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SYMMETRY OF A BOUNDARY INTEGRAL OPERATOR AND A CHARACTERIZATION OF A BALL

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ABSTRACT. If Ω is a ball in $\mathbb{R}^n$ $(n \geq 2)$, then the boundary integral operator of the double layer potential for the Laplacian is self-adjoint on $L^2(\partial\Omega)$. In this paper we prove that the ball is the only bounded Lipschitz domain on which the integral operator is self-adjoint.

1. Introduction

Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$ $(n \geq 2)$. The double layer potential of $f \in L^2(\partial\Omega)$ is defined by

$$\mathcal{D}_\Omega f(x) \;=\; \frac{1}{w_n}\int_{\partial\Omega} \frac{\langle y-x, \nu(y)\rangle}{|y-x|^n} f(y)d\sigma(y), \qquad x \in \mathbb{R}^n \setminus \partial\Omega,$$

where $d\sigma$ is the $(n-1)$-dimensional Hausdorff measure in $\mathbb{R}^n$, w_n is the surface measure of the unit sphere in $\mathbb{R}^n$ and $\nu(y)$ is the outward unit normal to $\partial\Omega$ at $y \in \partial\Omega$.

The double layer potential attracted much attention lately in connection with the Dirichlet problem for the Laplacian on Lipschitz domains; see, for example, [3] and [8]. In [8] it was shown that for every $f \in L^2(\partial\Omega)$, $\mathcal{D}_\Omega f$ has a nontangential limit at almost all points on $\partial\Omega$ and

$$\lim_{t\to 0^-} \mathcal{D}_\Omega f(x + t\nu(x)) \;=\; \frac{1}{2}f(x) + \mathcal{K}_\Omega f(x) \qquad \text{a.e. } x \in \partial\Omega,$$

where

$$\mathcal{K}_\Omega f(x) = \frac{1}{w_n}\; p.v. \int_{\partial\Omega} \frac{\langle y-x, \nu(y)\rangle}{|y-x|^n} f(y)d\sigma(y), \qquad x \in \partial\Omega.$$

It has also been shown [2] that the operator $\mathcal{K}_\Omega$ is a singular integral operator and bounded on $L^2(\partial\Omega)$. It is this operator $\mathcal{K}_\Omega$ that we are considering in this paper.

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If Ω is a ball in $\mathbb{R}^n$, then one can easily see that

$$\langle x-y, \nu(x)\rangle \;=\; \langle y-x, \nu(y)\rangle \tag{1.1}$$

for all $x, y \in \partial\Omega$; hence $\mathcal{K}_\Omega$ is self adjoint, i.e.,

$$\mathcal{K}_\Omega \;=\; \mathcal{K}_\Omega^*.$$

The question we consider in this paper is whether there is any other (symmetric) domain Ω on which $\mathcal{K}_\Omega$ is self-adjoint. We will prove the following result.

Theorem 1.1. *Let Ω be a bounded Lipschitz simply connected domain in $\mathbb{R}^n$ $(n \geq 2)$. If $\mathcal{K}_\Omega$ is self-adjoint on $L^2(\partial\Omega)$, then Ω is a ball.*

If $\mathcal{K}_\Omega$ is self-adjoint, then one can easily see that (1.1) holds for almost all $x, y \in \partial\Omega$. Hence Theorem 1.1 is a corollary of the following proposition.

Proposition 1.2. *Let Ω be a bounded Lipschitz domain. If* (1.1) *holds for almost all $x, y \in \partial\Omega$, then Ω is a ball.*

In the case when Ω is C^1, Boas [1] showed that (1.1) implies that Ω is a ball. He studied this problem in connection with the Bochner-Martinelli kernel. His proof uses the fact that at any point b on the boundary that has maximal distance from a subdimensional hyperplane the normal vector is necessarily orthogonal to this hyperplane. Since on a Lipschitz domain the normal vector at b may not exist, a different argument is needed to deal with the Lipschitz case.

When $n = 2$, one can say more about the operator $\mathcal{K}_\Omega$. If Ω is $B_r(a)$, the disk of radius r with the center a, then for $x, y \in \partial\Omega$,

$$\frac{\langle y-x, \nu(y)\rangle}{|y-x|^2} \;=\; \frac{1}{2r}.$$

Hence, for all $f \in L^2_0(\partial\Omega) := \; \{f \in L^2(\partial\Omega) \mid \int_{\partial\Omega} f d\sigma = 0\}$,

$$\mathcal{K}_\Omega f(x) \;=\; 0, \qquad x \in \partial\Omega.$$

This fact was used in [4] to derive a uniqueness theorem for the inverse conductivity problem. We will show:

Theorem 1.3. *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain. If $\mathcal{K}_\Omega f = 0$ for all $f \in L^2_0(\partial\Omega)$, then $n = 2$ and Ω is a disk.*

There is some related work in connection with the electrostatic theory. In [5], Mendez and Reichel proved that if $\rho =$ const is the only solution of the integral equation $-(1/2)\rho + \mathcal{K}_\Omega^*\rho = 0$ and Ω is a bounded convex Lipschitz domain, then Ω is a ball. Since the null space of $-(1/2)I + \mathcal{K}_\Omega^*$ is one dimensional [8], it follows that $\mathcal{K}_\Omega^* 1 = 1/2$ if and only if Ω is a ball. Since $\mathcal{K}_\Omega 1 = 1/2$ for any bounded Lipschitz domain Ω, Theorem 1.1 for convex domains follows from the result of Mendez and Reichel. However, our results apply to both convex and nonconvex domains. Other characterization of balls by means of layer potentials have been given in [6] and [7].

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2. Proofs

In this section, we prove Proposition 1.2 and Theorem 1.3.

Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$ and suppose (1.1) holds on $\partial\Omega \times \partial\Omega$. We first extend ν to a function $\tilde{\nu} : \partial\Omega \to S^{n-1}$, while preserving the property (1.1). Let

$$V := \{(x, y) \in \partial\Omega \times \partial\Omega \mid \nu(x), \nu(y) \text{ exist and } \langle x - y, \nu(x)\rangle = \langle y - x, \nu(y)\rangle\}.$$

Then V contains almost all points of $\partial\Omega \times \partial\Omega$ by our assumption. Thus almost all points of $\partial\Omega$ are contained in the set

$$W := \{y \in \partial\Omega \mid (x, y) \in V \quad \text{for a.e. } x \in \partial\Omega\}.$$

Moreover, $\partial\Omega$ cannot be contained in a hyperplane. Hence there exist $y_0, \cdots, y_n \in W$ such that $y_0, \cdots, y_n$ do not lie in a hyperplane. Define

$$S := \{x \in \partial\Omega \mid (x, y_j) \in V \quad \text{for all } j = 0, \cdots, n\}.$$

Then S contains almost all points of $\partial\Omega$. Hence for each $x \in \partial\Omega$ there exists a sequence $\{x_n\}$ in S converging to x. We may choose a subsequence, say $\{x_n\}$, such that $\{\nu(x_n)\}$ is also convergent, since S^{n-1} is compact. Define

$$\tilde{\nu}(x) := \lim_{n\to\infty} \nu(x_n).$$

Applying (1.1) to (x_n, y_j), we have that for all $n \in \mathbb{N}$

$$\langle x_n - y_j, \nu(x_n)\rangle = \langle y_j - x_n, \nu(y_j)\rangle, \qquad 0 \le j \le n.$$

Hence

$$\langle x - y_j, \tilde{\nu}(x)\rangle = \langle y_j - x, \nu(y_j)\rangle, \qquad 0 \le j \le n. \tag{2.1}$$

Note that $x, y_0, \cdots, y_n$ are not contained in a single hyperplane. Hence $y_0 - x, \cdots, y_n - x$ generate $\mathbb{R}^n$. Thus (2.1) implies that the definition of $\tilde{\nu}(x)$ is independent of the choice of $\{x_n\}$, and $\tilde{\nu}(x) = \nu(x)$ for all $x \in S$.

Lemma 2.1.

(i) *We have* $\langle x - y, \tilde{\nu}(x)\rangle = \langle y - x, \tilde{\nu}(y)\rangle$ *for all* $x, y \in \partial\Omega$.

(ii) *Let* $L^-(x) := \{x + t\tilde{\nu}(x) \mid t < 0\} \bigcap \partial\Omega$. *Then* $L^-(x) \neq \phi$ *for all* $x \in \partial\Omega$.

Proof. (i) Note that $(S\times S)\bigcap V$ contains almost all points of $\partial\Omega\times\partial\Omega$. For each pair $x,y\in\partial\Omega$ there exists a sequence $\{(x_n,y_n)\}$ in $(S\times S)\bigcap V$ converging to (x,y) such that

$$\tilde{\nu}(x)=\lim_{n\to\infty}\nu(x_n),\qquad \tilde{\nu}(y)=\lim_{n\to\infty}\nu(y_n).$$

Applying (1.1) to (x_n,y_n), we obtain

$$\langle x_n-y_n,\nu(x_n)\rangle=\langle y_n-x_n,\nu(y_n)\rangle\qquad\text{for all } n\in\mathbb{N}.$$

Hence

$$\langle x-y,\tilde{\nu}(x)\rangle=\langle y-x,\tilde{\nu}(y)\rangle. \tag{2.2}$$

(ii) For each $x\in S$, there exists a normal vector $\nu(x)$, and $\tilde{\nu}(x)=\nu(x)$. Define

$$L(x):=\{x+t\tilde{\nu}(x)\mid t\in\mathbb{R}\}.$$

Since there is a normal vector at x, there exists a point w_x in $(L(x)\bigcap\partial\Omega)\setminus\{x\}$ for which the line segment joining x and w_x is contained in $\bar{\Omega}$. Since $\nu(x)$ is the outward normal vector at x, we have

$$\nu(x)=\frac{x-w_x}{|x-w_x|}\ . \tag{2.3}$$

By (2.2),

$$\langle x-w_x,\frac{x-w_x}{|x-w_x|}\rangle=\langle w_x-x,\tilde{\nu}(w_x)\rangle.$$

Hence

$$|x-w_x|=\langle w_x-x,\tilde{\nu}(w_x)\rangle.$$

Since $|\tilde{\nu}(w_x)|=1$, we conclude that

$$\tilde{\nu}(w_x)=\frac{w_x-x}{|w_x-x|}=-\nu(x). \tag{2.4}$$

Now let $x,y\in S$. We have from (2.2) and (2.4)

$$\langle x-y,\nu(x)\rangle=\langle y-x,\nu(y)\rangle,$$
$$\langle w_x-w_y,-\nu(x)\rangle=\langle w_y-w_x,-\nu(y)\rangle.$$

Observe that

$$\langle x-w_x,\nu(x)\rangle+\langle w_x-y,\nu(x)\rangle=\langle y-w_y,\nu(y)\rangle+\langle w_y-x,\nu(y)\rangle,$$
$$\langle w_x-x,-\nu(x)\rangle+\langle x-w_y,-\nu(x)\rangle=\langle w_y-y,-\nu(y)\rangle+\langle y-w_x,-\nu(y)\rangle.$$

Adding the two equations, we obtain

$$\begin{aligned}2\langle x-w_x,\nu(x)\rangle+\langle w_x-y,\nu(x)\rangle+\langle x-w_y,-\nu(x)\rangle\\ = 2\langle y-w_y,\nu(y)\rangle+\langle w_y-x,\nu(y)\rangle+\langle y-w_x,-\nu(y)\rangle.\end{aligned}$$

From (2.2) and (2.4) we have

$$\langle w_x - y, \nu(x)\rangle = \langle y - w_x, -\nu(y)\rangle,$$
$$\langle x - w_y, -\nu(x)\rangle = \langle w_y - x, \nu(y)\rangle.$$

It then follows that

$$\langle x - w_x, \nu(x)\rangle = \langle y - w_y, \nu(y)\rangle.$$

Hence $|x - w_x|$ is a constant for all $x \in S$. Let C denote this constant.

Now we consider the points in $\partial\Omega \setminus S$. Let $x \in \partial\Omega$. Then

$$\tilde{\nu}(x) = \lim_{n\to\infty} \nu(x_n)$$

for some sequence $\{x_n\}$ in S converging to x. By (2.4),

$$w_{x_n} = x_n - C\nu(x_n) \qquad \text{for all } n \in \mathbb{N}.$$

Define

$$z := \lim_{n\to\infty} w_{x_n}.$$

Then $z \in \partial\Omega$ and

$$z = x - C\tilde{\nu}(x).$$

It follows that $L^-(x)$ is non-empty. □

Proof of Proposition 1.2. The single layer potential of a constant 1 is defined by

$$\mathcal{S}_\Omega 1(x) = \begin{cases} \dfrac{1}{(2-n)w_n}\displaystyle\int_{\partial\Omega} \frac{1}{|y-x|^{n-2}} d\sigma(y), & n \geq 3, \\ \dfrac{1}{2\pi}\displaystyle\int_{\partial\Omega} \log|y-x| d\sigma(y), & n = 2, \end{cases}$$

and satisfies

$$\lim_{t\to 0^-} \langle \bigtriangledown \mathcal{S}_\Omega 1(x + t\nu(x)), \nu(x)\rangle = -\frac{1}{2} + \mathcal{K}_\Omega^* 1(x),$$
$$\lim_{t\to 0^+} \langle \bigtriangledown \mathcal{S}_\Omega 1(x + t\nu(x)), \nu(x)\rangle = \frac{1}{2} + \mathcal{K}_\Omega^* 1(x), \qquad \text{a.e. } x \in \partial\Omega.$$

By the assumption, we have $\mathcal{K}_\Omega^* 1 = 1/2$. Hence $\mathcal{S}_\Omega 1$ is constant in the interior of Ω, and its exterior boundary gradient is equal to the unit normal vector $\nu(x)$. Let $u(x) := \mathcal{S}_\Omega 1(x)$ for $x \in \mathbb{R}^n \setminus \Omega$. By (2.2) we have

$$\tilde{\nu}(x) \cdot x - \tilde{\nu}(x) \cdot y - \tilde{\nu}(y) \cdot y + \tilde{\nu}(y) \cdot x = 0,$$
$$\tilde{\nu}(x) \cdot x - \tilde{\nu}(x) \cdot w_y - \tilde{\nu}(w_y) \cdot w_y + \tilde{\nu}(w_y) \cdot x = 0.$$

Adding these two equations and using (2.4), we get

$$2\tilde{\nu}(x) \cdot x - \tilde{\nu}(x) \cdot (y + w_y) - \tilde{\nu}(y) \cdot (y - w_y) = 0.$$

Since the last term is independent of y, it follows that

$$2\tilde{\nu}(x) \cdot x - \tilde{\nu}(x) \cdot (y + w_y) - C = 0.$$

and thus

$$2 \bigtriangledown u(x) \cdot x - \bigtriangledown u(x) \cdot (y + w_y) - C = 0.$$

Hence, for each y, the function $\bigtriangledown u(x) \cdot (y + w_y)$ is a harmonic function decaying at ∞ with the same boundary values. The maximum principle implies that this function is a harmonic function independent of y. From this we conclude that the function $y + w_y$ is independent of y. Indeed, if $y + w_y$ were dependent of y, then u would have a directional derivative that is constant and equal to zero, which is impossible because u is decaying at ∞. Since $|y - w_y|$ is constant, it follows that Ω is a ball. □

Proof of Theorem 1.3. Let $f \in L^2(\partial\Omega)$. Then

$$\begin{aligned}\mathcal{K}_\Omega f(x) &= \mathcal{K}_\Omega \left(f - \frac{1}{|\partial\Omega|} \int_{\partial\Omega} f(y) d\sigma(y) + \frac{1}{|\partial\Omega|} \int_{\partial\Omega} f(y) d\sigma(y) \right)(x) \\ &= \mathcal{K}_\Omega 1(x) \frac{1}{|\partial\Omega|} \int_{\partial\Omega} f(y) d\sigma(y).\end{aligned}$$

Note that

$$\mathcal{K}_\Omega 1(x) = \frac{1}{w_n}\ p.v. \int_{\partial\Omega} \frac{\langle y - x, \nu(y) \rangle}{|y - x|^n} d\sigma(y) = \frac{1}{2} \qquad \text{a.e. } x \in \partial\Omega.$$

Thus

$$\mathcal{K}_\Omega f(x) \ = \ \frac{1}{2|\partial\Omega|} \int_{\partial\Omega} f(y) d\sigma(y) \qquad \text{a.e. } x \in \partial\Omega$$

and hence

$$\frac{\langle y - x, \nu(y) \rangle}{|y - x|^n} \ = \ \frac{w_n}{2|\partial\Omega|} \qquad \text{a.e. } x, y \in \partial\Omega. \tag{2.5}$$

Applying Proposition 1.2 we obtain

$$\Omega \ = \ B_a(r)$$

for some $a \in \mathbb{R}^n$ and $r > 0$. Let

$$S := \{a + r(\cos\theta, \sin\theta, 0, \cdots, 0) \mid \theta \in [0, 2\pi]\}.$$

Then S is contained in $\partial\Omega$ and is a circle.

Therefore, if $x, y \in S$, then

$$\frac{\langle y - x, \nu(y) \rangle}{|y - x|^2} \ = \ \frac{1}{2} \frac{\langle y - x, \nu(y) - \nu(x) \rangle}{|y - x|^2} \ = \ \frac{1}{2r}.$$

It then follows from (2.5) that for all $x, y \in S$

$$\frac{1}{2r^{n-1}} = \frac{\langle y - x, \nu(y) \rangle}{|y - x|^n} = \frac{1}{|y - x|^{n-2}} \frac{1}{2r}.$$

Thus $n = 2$, and Ω is a disk. □

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