|d\phi|^2}{2}\right)\cdot\sum_{i=1}^m R^N(V,\phi_* e_i)\phi_* e_i, \quad V\in\Gamma(\phi^{-1}TN).\end{aligned} \tag{2.1}$$

We put

$$I(V,W) = \frac{\partial^2}{\partial s\partial t}\bigg|_{s,t=0} E_F(\phi_{s,t}).$$

An F-harmonic map ϕ is called F-stable, or stable, if $I(V,V)\geq 0$ for any compactly supported vector field V along ϕ, or equivalently, if the eigenvalues of the F-Jacobi operator $J_{F,\phi}$ are all nonnegative.

Remark 2.3. In the case of harmonic maps, the equation (2.1) reads

$$J_{F,\phi}(V) = \tilde\nabla^*\tilde\nabla V - \sum_{i=1}^m R^N(V,\phi_* e_i)\phi_* e_i =: J_{2,\phi}(V).$$

This is the Jacobi operator for harmonic maps. The operator $\tilde\nabla^*\tilde\nabla$ is often denoted by Δ and called the rough Laplacian.

Some geometric properties of F-harmonic maps are described in [1].

3. Stability of F-harmonic identity maps

Throughout this section, we assume that $F' + F'' \neq 0$ on $(0,\infty)$. This assumption ensures that the F-Jacobi operator is elliptic. We deal with the F-Jacobi operator of the identity map. When the identity map of M is F-stable, we say that M is F-stable (see [8]).

Theorem 3.1. *Let* M *be an* m*-dimensional compact Riemannian manifold and* $F:[0,\infty)\to[0,\infty)$ *a strictly increasing* C^2 *function such that* $mF''(m/2) + (2-m)F'(m/2)\geq 0$. *Then* M *is* F*-stable.*

Remark 3.2. In the case of harmonic maps, the assumption $mF''(m/2) + (2-m)F'(m/2) \geq 0$ implies that $m \leq 2$, since $F' = 1$ and $F'' = 0$.

In the case of p-harmonic maps, this assumption implies that $p \geq m$, since $F'(m/2) = m^{(p/2)-1}$ and $F''(m/2) = (p-2)m^{(p/2)-2}$. Moreover, for exponentially harmonic maps this assumption is always satisfied, since $F'(m/2) = F''(m/2) = e^{m/2}$. Therefore, the theorem is an extension of the results of [9] for p-harmonic maps and of [3] for exponentially harmonic maps.

Proof. Recall the formula of K.Yano (see [19])

$$\int_M g(J_{2,\mathrm{id}}(V), V)v_g = \int_M \left\{ \frac{1}{2}|\mathcal{L}_V g|^2 - (\operatorname{div} V)^2 \right\} v_g,$$

where $\mathcal{L}_V g$ is the Lie derivative of the metric g.

By the Cauchy-Schwarz inequality,

$$\begin{aligned} m|\mathcal{L}_V g|^2 &= m \sum_{i,j=1}^m \mathcal{L}_V g(e_i, e_j)^2 \\ &= m \sum_{i,j=1}^m \left(g(\nabla_{e_i} V, e_j) + g(e_i, \nabla_{e_j} V) \right)^2 \\ &\geq m \sum_{i=1}^m \left(g(\nabla_{e_i} V, e_i) + g(e_i, \nabla_{e_i} V) \right)^2 \\ &= 4m \sum_{i=1}^m g(\nabla_{e_i} V, e_i)^2 \geq 4 \left(\sum_{i=1}^m g(\nabla_{e_i} V, e_i) \right)^2 = 4(\operatorname{div} V)^2. \end{aligned}$$

Thus we have

$$\begin{aligned} \int_M g(J_{F,\mathrm{id}}(V), V)v_g &= F''\left(\frac{m}{2}\right) \int_M (\operatorname{div} V)^2 v_g \\ &\quad + F'\left(\frac{m}{2}\right) \int_M \sum_{i=1}^m \left\{ g(\nabla_{e_i} V, \nabla_{e_i} V) - g(R^M(V, e_i)e_i, V) \right\} v_g \\ &= F''\left(\frac{m}{2}\right) \int_M (\operatorname{div} V)^2 v_g + F'\left(\frac{m}{2}\right) \int_M g(J_{2,\mathrm{id}}(V), V) v_g \\ &= F''\left(\frac{m}{2}\right) \int_M (\operatorname{div} V)^2 v_g + F'\left(\frac{m}{2}\right) \int_M \left\{ \frac{1}{2}|\mathcal{L}_V g|^2 - (\operatorname{div} V)^2 \right\} v_g \\ &\geq \frac{1}{m} \left\{ mF''\left(\frac{m}{2}\right) + (2-m)F'\left(\frac{m}{2}\right) \right\} \int_M (\operatorname{div} V)^2 v_g \geq 0. \end{aligned}$$

Hence M is F-stable. □

Theorem 3.3. *Let M be an m-dimensional compact Riemannian manifold which supports a nonisometric, conformal vector field V, and let F :*

$[0,\infty) \to [0,\infty)$ be a strictly increasing C^2 function. Then M is F-stable if and only if F satisfies $mF''(m/2) + (2-m)F'(m/2) \geq 0$.

Proof. Since a vector field V on M is conformal if and only if $\mathcal{L}_V g = -(2/m)(\operatorname{div} V)g$, $(1/2)|\mathcal{L}_V g|^2 = (2/m)(\operatorname{div} V)^2$. Then we have

$$\begin{aligned}\int_M g(J_{F,\mathrm{id}}(V), V) v_g =& F''\left(\frac{m}{2}\right)\int_M (\operatorname{div} V)^2 v_g \\ &+ F'\left(\frac{m}{2}\right)\int_M \left\{\frac{1}{2}|\mathcal{L}_V g|^2 - (\operatorname{div} V)^2\right\} v_g \\ =& \frac{1}{m}\left\{mF''\left(\frac{m}{2}\right) + (2-m)F'\left(\frac{m}{2}\right)\right\}\int_M (\operatorname{div} V)^2 v_g.\end{aligned}$$

If V is nonisometric conformal, we have $\operatorname{div} V \not\equiv 0$. This completes the proof. □

Next we use methods of [9], [14], and [17] to establish the following theorem, which extends a theorem in [17] for p-harmonic maps.

THEOREM 3.4. *Let M be an m-dimensional compact Einstein manifold whose Ricci tensor equals ρg for some Einstein constant ρ. Let $F : [0,\infty) \to [0,\infty)$ be a strictly increasing C^2 function such that $F'(m/2) + F''(m/2) > 0$. Then M is F-stable if and only if F satisfies*

$$2\rho F'\left(\frac{m}{2}\right) \leq \left(F'\left(\frac{m}{2}\right) + F''\left(\frac{m}{2}\right)\right)\lambda_1,$$

where λ_1 is the smallest positive eigenvalue of the Laplacian for functions.

Proof. By the Kodaira-de Rham-Hodge decomposition, we have an orthogonal direct sum decomposition

$$\mathcal{X}(M) = \{V \in \mathcal{X}(M) \mid \operatorname{div} V = 0\} \oplus \{\operatorname{grad} f | f \in C^\infty(M)\},$$

where $\mathcal{X}(M)$ is the space of all smooth vector fields on M. The Laplacian Δ preserves this decomposition. Under the Einstein condition, $J_{F,\mathrm{id}}$ also preserves this decomposition, since R^M is a scalar multiple of the identity. Hence it suffices to show that the assertion holds separately on vector fields V with $\operatorname{div} V = 0$, and on the gradients $\operatorname{grad} f$.

For any vector field V such that $\operatorname{div} V = 0$, we have

$$\begin{aligned}
&\int_M g(J_{F,\mathrm{id}}(V), V)v_g \\
&\quad = -F''\left(\frac{m}{2}\right)\int_M g(\operatorname{grad}(\operatorname{div} V), V)v_g + F'\left(\frac{m}{2}\right)\int_M g(J_{2,\mathrm{id}}(V), V)v_g \\
&\quad = F'\left(\frac{m}{2}\right)\int_M \left\{\frac{1}{2}|\mathcal{L}_V g|^2 - (\operatorname{div} V)^2\right\} v_g \\
&\quad = \frac{1}{2}F'\left(\frac{m}{2}\right)\int_M |\mathcal{L}_V g|^2 v_g \geq 0.
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
\int_M g(J_{F,\mathrm{id}}(\operatorname{grad} f), \operatorname{grad} f)v_g &= -F''\left(\frac{m}{2}\right)\int_M g(\operatorname{grad}(\operatorname{div}\ \operatorname{grad} f), \operatorname{grad} f)v_g \\
&\quad + F'\left(\frac{m}{2}\right)\int_M g(J_{2,\mathrm{id}}(\operatorname{grad} f), \operatorname{grad} f)v_g \\
&= F''\left(\frac{m}{2}\right)\int_M g(\operatorname{grad}(\Delta f), \operatorname{grad} f)v_g \\
&\quad + F'\left(\frac{m}{2}\right)\int_M g(\operatorname{grad}(\Delta f) - 2c(\operatorname{grad} f), \operatorname{grad} f)v_g.
\end{aligned} \tag{3.1}$$

Now recall that

$$\int_M g(\operatorname{grad}(\Delta f), \operatorname{grad} f)v_g \geq \lambda_1 \int_M g(\operatorname{grad} f, \operatorname{grad} f)v_g$$

for every function f, and that there is some function f_1 satisfying $\Delta f_1 = \lambda_1 f_1$.

If M is F-stable, then (3.1) gives

$$\begin{aligned}
0 &\leq \int_M g(J_{F,\mathrm{id}}(\operatorname{grad} f_1), \operatorname{grad} f_1)v_g \\
&= \left(F'\left(\frac{m}{2}\right) + F''\left(\frac{m}{2}\right)\right)\lambda_1 \int_M g(\operatorname{grad} f_1, \operatorname{grad} f_1)v_g \\
&\qquad - 2F'\left(\frac{m}{2}\right)\rho \int_M g(\operatorname{grad} f_1, \operatorname{grad} f_1)v_g.
\end{aligned}$$

Therefore we have

$$2F'\left(\frac{m}{2}\right)\rho \leq \left(F'\left(\frac{m}{2}\right) + F''\left(\frac{m}{2}\right)\right)\lambda_1.$$

Conversely, if

$$2F'\left(\frac{m}{2}\right)\rho \leq \left(F'\left(\frac{m}{2}\right) + F''\left(\frac{m}{2}\right)\right)\lambda_1,$$

then (3.1) yields

$$\int_M g(J_{F,\mathrm{id}}(\operatorname{grad} f), \operatorname{grad} f)v_g \geq 0.$$

Thus M is F-stable. □

Corollary 3.5. *Let M be an m-dimensional compact Einstein manifold, and let $F : [0,\infty) \to [0,\infty)$ be a strictly increasing C^2 function. Then the following assertions hold:*

(1) *If $F''(m/2) \geq 0$ and M is stable, then M is F-stable.*
(2) *If $-F'(m/2) < F''(m/2) \leq 0$ and M is F-stable, then M is stable.*

Next we give the result for spherical space forms, which extends the results given in [13, Prop. 5.6] and [17, Th. 5.8] for harmonic maps and p-harmonic maps, respectively.

Proposition 3.6. *Let $F : [0,\infty) \to [0,\infty)$ be a strictly increasing C^2 function such that $F''(m/2) \geq 0$. Then every spherical space form $(S^m/G, g)$, where $G \neq \{e\}$ is a finite group acting fixed point freely on S^m, is F-stable. Here the metric g is the Riemannian metric on the quotient space S^m/G induced by the standard metric* can *of constant curvature one on S^m.*

Proof. Since $(S^m/G, g)$ is Einstein (i.e., the Ricci tensor ρ of g satisfies $\rho = (m-1)g$), the manifold $(S^m/G, g)$ is F-stable if and only if the smallest positive eigenvalue λ_1 of the Laplacian for functions is bigger than or equal to $(2(m-1)F'(m/2))/(F'(m/2)+F''(m/2))$. The eigenvalues of the Laplacian of (S^m, can) are $k(k+m-1)$, for $k = 0, 1, 2, \cdots$, and if $k \geq 2$, then

$$k(k+m-1) > 2(m-1) \geq 2(m-1)\frac{F'\left(\frac{m}{2}\right)}{F'\left(\frac{m}{2}\right) + F''\left(\frac{m}{2}\right)}.$$

Moreover, the eigenfunctions of the smallest positive eigenvalue m with $k = 1$ of (S^m, can) are given by $f \circ i_{S^m}$, where f is a linear map of $\mathbf{R}^{m+1}$ into $\mathbf{R}$ and i_{S^m} is the natural inclusion of S^m into $\mathbf{R}^{m+1}$. Therefore it suffices to show that every linear G-invariant function f on $\mathbf{R}^{m+1}$ must be zero. But this follows immediately from the assumption that G acts fixed point freely on S^m. Clearly, we have $f(x) = \langle x, y \rangle$ for $x \in \mathbf{R}^{m+1}$ and some $y \in \mathbf{R}^{m+1}$. The G-invariance of f implies that $\gamma \cdot y = y$ for all $\gamma \in G$. Hence, unless f vanishes, the point $y/|y| \in S^m$ must be a fixed point of G. □

Theorem 3.7. *Every compact Riemannian manifold of constant curvature, except the standard unit sphere (S^m, can) $(m \geq 3)$, is F-stable for some strictly increasing and convex C^2 function $F : [0,\infty) \to [0,\infty)$.*

Proof. Since every compact Riemannian manifold of positive constant curvature is as in Proposition 3.6 (see [18, Lemma 5.1.1]) and every compact Riemannian manifold of constant nonpositive curvature is F-stable if $F'' \geq 0$ (see [1, Theorem 6.2]), the assertion follows. □

4. F-strongly unstable manifolds

First we recall the definitions of superstrongly unstable manifolds, strongly unstable manifolds and unstable manifolds.

Definition 4.1. An m-dimensional Riemannian manifold M with a Riemannian metric $\langle , \rangle_M$ is said to be *superstrongly unstable* (SSU), if there exists an isometric immersion in $\mathbb{R}^r$ such that, for any unit tangent vector X to M at any point $x \in M$, the following functional is always negative:

$$\langle Q_x^M(X), X\rangle_M = \sum_{\alpha=1}^{m} \Big(2\langle B(X, v_\alpha), B(X, v_\alpha)\rangle_{\mathbb{R}^r} - \langle B(X, X), B(v_\alpha, v_\alpha)\rangle_{\mathbb{R}^r} \Big). \tag{4.1}$$

Here B is the second fundamental form of the immersion, and $\{v_\alpha\}_{\alpha=1}^m$ is a local orthonormal frame field on M near x.

Definition 4.2. A compact Riemannian manifold M is *strongly unstable* (SU) if M is neither a domain nor a target of any nonconstant stable harmonic map. A compact Riemannian manifold M is *unstable* (U) if the identity map of M is unstable.

Remark 4.3. For compact irreducible symmetric spaces, the notions SSU, SU, and U are equivalent (see [6] and [10]).

We now introduce the notions of F-strongly unstable manifolds and F-unstable manifolds.

Definition 4.4. A compact Riemannian manifold M is *F-strongly unstable* (F-SU) if M is neither a domain nor a target of any nonconstant stable F-harmonic map. A compact Riemannian manifold M is *F-unstable* (F-U) if the identity map of M is F-unstable.

We will prove the following theorem, which is one of our main results.

Theorem 4.5. *Let M be an SSU manifold. Then there exists a strictly increasing and strictly convex C^2 function $F : [0, \infty) \to [0, \infty)$ such that M is F-SU.*

Remark 4.6. In the case $F(t) = t$ (so that F-SU manifolds are SU) we know that SSU manifolds are F-SU. However, note that the function F in the theorem must be strictly convex.

In order to prove the theorem, we derive average variational formulas, as in [15]. We assume throughout that $\phi : M \to N$ is an F-harmonic map from an m-dimensional Riemannian manifold into an n-dimensional Riemannian manifold.

We can isometrically immerse N into $\mathbb{R}^r$ with second fundamental form B. Let $\{e_i\}_{i=1}^m$, V, and $V^\top$ denote a local orthonormal frame field on M, a unit vector in $\mathbb{R}^r$ and the tangential projection of V onto N, respectively. We can choose an adopted orthonormal basis $\{V_p\}_{p=1}^r$ in $\mathbb{R}^r$ such that $\{V_p\}_{p=1}^n$ is tangent to N. Denote by $f_t^{V_p^\top}$ the flow generated by $V_p^\top$. Then apply the second variational formula with $\phi_t = f_t^{V_p^\top} \circ \phi$, $\phi_0 = \phi$, and $s = t$, and over $p = 1, \ldots, r$:

$$
\begin{aligned}
\sum_{p=1}^r \frac{d^2}{dt^2} E_F(f_t^{V_p^\top} \circ \phi)|_{t=0} &= \sum_{p=1}^r \int_M \Bigg\{ F''\left(\frac{|d\phi|^2}{2}\right) \left(\sum_{i=1}^m \langle \tilde{\nabla}_{e_i} V_p^\top, \phi_* e_i \rangle \right)^2 \\
&\quad + F'\left(\frac{|d\phi|^2}{2}\right) \sum_{i=1}^m \left(|\tilde{\nabla}_{e_i} V_p^\top|^2 - \langle R^N(V_p^\top, \phi_* e_i)\phi_* e_i, V_p^\top \rangle \right) \Bigg\} dv_g.
\end{aligned} \tag{4.2}
$$

As V_p is parallel in $\mathbb{R}^r$, we have

$$
\begin{aligned}
\tilde{\nabla}_{e_i} V_p^\top &= {}^N\nabla_{\phi_* e_i} V_p^\top = ({}^R\nabla_{\phi_* e_i} V_p^\top)^\top = ({}^R\nabla_{\phi_* e_i} (V_p - V_p^\perp))^\top \\
&= -({}^R\nabla_{\phi_* e_i} V_p^\perp)^\top = A^{V_p^\perp}(\phi_* e_i),
\end{aligned}
$$

and so

$$
\langle \tilde{\nabla}_{e_i} V_p^\top, \phi_* e_i \rangle = \langle A^{V_p^\perp}(\phi_* e_i), \phi_* e_i \rangle = \langle B(\phi_* e_i, \phi_* e_i), V_p^\top \rangle.
$$

Thus

$$
\sum_{p=1}^r \left(\sum_{i=1}^m \langle \tilde{\nabla}_{e_i} V_p^\top, \phi_* e_i \rangle \right)^2 = \left| \sum_{i=1}^m B(\phi_* e_i, \phi_* e_i) \right|^2. \tag{4.3}
$$

We have also

$$
\begin{aligned}
\sum_{p=1}^r \left| \tilde{\nabla}_{e_i} V_p^\top \right|^2 &= \sum_{p=1}^r \left| A^{V_p^\perp}(\phi_* e_i) \right|^2 = \sum_{p=1}^r \sum_{q=1}^n \langle A^{V_p^\perp}(\phi_* e_i), V_q \rangle^2 \\
&= \sum_{p=1}^r \sum_{q=1}^n \langle B(\phi_* e_i, V_q), V_p^\top \rangle^2 = \sum_{q=1}^n |B(\phi_* e_i, V_q)|^2
\end{aligned} \tag{4.4}
$$

From (4.2)–(4.4) and the Gauss equation, we obtain the following result.

Theorem 4.7 (Average second variational formula on the target).

$$\sum_{p=1}^{r} \frac{d^2}{dt^2} E_F(f_t^{V_p^\top} \circ \phi)|_{t=0} = \int_M \Bigg\{ F''\left(\frac{|d\phi|^2}{2}\right) \left| \sum_{i=1}^{m} B(\phi_* e_i, \phi_* e_i) \right|^2 + F'\left(\frac{|d\phi|^2}{2}\right) \sum_{i=1}^{m} \langle Q^N(\phi_* e_i), \phi_* e_i \rangle \Bigg\} dv_g. \tag{4.5}$$

Similarly, we can isometrically immerse M into $\mathbb{R}^r$. Let $\{V_p^\top\}_{p=1}^r$ be the tangential projection of an orthonormal frame field $\{V_p\}_{p=1}^r$ in $\mathbb{R}^r$ onto M. Denote by $f_t^{V_p^\top}$ the flow generated by $V_p^\top$, apply the second variational formula with $\phi_t = \phi \circ f_t^{V_p^\top}$, $\phi_0 = \phi$ and $s = t$ and sum over $p = 1, \dots, r$. For convenience, we choose $(V_1, \cdots, V_m) = (e_1, \cdots, e_m)$ to be tangent to M, $(V_{m+1}, \cdots, V_r) = (\nu_1, \cdots, \nu_{r-m})$ to be normal to M, and $\nabla e_i|_x$ at $x \in M$. We have

$$\sum_{p=1}^{r} \frac{d^2}{dt^2} E_F(\phi \circ f_t^{V_p^\top})|_{t=0} = \sum_{p=1}^{r} \int_M \Bigg\{ F''\left(\frac{|d\phi|^2}{2}\right) \left(\sum_{i=1}^{m} \langle \tilde{\nabla}_{e_i} \phi_* V_p^\top, \phi_* e_i \rangle \right)^2 + F'\left(\frac{|d\phi|^2}{2}\right) \sum_{i=1}^{m} \left(|\tilde{\nabla}_{e_i} \phi_* V_p^\top|^2 - \langle R^N\left(\phi_* V_p^\top, \phi_* e_i\right) \phi_* e_i, \phi_* V_p^\top \rangle \right) \Bigg\} dv_M. \tag{4.6}$$

Since $V_p^\top = V_p - V_p^\perp$ and V_p are parallel in $\mathbb{R}^r$, we have

$$\begin{aligned} \sum_{p=1}^{r} \left(\sum_{i=1}^{m} \langle \tilde{\nabla}_{e_i} \phi_* V_p^\top, \phi_* e_i \rangle \right)^2 &= \sum_{p=1}^{r} \left(\sum_{i=1}^{m} \langle (\tilde{\nabla}_{e_i} d\phi) V_p^\top - \phi_* \nabla_{e_i} V_p^\top, \phi_* e_i \rangle \right)^2 \\ &= \sum_{p=1}^{r} \left(\sum_{i=1}^{m} \langle (\tilde{\nabla}_{e_i} d\phi) V_p^\top - \phi_* \nabla_{e_i} V_p^\perp, \phi_* e_i \rangle \right)^2 \\ &= \sum_{p=1}^{m} \left(\sum_{i=1}^{m} \langle (\tilde{\nabla}_{e_i} d\phi) e_p, \phi_* e_i \rangle \right)^2 + \sum_{\alpha=1}^{r-m} \left(\sum_{i=1}^{m} \langle \phi_* A^{\nu_\alpha} e_i, \phi_* e_i \rangle \right)^2 \\ &= \frac{1}{4} |d|d\phi|^2|^2 + \sum_{\alpha=1}^{r-m} \left(\sum_{i=1}^{m} \langle \phi_* A^{\nu_\alpha} e_i, \phi_* e_i \rangle \right)^2, \end{aligned} \tag{4.7}$$

where A^{ν_α} is the Weingarten map of M in $\mathbb{R}^r$ in the normal direction ν_α.

It follows from (4.5) in [6] that

$$\sum_{i=1}^{m}\nabla_{e_i}\nabla_{e_i}(d\phi(V_p^\top)) = \sum_{i=1}^{m}(\nabla_{e_i}\nabla_{e_i}d\phi)(V_p^\top) + 2\sum_{i=1}^{m}(\nabla_{e_i}d\phi)(\nabla_{e_i}V_p^\top) + \sum_{i=1}^{m}\phi_*\nabla_{e_i}\nabla_{e_i}V_p^\top.$$

From the Weitzenböck formula [3] we get

$$\sum_{i=1}^{m}(\nabla_{e_i}\nabla_{e_i}d\phi)(V_p^\top) = -\sum_{i=1}^{m}R^N(\phi_*V_p^\top,\phi_*e_i)\phi_*e_i + \phi_*\operatorname{Ric}^M(V_p^\top) - \triangle_H(d\phi)(V_p^\top),$$

where $\triangle_H$ denotes the Hodge-Laplacian on 1-form. Hence,

$$\begin{aligned}
&\sum_{p=1}^{r}\sum_{i=1}^{m}\left\{\langle\nabla_{e_i}\phi_*V_p^\top,\nabla_{e_i}\phi_*V_p^\top\rangle - \langle R^N(\phi_*V_p^\top,\phi_*e_i)\phi_*e_i,\phi_*V_p^\top\rangle\right\}\\
&=\sum_{p=1}^{r}\left\{\frac{1}{2}\triangle|d\phi|^2 - \sum_{i=1}^{m}\langle\nabla_{e_i}\nabla_{e_i}\phi_*V_p^\top,\phi_*V_p^\top\rangle - \sum_{i=1}^{m}\langle R^N(\phi_*V_p^\top,\phi_*e_i)\phi_*e_i,\phi_*V_p^\top\rangle\right\}\\
&=\sum_{p=1}^{r}\langle -2\sum_{i=1}^{m}(\nabla_{e_i}d\phi)(\nabla_{e_i}V_p^\top) - \sum_{i=1}^{m}\phi_*\nabla_{e_i}\nabla_{e_i}V_p^\top - \phi_*\operatorname{Ric}^M(V_p^\top) + \triangle_H(d\phi)(V_p^\top), d\phi(V_p^\top)\rangle + \frac{1}{2}\triangle|d\phi|^2 \qquad (4.8)\\
&=\sum_{i=1}^{m}\langle\phi_*Q^N(e_i),\phi_*e_i\rangle + \langle d(\triangle_H\phi)(e_i),\phi_*e_i\rangle + \frac{1}{2}\triangle|d\phi|^2.
\end{aligned}$$

From (4.6)–(4.8) and the F-harmonicity we obtain

$$\begin{aligned}
&\sum_{p=1}^{r}\frac{d^2}{dt^2}E_F(\phi_t)|_{t=0}\\
&=\int_M\left\{F''\left(\frac{|d\phi|^2}{2}\right)\left(\frac{1}{4}|d|d\phi|^2|^2 + \sum_{\alpha=1}^{r-m}\left(\sum_{i=1}^{m}\langle\phi_*A^{\nu_\alpha}e_i,\phi_*e_i\rangle\right)^2\right)\right.\\
&\quad + F'\left(\frac{|d\phi|^2}{2}\right)\left(\sum_{i=1}^{m}\langle\phi_*Q^N(e_i),\phi_*e_i\rangle\right.\\
&\quad \left.\left.+\langle d(\triangle_H\phi)(e_i),\phi_*e_i\rangle + \frac{1}{2}\triangle|d\phi|^2\right)\right\}dv_g
\end{aligned}$$

$$= \int_M \left\{ F''\left(\frac{|d\phi|^2}{2}\right)\left(\frac{1}{4}|d|d\phi|^2|^2 + \sum_{\alpha=1}^{r-m}\left(\sum_{i=1}^{m}\langle \phi_* A^{\nu_\alpha} e_i, \phi_* e_i\rangle\right)^2\right)\right.$$
$$\left. + F'\left(\frac{|d\phi|^2}{2}\right)\sum_{i=1}^{m}\langle \phi_* Q^N(e_i), \phi_* e_i\rangle - F''\left(\frac{|d\phi|^2}{2}\right)\frac{1}{4}|d|d\phi|^2|^2 \right\} dv_g.$$

Hence we have the following result:

Theorem 4.8 (Average second variational formula on the domain).

$$\sum_{p=1}^{r}\frac{d^2}{dt^2}E_F(\phi\circ f_t^{V_p^\top})|_{t=0} = \int_M \left\{ F''\left(\frac{|d\phi|^2}{2}\right)\sum_{\alpha=1}^{r-m}\left(\sum_{i=1}^{m}\langle \phi_* A^{\nu_\alpha} e_i, \phi_* e_i\rangle\right)^2\right.$$
$$\left. + F'\left(\frac{|d\phi|^2}{2}\right)\sum_{i=1}^{m}\langle \phi_* Q^N(e_i), \phi_* e_i\rangle \right\} dv_g. \tag{4.9}$$

The following lemma is essential in our argument.

Lemma 4.9. *For any constant $a > 0$, there is a strictly increasing and convex C^2 function $F : [0,\infty) \to [0,\infty)$ such that $t \cdot F''(t) < a \cdot F'(t)$ for any $t > 0$.*

Proof. The following functions have the desired properties:

(i) $F_1(t) = t^{b+1}$, $0 < b < a$,

(ii) $F_{2,n}(t) = \sum_{i=1}^{n} a_i t^i$, $n < a+1$, $a_1 > 0$, $a_i \geq 0$ $(i = 2, \cdots, n)$,

(iii) $F_3(t) = \int_0^t e^{\int_0^s G(u)du} ds$, where $G(u)$ is a continuous function and $u \cdot G(u) < a$.

□

From this lemma we obtain the following result concerning the relations (4.5) and (4.9).

Lemma 4.10. *Let M be an SSU manifold. Then there exists a strictly increasing and convex C^2 function $F : [0,\infty) \to [0,\infty)$ such that*

$$\sum_{p=1}^{r}\frac{d^2}{dt^2}E_F(f_t^{V_p^\top}\circ\phi)|_{t=0} < 0 \qquad \text{for any } F\text{-harmonic maps } \phi \text{ from } M,$$
$$\sum_{p=1}^{r}\frac{d^2}{dt^2}E_F(\phi\circ f_t^{V_p^\top})|_{t=0} < 0 \qquad \text{for any } F\text{-harmonic maps } \phi \text{ into } M.$$

Proof. Set

$$a = \min_{X\in UM}\frac{-\langle Q_x^M(X), X\rangle_M}{2|B(X,X)|^2_{\mathbf{R}^r}} > 0.$$

By Lemma 4.9 there exists a strictly increasing and convex C^2 function $F : [0,\infty) \to [0,\infty)$ such that

$$t \cdot F''(t) < \min_{X \in UM} \frac{-\langle Q_x^M(X), X\rangle_M}{2|B(X,X)|^2_{\mathbf{R}^r}} \cdot F'(t) \text{ for any } t > 0. \tag{4.10}$$

Let $\{v_\alpha\}_{\alpha=1}^n$ be a local orthonormal frame field on M and let $\phi_* e_i = \sum_{\alpha=1}^n a_i^\alpha v_\alpha$. We can choose $\{v_\alpha\}_{\alpha=1}^n$ so that $\sum_{i=1}^m a_i^\alpha a_i^\beta = 0$ if $\alpha \neq \beta$. Let $C_\alpha = \sum_{i=1}^m (a_i^\alpha)^2$ and $B(v_\alpha, v_\beta) = B_{\alpha\beta}$. Then $|d\phi|^2 = \sum_{\gamma=1}^n C_\gamma$ and

$$\begin{aligned}
&F''\left(\frac{|d\phi|^2}{2}\right)\left|\sum_{i=1}^m B(\phi_* e_i, \phi_* e_i)\right|^2 + F'\left(\frac{|d\phi|^2}{2}\right)\sum_{i=1}^m \langle Q^N(\phi_* e_i), \phi_* e_i\rangle \\
=&F''\left(\frac{|d\phi|^2}{2}\right)\left(\sum_{\alpha,\beta=1}^n \sum_{i=1}^m a_i^\alpha a_i^\beta B_{\alpha\beta}\right)^2 \\
&\quad + F'\left(\frac{|d\phi|^2}{2}\right)\sum_{\gamma=1}^n \left(\sum_{i=1}^m 2\left(\sum_{\alpha=1}^n a_i^\alpha B_{\alpha\gamma}\right)^2 - \sum_{\alpha,\beta=1}^n \sum_{i=1}^m a_i^\alpha a_i^\beta \langle B_{\alpha\beta}, B_{\gamma\gamma}\rangle\right) \\
=&F''\left(\frac{|d\phi|^2}{2}\right)\sum_{\alpha,\beta=1}^n C_\alpha C_\beta \langle B_{\alpha\alpha}, B_{\beta\beta}\rangle \\
&\quad + F'\left(\frac{|d\phi|^2}{2}\right)\sum_{\alpha,\beta=1}^n C_\alpha (2B_{\alpha\beta}^2 - \langle B_{\alpha\alpha}, B_{\beta\beta}\rangle) \\
=&\sum_{\alpha=1}^n C_\alpha \left(F''\left(\frac{|d\phi|^2}{2}\right)\sum_{\beta=1}^n C_\beta \langle B_{\alpha\alpha}, B_{\beta\beta}\rangle\right. \\
&\quad \left. + F'\left(\frac{|d\phi|^2}{2}\right)\sum_{\beta=1}^n (2B_{\alpha\beta}^2 - \langle B_{\alpha\alpha}, B_{\beta\beta}\rangle)\right) \\
\le&\sum_{\alpha=1}^n C_\alpha \left(F''\left(\frac{|d\phi|^2}{2}\right)|d\phi|^2 B_{\alpha\alpha}^2 + F'\left(\frac{|d\phi|^2}{2}\right)\sum_{\beta=1}^n (2B_{\alpha\beta}^2 - \langle B_{\alpha\alpha}, B_{\beta\beta}\rangle)\right).
\end{aligned}$$

Hence by (4.10) we have

$$\sum_{p=1}^r \frac{d^2}{dt^2} E_F(f_t^{V_p^\top} \circ \phi)|_{t=0} < 0.$$

Similarly, for each $1 \le \alpha \le r - m$, choose a corresponding local orthonormal basis $\{e_i^\alpha\}_{i=1}^n$ in M such that A^{ν_α} is diagonizable, and let $B_{ij}^\alpha = \langle A^{\nu_\alpha}(e_i^\alpha), e_j^\alpha\rangle$.

Then

$$\begin{aligned}
&F''\left(\frac{|d\phi|^2}{2}\right)\sum_{\alpha=1}^{r-m}\left(\sum_{i=1}^{m}\langle\phi_* A^{\nu_\alpha}e_i,\phi_* e_i\rangle\right)^2+F'\left(\frac{|d\phi|^2}{2}\right)\sum_{i=1}^{m}\langle\phi_* Q^N(e_i),\phi_* e_i\rangle\\
&=\sum_{\alpha=1}^{r-m}\left(F''\left(\frac{|d\phi|^2}{2}\right)\left(\sum_{i=1}^{m}B^\alpha_{ii}|\phi_* e_i^\alpha|^2\right)^2\right.\\
&\qquad\left.+\,F'\left(\frac{|d\phi|^2}{2}\right)\sum_{i,j=1}^{m}(2(B^\alpha_{ij})^2-B^\alpha_{ii}B^\alpha_{jj})|\phi_* e_i^\alpha|^2\right)\\
&\le\sum_{\alpha=1}^{r-m}\sum_{i,j=1}^{m}|\phi_* e_i^\alpha|^2\left(F''\left(\frac{|d\phi|^2}{2}\right)\cdot|d\phi|^2\cdot B^\alpha_{ii}B^\alpha_{jj}\right.\\
&\qquad\left.+F'\left(\frac{|d\phi|^2}{2}\right)\sum_{l=1}^{m}(2(B^\alpha_{il})^2-B^\alpha_{ii}B^\alpha_{ll})\right).
\end{aligned}$$

Hence by (4.10) we have

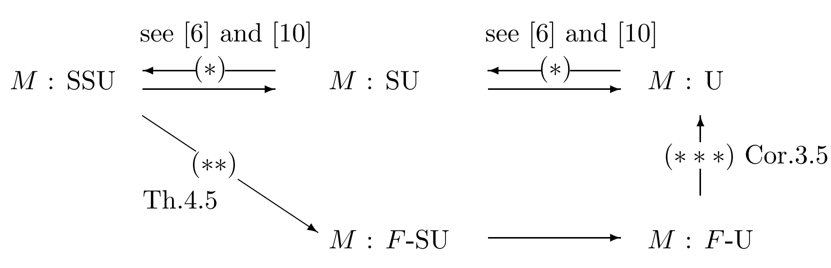
$$\sum_{p=1}^{r}\frac{d^2}{dt^2}E_F(\phi\circ f_t^{V_p^\top})|_{t=0}<0. \qquad \square$$

Proof of Theorem 4.5. The assertion follows immediately from Lemma 4.9 and Lemma 4.10. □

Corollary 4.11. *Let M be a compact irreducible symmetric space. Then there exists a strictly increasing and strictly convex C^2 function $F:[0,\infty)\to[0,\infty)$ such that M is F-SU if and only if M is F-U.*

Proof. This follows immediately from Corollary 3.5, Theorem 4.5 and the results of [10], [12], and [16] (see Theorem 4.14). □

The following diagram summarizes our results:



The arrows marked by asterisks hold under the following conditions:

(∗) If M is a compact irreducible symmetric space.
(∗∗) If F is as in Theorem 4.5.
(∗ ∗ ∗) If F is convex, and M is a compact irreducible symmetric space.

For compact irreducible F-U symmetric spaces we obtain, using the results of [6], [10], [12] and Theorem 3.4:

THEOREM 4.12. *Let $F : [0,\infty) \to [0,\infty)$ be a strictly increasing and convex C^2 function. Then M is a compact irreducible F-U symmetric space if and only if M is as given in Table 1, with $vF''(w) < F'(w)$.*

TABLE 1

		v	w
(1)	simply connected simple Lie groups		
	$(A_l)_{l\geq 1}$	l^2+2l	$\frac{l^2+2l}{2}$
	$(C_l)_{l\geq 2}$	$2l+1$	$\frac{2l^2+l}{2}$
(2)	$SU(2n)/Sp(n),\ n \geq 3$	$\frac{2n^2-n-1}{n+1}$	$\frac{2n^2-n-1}{2}$
(3)	spheres $S^k,\ k\geq 3$	$\frac{k}{k-2}$	$\frac{k}{2}$
(4)	quaternionic Grassmannians		
	$Sp(l+n)/Sp(l)\times Sp(n),\ l\geq n\geq 1$	$l+n$	$2ln$
(5)	E_6/F_4	$13/5$	7
(6)	Cayley plane $F_4/Spin(9)$	2	8

We next consider the case of p-harmonic maps, i.e., when $F(t) = (2t)^{p/2}/p$.

COROLLARY 4.13. *M is a compact irreducible p-U symmetric space ($p \geq 2$) if and only if M is as given in Table 2 below.*

Proof. Note that in the case where F is convex, every F-U manifold is U (see Corollary 3.5(i)). On every compact irreducible symmetric space M with the Cartan-Killing metric,

$$\frac{2F'\left(\frac{m}{2}\right)}{F'\left(\frac{m}{2}\right)+F''\left(\frac{m}{2}\right)}\cdot\frac{c}{m} = \frac{F'\left(\frac{m}{2}\right)}{F'\left(\frac{m}{2}\right)+F''\left(\frac{m}{2}\right)},$$

TABLE 2

(1)	simply connected simple Lie groups $(A_l)_{l\geq 1}$ where $p<3$ $(C_l)_{l\geq 2}$ where $l>p-2$
(2)	$SU(2n)/Sp(n),\ n\geq 3$ where $n>p-3$
(3)	spheres $S^k,\ k\geq 3$ where $k>p$
(4)	quaternionic Grassmannians $Sp(l+n)/Sp(l)\times Sp(n),\ l\geq n\geq 1$ where $(p-2)(l+n)<4ln$
(5)	E_6/F_4 where $p<76/13$
(6)	Cayley plane $F_4/\mathrm{Spin}(9)$, where $p<10$

where c is the scalar curvature of M and $m=\dim M$. Since $\dim A_l = l^2+2l$, $\dim B_2 = 10$ and $\dim C_l = 2l^2+l$, we see that

$$\lambda_1(A_l)_{l\geq 1} = \frac{l^2+2l}{l^2+2l+1} < \frac{F'\left(\frac{m}{2}\right)}{F'\left(\frac{m}{2}\right)+F''\left(\frac{m}{2}\right)}$$
$$\text{if and only if } (l^2+2l)\cdot F''((l^2+2l)/2) < F'((l^2+2l)/2),$$
$$\lambda_1(C_l)_{l\geq 2} = \frac{2l+1}{2l+2} < \frac{F'\left(\frac{m}{2}\right)}{F'\left(\frac{m}{2}\right)+F''\left(\frac{m}{2}\right)}$$
$$\text{if and only if } \ (2l+1)\cdot F''((2l^2+l)/2) < F'((2l^2+l)/2),$$

which gives entry (1) of Table 1. Similar computations give entries (2)–(6) by using the fact that

$$\lambda_1(SU(2n)/Sp(n))_{n\geq 3} = \frac{2n^2-n-1}{2n^2},$$
$$\dim SU(2n)/Sp(n) = 2n^2-n-1,$$
$$\lambda_1(S^k) = \frac{k}{2(k-1)}, \qquad \dim S^k = k,$$
$$\lambda_1(Sp(l+n)/Sp(l)\times Sp(n)) = \frac{l+n}{l+n+1},$$
$$\dim Sp(l+n)/Sp(l)\times Sp(n) = 4ln,$$
$$\lambda_1(E_6/F_4) = \frac{13}{18},\ \dim E_6/F_4 = 14,$$

$$\lambda_1(F_4/\mathrm{Spin}(9)) = \frac{2}{3}, \qquad \dim F_4/\mathrm{Spin}(9) = 16. \qquad \square$$

In the case where $F(t) = t$, the above theorem contains the following result of Howard-Wei and Ohnita:

THEOREM 4.14 ([6], [10], [12] and [16]). *Let M be a compact irreducible symmetric space. The following statements are equivalent:*

(a) *M is SSU.*
(b) *M is SU.*
(c) *M is U.*
(d) *M is one of the following:*
  (1) *one of the simply connected simple Lie groups $(A_l)_{l\geq 1}$ and $(C_l)_{l\geq 2}$;*
  (2) *$SU(2n)/Sp(n), n \geq 3$;*
  (3) *a sphere $S^k, k \geq 3$;*
  (4) *a quaternionic Grassmannian $Sp(l+n)/Sp(l)\times Sp(n)$, $l \geq n \geq 1$;*
  (5) *E_6/F_4;*
  (6) *the Cayley plane $F_4/Spin(9)$.*

5. Nonexistence of F-harmonic maps

In this section, we prove nonexistence theorems for nonconstant F-harmonic maps by adapting the techniques in [17] and [7]. We first derive the Bochner formula.

THEOREM 5.1 (Bochner formula).

$$\begin{aligned}\triangle F\left(\frac{|d\phi|^2}{2}\right) &= F'\left(\frac{|d\phi|^2}{2}\right)\Bigg\{-\langle \triangle_H d\phi, d\phi\rangle + |\nabla d\phi|^2 \\ &\quad - \sum_{ij}\langle R^N(\phi_* e_i, \phi_* e_j)\phi_* e_j, \phi_* e_i\rangle + \sum_i \langle \phi_* \operatorname{Ric}^M e_i, \phi_* e_i\rangle\Bigg\} \\ &\quad + F''\left(\frac{|d\phi|^2}{2}\right)\cdot |d\phi|^2 \cdot |\nabla|d\phi||^2\end{aligned}$$

Proof. We have

$$\begin{aligned}\triangle F\left(\frac{|d\phi|^2}{2}\right) &= F''\left(\frac{|d\phi|^2}{2}\right)\cdot \langle \nabla d\phi, d\phi\rangle^2 \\ &\quad + F'\left(\frac{|d\phi|^2}{2}\right)\cdot \langle \triangle d\phi, d\phi\rangle + F'\left(\frac{|d\phi|^2}{2}\right)\cdot |\nabla d\phi|^2 \\ &= F''\left(\frac{|d\phi|^2}{2}\right)\cdot |d\phi|^2 \cdot |\nabla|d\phi||^2 \\ &\quad + F'\left(\frac{|d\phi|^2}{2}\right)\Bigg\{ - \langle \triangle_H d\phi, d\phi\rangle + |\nabla d\phi|^2\end{aligned}$$

$$- \sum_{ij} \langle R^N(\phi_* e_i, \phi_* e_j)\phi_* e_j, \phi_* e_i \rangle + \sum_i \langle \phi_* \operatorname{Ric}^M e_i, \phi_* e_i \rangle \Bigg\}.$$

$\square$

THEOREM 5.2. *Let $F : [0, \infty) \to [0, \infty)$ be a strictly increasing and strictly convex C^2 function. Let $\phi : M \to N$ be an F-harmonic map, and suppose that $\operatorname{Ric}^M \geq 0$ and $R^N \leq 0$. Then we have:*

(1) *ϕ must be constant or totally geodesic.*

Furthermore, if, in addition, $F'(0) = 0$, then we have:

(2) *If $\operatorname{Ric}^M > 0$ at some point, then ϕ must be a constant map.*

(3) *If $R^N > 0$, then ϕ must be either a constant map or a mapping of rank one, that is, whose image is a closed geodesic.*

Proof. Integrating the Bochner formula and observing that, by the F-harmonicity,

$$\int_M F'\left(\frac{|d\phi|^2}{2}\right) \langle \triangle_H d\phi, d\phi \rangle v_g = \int_M \langle \delta d\phi, \delta \left(F'\left(\frac{|d\phi|^2}{2}\right) d\phi \right) \rangle v_g = 0$$

we have

$$\begin{aligned} 0 &\leq \int_M F'\left(\frac{|d\phi|^2}{2}\right) \cdot |\nabla d\phi|^2 v_g \\ &= \int_M F'\left(\frac{|d\phi|^2}{2}\right) \langle R^N(\phi_* e_i, \phi_* e_j)\phi_* e_j, \phi_* e_i \rangle v_g \\ &\quad - \int_M F'\left(\frac{|d\phi|^2}{2}\right) \sum_i \langle \phi_* \operatorname{Ric}^M e_i, \phi_* e_i \rangle v_g \\ &\quad - \int_M F''\left(\frac{|d\phi|^2}{2}\right) \cdot |\nabla |d\phi||^2 \cdot |d\phi|^2 v_g \leq 0. \end{aligned} \tag{5.1}$$

Thus, each nonpositive term is zero. We set $B = \{x \in M : |d\phi(x)| > 0\}$. If ϕ is not constant, then B is a nonempty open subset of M. In view of the inequality on the left of (5.1), ϕ is totally geodesic and $|d\phi|$ is constant on B. Hence B is also closed in M, so $B = M$. Therefore, ϕ is totally geodesic in M.

Next we assume that $F'(0) = 0$. Since the function F is strictly convex, this assumption implies that if $F'(t) = 0$ then $t = 0$. If $\operatorname{Ric}^M > 0$ at some point, then $F'\left(\frac{|d\phi|^2}{2}\right) = 0$, i.e., $|d\phi| = 0$ at that point. If $d\phi \not\equiv 0$, then B is a nonempty open subset of M. In view of the last integral in (5.1), $|d\phi|$ is constant on B. Hence B is also closed, so $B = M$, which is a contradiction.

If $R^N < 0$, then $F'\left(\frac{|d\phi|^2}{2}\right) = 0$ or $\langle R^N(\phi_* e_i, \phi_* e_j)\phi_* e_j, \phi_* e_i \rangle = 0$. In this case, the equation $\langle R^N(\phi_* e_i, \phi_* e_j)\phi_* e_j, \phi_* e_i \rangle = 0$ implies that the rank of ϕ

is either zero, and hence ϕ is constant, or one, in which case the image of a totally geodesic ϕ is a closed geodesic and the rank is constant and equal to one. □

We next study F-harmonic maps to manifolds which have convex functions. The following lemma is essential in our argument.

LEMMA 5.3. *Let $\phi : M \to N$ be a C^1 map between Riemannian manifolds and f a real valued C^2 function on N. Let $F : [0,\infty) \to [0,\infty)$ be a strictly increasing C^2 function. Then, for every C^1 function η on M, we have*

$$\langle F'\left(\frac{|d\phi|^2}{2}\right) d(f\circ\phi), d\eta\rangle = -F'\left(\frac{|d\phi|^2}{2}\right)\mathrm{Trace}(\nabla df)(d\phi, d\phi)\eta + \langle \nabla(\eta\cdot(\mathrm{grad}\, f)\circ\phi), F'\left(\frac{|d\phi|^2}{2}\right) d\phi\rangle.$$

Proof. Let $\{e_i\}$ be an orthonormal frame around some point of M which satisfies $\nabla e_i = 0$ at that point. We then compute:

$$\begin{aligned}
&\langle \nabla(\eta\cdot(\mathrm{grad}\, f)\circ\phi), F'\left(\frac{|d\phi|^2}{2}\right) d\phi\rangle \\
&= \sum_i \langle \nabla_{e_i}(\eta\cdot(\mathrm{grad}\, f)\circ\phi), F'\left(\frac{|d\phi|^2}{2}\right) d\phi(e_i)\rangle \\
&= \sum_i \langle d\eta(e_i)((\mathrm{grad}\, f)\circ\phi), F'\left(\frac{|d\phi|^2}{2}\right) d\phi(e_i)\rangle \\
&\qquad + \sum_i \eta F'\left(\frac{|d\phi|^2}{2}\right)\langle \nabla_{d\phi(e_i)}((\mathrm{grad}\, f)\circ\phi), d\phi(e_i)\rangle \\
&= \langle F'\left(\frac{|d\phi|^2}{2}\right) d(f\circ\phi), d\eta\rangle + \eta F'\left(\frac{|d\phi|^2}{2}\right)\mathrm{Trace}(\nabla df)(d\phi, d\phi).
\end{aligned}$$

This completes the proof. □

Using this lemma, we can now prove the following theorem.

THEOREM 5.4. *Let M be a compact connected Riemannian manifold and N a Riemannian manifold admitting a strictly convex function on N. Let $F : [0,\infty) \to [0,\infty)$ be a strictly increasing C^2 function. Then every F-harmonic map ϕ from M to N must be a constant map.*

REMARK 5.5. This theorem is an extension of results obtained in [4], [2] and [7] for harmonic maps and p-harmonic maps, respectively.

Proof. Let f be a real valued strictly convex function on N. Taking $\eta \equiv 1$ in the above lemma and integrating on M, we obtain, via the first variational formula for F-harmonic maps, the equation

$$\int_M F'\left(\frac{|d\phi|^2}{2}\right)\operatorname{Trace}(\nabla df)(d\phi, d\phi)v_g = 0.$$

Hence we have $d\phi = 0$ everywhere on M, which completes the proof. □

We next consider the case where the domain manifold is complete, noncompact and connected. Using Lemma 5.3, we can prove Liouville type theorems.

PROPOSITION 5.6. *Let M be a complete and noncompact connected Riemannian manifold and N a Riemannian manifold which possesses a strictly convex function f on N such that the uniform norm $\|df\|_\infty$ is bounded. Let $F : [0,\infty) \to [0,\infty)$ be a strictly increasing C^2 function. Then every F-harmonic map ϕ from M to N with finite $\int_M F'\left(\frac{|d\phi|^2}{2}\right)\cdot |d\phi| v_g$ must be a constant map.*

Proof. For every $R > 0$ we can find a Lipschitz continuous function η on M such that $\eta(x) = 1$ for $x \in B_R$, $\eta(x) = 0$ for $x \in M \setminus B_{2R}, 0 \le \eta \le 1$, and $|d\eta| \le C/R$ with a number $C > 0$ which is independent of R. Here B_R denotes a geodesic ball with radius R and with fixed point x_0.

By Lemma 5.3 we have

$$\begin{aligned}\int_M F'\left(\frac{|d\phi|^2}{2}\right)&\operatorname{Trace}(\nabla df)(d\phi, d\phi)\eta v_g \\ &= -\int_M F'\left(\frac{|d\phi|^2}{2}\right)\langle d(f\circ\phi), d\eta\rangle v_g \\ &\le \int_M F'\left(\frac{|d\phi|^2}{2}\right)\cdot \|df\|_\infty \cdot |d\phi|\cdot|d\eta| v_g.\end{aligned}$$

Since $\|df\|_\infty$ is bounded and $\int_M F'\left(\frac{|d\phi|^2}{2}\right)\cdot |d\phi| v_g < \infty$, we obtain

$$\int_{B_R} F'\left(\frac{|d\phi|^2}{2}\right)\operatorname{Trace}(\nabla df)(d\phi, d\phi)v_g \le \frac{C}{R}\int_M F'\left(\frac{|d\phi|^2}{2}\right)\cdot |d\phi| v_g.$$

Letting $R \to \infty$, we have $d\phi = 0$, which completes the proof. □

We can construct a smooth and strictly convex function whose uniform norm is bounded on a simply connected manifold with nonpositive sectional curvature (see [7]). Hence we have the following result.

THEOREM 5.7. *Let M be a complete and noncompact connected Riemannian manifold and N a simply connected Riemannian manifold with nonpositive sectional curvature. Let $F : [0,\infty) \to [0,\infty)$ be a strictly increasing C^2 function. Then every F-harmonic map ϕ from M to N with finite $\int_M F'\left(\frac{|d\phi|^2}{2}\right)\cdot |d\phi| v_g$ must be a constant map.*

Next we consider the case where $N = \mathbb{R}$. In this case we can deal with F-subharmonic functions. We call a function ϕ on M F-subharmonic if and only if ϕ satisfies the inequality

$$\mathrm{Trace}\nabla\left(F'\left(\frac{|d\phi|^2}{2}\right)d\phi\right) \geq 0.$$

THEOREM 5.8. *Let M be a complete and noncompact connected Riemannian manifold. Let $F : [0,\infty) \to [0,\infty)$ be a strictly increasing C^2 function. Then every F-subharmonic function ϕ from M with finite $\int_M F'\left(\frac{|d\phi|^2}{2}\right) \cdot |d\phi| v_g$ must be a constant map.*

Proof. Note that there is a non-decreasing strictly convex function f with bounded derivative on the real line. Then we get

$$\int_M F'\left(\frac{|d\phi|^2}{2}\right)\mathrm{Trace}(\nabla df)(d\phi, d\phi)\eta v_g \leq -\int_M F'\left(\frac{|d\phi|^2}{2}\right)\langle d(f\circ\phi), d\eta\rangle v_g$$

for every non-negative function η with compact support. The proof is now completed in the same way as that of Theorem 5.6. □

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