Illinois Journal of Mathematics
Volume 45, Number 2, Summer 2001, Pages 631–640
S 0019-2082

A PRIORI ESTIMATES FOR SCHRÖDINGER TYPE MULTIPLIERS

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ABSTRACT. We present an elementary proof of two a priori estimates for Schrödinger type multipliers on the circle. The first is an $L^4 - L^2$ inequality of Bourgain, while the second is a new $L^6 - L^{3/2}$ inequality. Estimates of this type are useful for the study of the Cauchy problem for Schrödinger type equations. The proofs are based on a counting argument and standard real and harmonic analysis techniques.

1. Introduction and results

In the first part of this work we prove the following estimate that arises in the study of the Cauchy problem for Schrödinger type equations.

THEOREM 1.1. *Let $(x,t) \in \mathbb{T} \times \mathbb{R}$ and let $(\xi, \tau) \in \mathbb{Z} \times \mathbb{R}$ be the dual variables. Let ν be a positive even integer. Then there is a constant $c_\nu > 0$ such that*

$$\|f\|_{L^4(\mathbb{T}\times\mathbb{R})} \le c_\nu \|(1 + |\tau - \xi^\nu|)^{\frac{\nu+1}{4\nu}} \hat{f}\|_{L^2(\mathbb{Z}\times\mathbb{R})}, \tag{1.1}$$

for any test function f on $\mathbb{T} \times \mathbb{R}$.

We immediately have the following dual estimate.

COROLLARY 1.2. *For any test function f we have*

$$\|(1 + |\tau - \xi^\nu|)^{-\frac{\nu+1}{4\nu}} \hat{f}\|_{L^2(\mathbb{Z}\times\mathbb{R})} \le c_\nu \|f\|_{L^{4/3}(\mathbb{T}\times\mathbb{R})}. \tag{1.2}$$

The quadratic case ($\nu = 2$) was proved by Bourgain in [B1]. The general case is stated without proof in [B3]. Our proof is motivated by the work of Fang and Grillakis [FG] and Zygmund [Z], and we believe that it is more transparent.

Received February 28, 2000.
2000 *Mathematics Subject Classification.* Primary 42B15. Secondary 35G25.
The authors were partially supported by the NSF under grant number DMS-9970857, and the Faculty Research Program of the University of Notre Dame.

In the second part of this work, using similar ideas, we prove the following new result:

THEOREM 1.3. *Let $(x,t) \in \mathbb{T} \times \mathbb{R}$ and $(\xi,\tau) \in \mathbb{Z} \times \mathbb{R}$ be the dual variables. Let ν be a positive even integer. Then there is a constant $c_\nu > 0$ such that*

$$\|f\|_{L^6(\mathbb{T}\times\mathbb{R})} \le c_\nu \|(1+|\tau-\xi^\nu|)^{\frac{\nu+1}{6\nu}} \hat{f}\|_{L^{3/2}(\mathbb{Z}\times\mathbb{R})}, \tag{1.3}$$

for any test function f on $\mathbb{T} \times \mathbb{R}$.

Similarly, dualizing (1.3) gives

COROLLARY 1.4. *For any test function f we have*

$$\|(1+|\tau-\xi^\nu|)^{-\frac{\nu+1}{6\nu}} \hat{f}\|_{L^3(\mathbb{Z}\times\mathbb{R})} \le c_\nu \|f\|_{L^{6/5}(\mathbb{T}\times\mathbb{R})}. \tag{1.4}$$

It is possible to compute explicitly the constants in Theorems 1.1 and 1.3. For example, in Theorem 1.1 one obtains

$$c_\nu \approx 3^{\frac{1}{4\nu}} \left(1 - 2^{\frac{1-\nu}{4\nu}}\right)^{-1/2}.$$

A natural question is to find the best constants for the above inequalities and to investigate their geometric significance. This question seems particularly interesting in higher dimensions.

Interpolating between (1.1) and (1.3) it is possible to obtain some intermediate $L^p - L^q$ estimates. One may ask what is the largest value of p for which an $L^p - L^2$ estimate holds for the above multipliers. The counterexample of Bourgain (see [B1]) shows that one cannot have an $L^6 - L^2$ estimate in the quadratic case $\nu = 2$. However, Bourgain conjectures that $L^{6-\epsilon} - L^2$ estimates should hold for small $\epsilon > 0$. Similarly, based on Theorem 1.3, one may conjecture that the corresponding $L^6 - L^{2-\epsilon}$ estimates hold for any small $\epsilon > 0$.

Inequalities of the type (1.1) – (1.4) are closely related to the periodic analogues of Strichartz inequalities. For a detailed discussion of these inequalities in the periodic case see Lecture 2 in [B3]. For nonperiodic Strichartz type inequalities and their applications to the wellposedness of the Cauchy problem for nonlinear pde's see Ginibre [G], Ginibre and Velo [GV], [HM], Kenig, Ponce and Vega [KPV1], [KPV2], [KPV3], Ponce [P], Sogge [So], Strichartz [Str], Stein [St], Vega [V], and the references in these works.

In the proof of the theorems, we follow the approach of Fang and Grillakis developed for the Boussinesq equation (see [FG]). In the next section we prove Theorem 1.1. Using a dyadic decomposition we reduce its proof to bilinear estimates (see Lemma 2.2). The main ingredient in the proof of these estimates is a counting argument (see Lemma 2.3) together with standard techniques involving the inverse Fourier transform, Plancherel's equality and Jensen's inequality. The proof of Theorem 1.3 is analogous, and is based on the same counting lemma. The main difference is the use of the Hausdorff-Young inequality which leads to the $L^{3/2}$ norm on the right hand side of (1.3).

2. Proof of Theorem 1.1

We may assume that

$$\operatorname{supp} \hat{f} \subseteq \{(\xi,\tau) : \tau - \xi^\nu \geq 0\}.$$

Otherwise we decompose $\hat{f}$ into a sum of two functions one supported in the above set and the other in the set $\{(\xi,\tau) : \tau - \xi^\nu \leq 0\}$. In both cases the proof is similar.

It will be convenient to introduce a dyadic decomposition of the frequency (ξ,τ)-space. For this we need the following lemma.

LEMMA 2.1. *There is a function* $\varphi \in C^\infty(\mathbb{R})$ *supported in the interval* $[1/2, 2]$ *such that*

$$\varphi(x) + \varphi(2x) = 1 \quad \textit{for all } 1/2 \leq x \leq 1,$$

and therefore

$$\sum_{j=-\infty}^{\infty} \varphi\left(\frac{x}{2^j}\right) = 1 \ \ \textit{for all } x > 0.$$

Proof. Observe that the function φ_0 defined by

$$\varphi_o(x) = e e^{-\frac{1}{x}}\left(1 - e^{-\frac{1}{1-x}}\right), \ \ 0 \leq x \leq 1,$$

is in $C^\infty[0,1]$, vanishes to infinite order at $x = 0$, and that $\varphi_o - 1$ vanishes to infinite order at $x = 1$. Define

$$\varphi(x) = \begin{cases} \varphi_o(2x-1), & \frac{1}{2} \leq x \leq 1, \\ 1 - \varphi_o(x-1), & 1 \leq x \leq 2, \\ 0, & \text{elsewhere.} \end{cases}$$

One readily checks that φ has the desired properties. □

REMARK. In the region $\{(\xi,\tau) : \tau - \xi^\nu \leq 0\}$ the appropriate cut-off function is similar to the one given above, but now supported in $[-2, -1/2]$.

Let

$$\varphi_j(x) = \varphi\left(\frac{x}{2^j}\right) \ \text{ and } \ \varphi_0(x) = 1 - \sum_{j=1}^{\infty} \varphi_j(x),$$

and define

$$\hat{f}_j(\xi,\tau) \doteq \varphi_j(\tau - \xi^\nu)\hat{f}(\xi,\tau).$$

Then

$$f = \sum_{j=0}^{\infty} f_j, \quad \text{supp}\hat{f}_o \subseteq \{(\xi, \tau) : 0 \le \tau - \xi^\nu \le 2\},$$

and

$$\text{supp}\hat{f}_j \subseteq \left\{(\xi, \tau) : 2^{j-1} \le \tau - \xi^\nu \le 2^{j+1}\right\}, \quad j = 1, 2, \cdots \tag{2.1}$$

We have

$$\|f\|_{L^4}^2 = \|f \cdot f\|_{L^2} = \| \sum_{j,k=0}^{\infty} f_j f_k \|_{L^2} \le \sum_{j,k=0}^{\infty} \|f_j f_k\|_{L^2},$$

where the last step is a consequence of Minkowski's inequality and Fatou's lemma. It therefore suffices to show:

Lemma 2.2. *There is a positive constant c such that*

$$\|f_j f_k\|_{L^2(\mathbb{T}\times\mathbb{R})} \le \frac{c}{2^{\frac{\nu-1}{4\nu}|j-k|}} \|(1 + |\tau - \xi^\nu|)^{\frac{\nu+1}{4\nu}} \hat{f}_j\|_{L^2(\mathbb{Z}\times\mathbb{R})} \cdot \|(1 + |\tau - \xi^\nu|)^{\frac{\nu+1}{4\nu}} \hat{f}_k\|_{L^2(\mathbb{Z}\times\mathbb{R})}.$$

Next, assuming this lemma we proceed to prove Theorem 1.1. We have

$$\begin{aligned}
\|f\|_{L^4}^2 &\le \sum_{j,k=0}^{\infty} \frac{c}{2^{\frac{\nu-1}{4\nu}|j-k|}} \|(1 + |\tau - \xi^\nu|)^{\frac{\nu+1}{4\nu}} \hat{f}_j\|_{L^2} \|(1 + |\tau - \xi^\nu|)^{\frac{\nu+1}{4\nu}} \hat{f}_k\|_{L^2} \\
&\le c\left(\sum_{j,k=0}^{\infty} \frac{1}{2^{\frac{\nu-1}{4\nu}|j-k|}} \|(1 + |\tau - \xi^\nu|)^{\frac{\nu+1}{4\nu}} \hat{f}_j\|_{L^2}^2 \right)^{1/2} \\
&\qquad \cdot \left(\sum_{j,k=0}^{\infty} \frac{1}{2^{\frac{\nu-1}{4\nu}|j-k|}} \|(1 + |\tau - \xi^\nu|)^{\frac{\nu+1}{4\nu}} \hat{f}_k\|_{L^2}^2 \right)^{1/2}
\end{aligned}$$

Since

$$\sum_{k=0}^{\infty} 2^{-\frac{\nu-1}{4\nu}|j-k|} \le 2 \sum_{m=0}^{\infty} \left(2^{-\frac{\nu-1}{4\nu}}\right)^m \le \frac{2}{1 - 2^{\frac{1-\nu}{4\nu}}}$$

we obtain

$$\begin{aligned}
\|f\|_{L^4}^2 &\le c \sum_{j=0}^{\infty} \|(1 + |\tau - \xi^\nu|)^{\frac{\nu+1}{4\nu}} \hat{f}_j\|_{L^2}^2 \\
&= c \sum_{j=0}^{\infty} \sum_{\xi\in\mathbb{Z}} \int_{\mathbb{R}} \left(1 + |\tau - \xi^\nu|\right)^{\frac{\nu+1}{2\nu}} \left(\phi_j(\tau - \xi^\nu)\right)^2 |\hat{f}(\tau, \xi)|^2 \, d\tau \\
&\le c \sum_{\xi\in\mathbb{Z}} \int_{\mathbb{R}} \left(1 + |\tau - \xi^\nu|\right)^{\frac{\nu+1}{2\nu}} \left(\sum_{j=0}^{\infty} \phi_j(\tau - \xi^\nu)\right)^2 |\hat{f}(\tau, \xi)|^2 \, d\tau \\
&\le c\|(1 + |\tau - \xi^\nu|)^{\frac{\nu+1}{4\nu}} \hat{f}\|_{L^2}^2,
\end{aligned}$$

which completes the proof of Theorem 1.1.

Proof of Lemma 2.2. By symmetry we may assume that $k \le j$. Using the inverse Fourier transform we write

$$f_j f_k(x,t) = \int_{\mathbb{R}^2} \sum_{\xi_1,\xi_2 \in \mathbb{Z}} e^{i[t(\tau_1+\tau_2)+x(\xi_1+\xi_2)]} \hat{f}_j(\xi_1,\tau_1)\hat{f}_k(\xi_2,\tau_2)\, d\tau_1 d\tau_2.$$

Introducing the change of variables

$$\tau = \tau_1 + \tau_2, \qquad q = \tau_2 - \xi_2^\nu,$$

and letting $\xi = \xi_1 + \xi_2$, we write $f_j f_k$ in the form

$$f_j f_k(x,t) = \int_{\mathbb{R}} \sum_{\xi \in \mathbb{Z}} e^{i(t\tau + x\xi)} \hat{G}_{jk}(\xi,\tau)\, d\tau,$$

where

$$\hat{G}_{jk}(\xi,\tau) = \int_{\mathbb{R}} \sum_{\xi_2 \in \mathbb{Z}} \hat{f}_j(\xi - \xi_2, \tau - q - \xi_2^\nu)\hat{f}_k(\xi_2, q + \xi_2^\nu)\, dq.$$

Observe that the restriction on the support of $\hat{f}_l$ in (2.1) implies that q and ξ_2 must satisfy the relations

$$q \in \Delta_k = [2^{k-1}, 2^{k+1}] \quad \text{and} \quad \xi_2 \in \Lambda_j(\tau,\xi,q),$$

where

$$\Lambda_j(\tau,\xi,q) = \left\{ \xi_2 \in \mathbb{Z} : \tau - q - 2^{j+1} \le \xi_1^\nu + \xi_2^\nu \le \tau - q - 2^{j-1}, \quad \xi_1 + \xi_2 = \xi \right\}.$$

The following estimate is crucial in what follows.

Lemma 2.3. *There exists a constant c independent of j such that*

$$\sup_{\tau,\xi,q} \operatorname{card}\Big(\Lambda_j(\tau,\xi,q)\Big) \le c\, 2^{\frac{j}{\nu}}.$$

The proof of Lemma 2.3 is given at the end of this section. Assuming the result for the moment and using Plancherel's equality and Jensen's inequality, we get

$$\begin{aligned}
\|f_j f_k\|_{L^2}^2 = \|\hat{G}_{jk}\|_{L^2}^2 &\le \int_{\mathbb{R}} \sum_{\xi \in \mathbb{Z}} \left(\int_{\Delta_k} \sum_{\xi_2 \in \Lambda_j} |\hat{f}_j \hat{f}_k|\, dq \right)^2 d\tau \\
&\le \int_{\mathbb{R}} \sum_{\xi \in \mathbb{Z}} \operatorname{meas}(\Delta_k) \int_{\Delta_k} \left(\sum_{\xi_2 \in \Lambda_j} |\hat{f}_j \hat{f}_k| \right)^2 dq\, d\tau \\
&\le c \int_{\mathbb{R}} \sum_{\xi \in \mathbb{Z}} 2^k\, 2^{\frac{j}{\nu}} \int_{\Delta_k} \sum_{\xi_2 \in \Lambda_j} |\hat{f}_j \hat{f}_k|^2\, dq\, d\tau
\end{aligned}$$

$$\leq c\,2^k\,2^{\frac{j}{\nu}} \int_{\mathbb{R}} \sum_{\xi\in\mathbb{Z}} \int_{\mathbb{R}} \sum_{\xi_2\in\mathbb{Z}} |\hat{f}_j(\xi-\xi_2, \tau-q-\xi_2^\nu)|^2 |\hat{f}_k(\xi_2, q+\xi_2^\nu)|^2\, dq\, d\tau$$

$$= c\,2^k\,2^{\frac{j}{\nu}} \sum_{\xi\in\mathbb{Z}} \sum_{\xi_2\in\mathbb{Z}} \int_{\mathbb{R}} |\hat{f}_j(\xi-\xi_2, \eta_1)|^2\, d\eta_1 \cdot \int_{\mathbb{R}} |\hat{f}_k(\xi_2, \eta_2)|^2\, d\eta_2$$

$$= c\,2^k\,2^{\frac{j}{\nu}} \|\hat{f}_j\|_{L^2}^2 \|\hat{f}_k\|_{L^2}^2.$$

Therefore, since $\tau - \xi^\nu \simeq 2^j$, we get

$$\|f_j f_k\|_{L^2} \leq \frac{c}{2^{\frac{\nu-1}{4\nu}(j-k)}}\, 2^{\frac{\nu+1}{4\nu} j}\, 2^{\frac{\nu+1}{4\nu} k}\, \|\hat{f}_j\|_{L^2} \|\hat{f}_k\|_{L^2}$$

$$\simeq \frac{c}{2^{\frac{\nu-1}{4\nu}(j-k)}} \|(1+|\tau-\xi^\nu|)^{\frac{\nu+1}{4\nu}} \hat{f}_j\|_{L^2} \|(1+|\tau-\xi^\nu|)^{\frac{\nu+1}{4\nu}} \hat{f}_k\|_{L^2},$$

and Lemma 2.2 follows. □

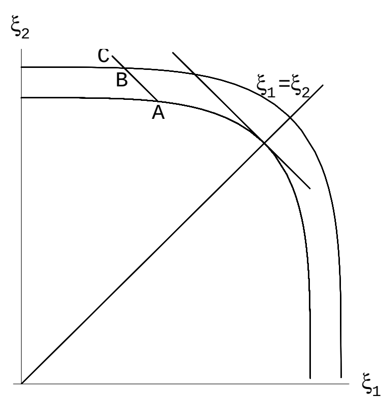
Proof of Lemma 2.3. Let $a = \tau - q - 2^{j+1}$ and consider the set

$$\Lambda_j(a,\xi) = \left\{ \xi_2 \in \mathbb{Z} : \quad a \leq \xi_1^\nu + \xi_2^\nu \leq a + \frac{3}{2}\, 2^j, \quad \xi_1 + \xi_2 = \xi \right\}. \tag{2.2}$$

To prove the lemma it suffices to estimate the length of the largest straight line segment of slope -1 intersecting the region between the two level curves

$$\xi_1^\nu + \xi_2^\nu = a \quad \text{and} \quad \xi_1^\nu + \xi_2^\nu = a + \frac{3}{2}\, 2^j.$$

By symmetry it suffices to consider the region between the diagonal $\xi_1 = \xi_2$ and the ξ_2–axis. Observe that the diagonal intersects the level curve $\xi_1^\nu + \xi_2^\nu = a$ at the point $((a/2)^{1/\nu}, (a/2)^{1/\nu})$.



Let s be any number in the interval $[0, (a/2)^{1/\nu}]$. Clearly the point $A = (s, (a-s^\nu)^{1/\nu})$ lies on the level curve $\xi_1^\nu + \xi_2^\nu = a$, and the equation of the line through A with slope -1 is

$$\xi_2 = -\xi_1 + s + (a - s^\nu)^{1/\nu}.$$

Consider the function

$$h(\xi_1) = \xi_1^\nu + \left[- \xi_1 + s + (a - s^\nu)^{1/\nu} \right]^\nu - a - \frac{3}{2}\, 2^j.$$

Observe that we have $h(\xi_1) = 0$ if and only if the point $B = (\xi_1, -\xi_1 + s + (a - s^\nu)^{1/\nu})$ lies on the outer level curve $\xi_1^\nu + \xi_2^\nu = a + \frac{3}{2}\, 2^j$. Furthermore, observe that

$$h(s) = -\frac{3}{2}\, 2^j. \tag{2.3}$$

On the other hand we have

$$h\left(s - \left(\frac{3}{2}\, 2^j \right)^{1/\nu} \right) \geq \frac{3}{2}\, 2^j. \tag{2.4}$$

In fact,

$$\begin{aligned} h\left(s - \left(\frac{3}{2}\, 2^j \right)^{1/\nu} \right) &= \left(s - \left(\frac{3}{2}\, 2^j \right)^{1/\nu} \right)^\nu \\ &\quad + \left(- s + \left(\frac{3}{2}\, 2^j \right)^{1/\nu} + s + (a - s^\nu)^{1/\nu} \right)^\nu - a - \frac{3}{2}\, 2^j. \end{aligned}$$

Since $(a - s^\nu)^{1/\nu} \geq s$, using the binomial formula and the fact that ν is even, we obtain

$$h\left(s - \left(\frac{3}{2}\, 2^j \right)^{1/\nu} \right) \geq s^\nu + \frac{3}{2}\, 2^j + (a - s^\nu) + \frac{3}{2}\, 2^j - a - \frac{3}{2}\, 2^j = \frac{3}{2}\, 2^j,$$

which gives (2.4).

From (2.3) and (2.4) we conclude that the distance between the points A and B is smaller than the distance between A and the point

$$C = \left(s - \left(\frac{3}{2}\, 2^j \right)^{1/\nu}, (a - s^\nu)^{1/\nu} + \left(\frac{3}{2}\, 2^j \right)^{1/\nu} \right).$$

Therefore

$$d(A, B) \leq d(A, C) = \sqrt{2} \left(\frac{3}{2}\, 2^j \right)^{1/\nu},$$

which proves Lemma 2.3. □

3. Proof of Theorem 1.3

The proof will be structured in a way similar to that of Theorem 1.1. Writing $f = \sum_{j=0}^\infty f_j$ as in the proof of Theorem 1.1, we have

$$\|f\|_{L^6}^2 = \|f \cdot f\|_{L^3} \leq \sum_{j,k=0}^\infty \|f_j f_k\|_{L^3}.$$

It therefore suffices to show:

LEMMA 3.1. *There is a positive constant c such that*

$$\|f_j f_k\|_{L^3(\mathbb{T}\times\mathbb{R})} \le \frac{c}{2^{\frac{\nu-1}{6\nu}|j-k|}} \|(1+|\tau-\xi^\nu|)^{\frac{\nu+1}{6\nu}} \hat{f}_j\|_{L^{3/2}} \cdot \|(1+|\tau-\xi^\nu|)^{\frac{\nu+1}{6\nu}} \hat{f}_k\|_{L^{3/2}}.$$

Next, assuming this lemma, we proceed to prove Theorem 1.3. Using Cauchy-Schwarz, we have

$$\|f\|_{L^6}^2 \le c\Bigg(\sum_{j,k=0}^{\infty} \frac{1}{2^{\frac{\nu-1}{6\nu}|j-k|}} \|(1+|\tau-\xi^\nu|)^{\frac{\nu+1}{6\nu}} \hat{f}_j\|_{L^{3/2}}^2 \Bigg)^{1/2} \cdot \Bigg(\sum_{j,k=0}^{\infty} \frac{1}{2^{\frac{\nu-1}{6\nu}|j-k|}} \|(1+|\tau-\xi^\nu|)^{\frac{\nu+1}{6\nu}} \hat{f}_k\|_{L^{3/2}}^2 \Bigg)^{1/2}.$$

Since

$$\sum_{k=0}^{\infty} 2^{-\frac{\nu-1}{6\nu}|j-k|} \le 2\sum_{m=0}^{\infty}\left(2^{-\frac{\nu-1}{6\nu}}\right)^m \le \frac{2}{1-2^{\frac{1-\nu}{6\nu}}},$$

we obtain

$$\begin{aligned}
\|f\|_{L^6}^2 &\le c\sum_{j=0}^{\infty} \|(1+|\tau-\xi^\nu|)^{\frac{\nu+1}{6\nu}} \hat{f}_j\|_{L^{3/2}}^2 \\
&= c\sum_{j=0}^{\infty}\left(\sum_{\xi\in\mathbb{Z}} \int_{\mathbb{R}} \left(1+|\tau-\xi^\nu|\right)^{\frac{\nu+1}{6\nu}\cdot\frac{3}{2}} \left(\phi_j(\tau-\xi^\nu)\right)^{3/2} |\hat{f}(\tau,\xi)|^{3/2}\, d\tau \right)^{4/3} \\
&\le c\left(\sum_{\xi\in\mathbb{Z}} \int_{\mathbb{R}} \left(1+|\tau-\xi^\nu|\right)^{\frac{\nu+1}{4\nu}} \left(\sum_{j=0}^{\infty}\phi_j(\tau-\xi^\nu)\right)^{3/2} |\hat{f}(\tau,\xi)|^{3/2}\, d\tau \right)^{4/3} \\
&\le c\|(1+|\tau-\xi^\nu|)^{\frac{\nu+1}{6\nu}} \hat{f}\|_{L^{3/2}}^2,
\end{aligned}$$

where the second last step follows from the inequality $\sum_{j=0}^{\infty} a_j^p \le \left(\sum_{j=0}^{\infty} a_j\right)^p$, which is valid for any $p \ge 1$ and any sequence of nonnegative numbers a_j. This completes the proof of Theorem 1.3.

Proof of Lemma 3.1. Proceeding as in the proof of the corresponding Lemma 2.2, we write $f_j f_k$ using the inverse Fourier transform. Then applying the counting Lemma 2.3, the Hausdorff-Young inequality and Jensen's inequality gives

$$\begin{aligned}
\|f_j f_k\|_{L^3}^{3/2} \le \|\hat{G}_{jk}\|_{L^{3/2}}^{3/2} &\le \int_{\mathbb{R}} \sum_{\xi\in\mathbb{Z}} \Big(\int_{\Delta_k} \sum_{\xi_2\in\Lambda_j} |\hat{f}_j \hat{f}_k|\, dq \Big)^{3/2} d\tau \\
&\le \int_{\mathbb{R}} \sum_{\xi\in\mathbb{Z}} \operatorname{meas}(\Delta_k)^{1/2} \int_{\Delta_k} \Big(\sum_{\xi_2\in\Lambda_j} |\hat{f}_j \hat{f}_k| \Big)^{3/2} dq\, d\tau \\
&\le c \int_{\mathbb{R}} \sum_{\xi\in\mathbb{Z}} 2^{\frac{k}{2}}\, 2^{\frac{j}{2\nu}} \int_{\Delta_k} \sum_{\xi_2\in\Lambda_j} |\hat{f}_j \hat{f}_k|^{3/2}\, dq\, d\tau \\
&\le c\, 2^{\frac{k}{2}}\, 2^{\frac{j}{2\nu}} \int_{\mathbb{R}^2} \sum_{\xi,\xi_2\in\mathbb{Z}} |\hat{f}_j(\xi-\xi_2, \tau-q-\xi_2^\nu)|^{3/2} |\hat{f}_k(\xi_2, q+\xi_2^\nu)|^{3/2}\, dq\, d\tau \\
&= c\, 2^{\frac{k}{2}}\, 2^{\frac{j}{2\nu}} \|\hat{f}_j\|_{L^{3/2}}^{3/2} \|\hat{f}_k\|_{L^{3/2}}^{3/2}.
\end{aligned}$$

Therefore, since $\tau - \xi^\nu \simeq 2^j$, we get

$$\begin{aligned}
\|f_j f_k\|_{L^3} &\le \frac{c}{2^{\frac{\nu-1}{6\nu}(j-k)}}\, 2^{\frac{\nu+1}{6\nu} j}\, 2^{\frac{\nu+1}{6\nu} k}\, \|\hat{f}_j\|_{L^{3/2}} \|\hat{f}_k\|_{L^{3/2}} \\
&\simeq \frac{c}{2^{\frac{\nu-1}{6\nu}(j-k)}} \|(1+|\tau-\xi^\nu|)^{\frac{\nu+1}{6\nu}} \hat{f}_j\|_{L^{3/2}} \|(1+|\tau-\xi^\nu|)^{\frac{\nu+1}{6\nu}} \hat{f}_k\|_{L^{3/2}}
\end{aligned}$$

and Lemma 3.1 follows. □

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