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THE TRANSVERSE GEOMETRY OF G-MANIFOLDS AND RIEMANNIAN FOLIATIONS

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Abstract. Given a compact Riemannian manifold on which a compact Lie group acts by isometries, it is shown that there exists a Riemannian foliation whose leaf closure space is naturally isometric (as a metric space) to the orbit space of the group action. Furthermore, this isometry (and foliation) may be chosen so that a leaf closure is mapped to an orbit with the same volume, even though the dimension of the orbit may be different from the dimension of the leaf closure. Conversely, given a Riemannian foliation, there is a metric on the basic manifold (an $O(q)$-manifold associated to the foliation) such that the leaf closure space is isometric to the $O(q)$-orbit space of the basic manifold via an isometry that preserves the volume of the leaf closures of maximal dimension. Thus, the orbit space of any Riemannian G-manifold is isometric to the orbit space of a Riemannian $O(q)$-manifold via an isometry that preserves the volumes of orbits of maximal dimension. Consequently, the spectrum of the Laplacian restricted to invariant functions on any G-manifold may be identified with the spectrum of the Laplacian restricted to invariant functions on a Riemannian $O(q)$-manifold. Other similar results concerning the spectrum of differential operators on sections of vector bundles over Riemannian foliations and G-manifolds are discussed.

1. Introduction

Let M be a compact, Riemannian n-manifold on which a compact Lie group G acts by orientation-preserving isometries. In this paper, we construct a Riemannian $SO(n)$-manifold W that has the following properties. First, $W \diagup SO(n)$ is isometric as a metric space to $M \diagup G$. Next, the metric on W may be chosen so that the isometry preserves the volumes of orbits of maximal dimension, even though the dimensions of corresponding orbits may be different. To prove this, in Section 2 we construct a transversally-oriented, codimension n, Riemannian foliation whose leaf closure space is naturally

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isometric to the orbit space of the group action. The metric on the foliation may be chosen so that a leaf closure is mapped to an orbit with the same volume, even though the dimension of the orbit is different from the dimension of the leaf closure. In Section 2, we show that given any transversally-oriented, Riemannian foliation of codimension q, there is a metric on the basic manifold (an $SO(q)$-manifold associated to the foliation) such that the leaf closure space is isometric to the $SO(q)$-orbit space of the basic manifold via an isometry that preserves the volume of the leaf closures of maximal dimension. By combining these results, we are able to construct the Riemannian $SO(n)$-manifold W that has the characteristics mentioned above (see Section 4). Another consequence is the fact that the leaf closure space of any transversally-oriented, Riemannian foliation of codimension q is isometric to the leaf closure space of another Riemannian foliation that is constructed by suspending a dense subgroup of $SO(q)$; again the isometry preserves the volumes of leaf closures of maximal dimension.

We remark that in cases where the group action does not necessarily preserve orientation and where the foliation is not transversally orientable, results similar to those in the first paragraph are true with the group $SO(*)$ replaced by $O(*)$.

The consequences of the existence of these "transverse isometries" are discussed in Sections 5 and 6. The invariant differential forms on a G-manifold correspond precisely to the basic forms (pullbacks of forms on the local quotients) on the Riemannian foliation constructed in Section 2; in fact, the natural correspondence gives an isometry between the L^2 spaces. This isometry intertwines the invariant Laplacian on the G-manifold and the basic Laplacian on the Riemannian foliation. Analogously, the spectrum of the basic Laplacian on functions on a Riemannian foliation (or the spectrum of the Laplacian on invariant functions on a G-manifold) may be identified with the spectrum of the Laplacian restricted to invariant functions on a Riemannian $O(q)$-manifold. Also, one may write an invariant heat kernel on a G-manifold in terms of a basic heat kernel on a Riemannian foliation, and vice versa (see Corollary 5.2 and Corollary 5.4). Other similar results concerning the spectrum of differential operators on sections of vector bundles over Riemannian foliations and G-manifolds are discussed in Section 6. For example, given a G-equivariant, elliptic differential operator D on sections of a G-equivariant vector bundle over a G-manifold M, there exists an $O(n)$-equivariant vector bundle E' over an $O(n)$-manifold M', an $O(n)$-equivariant, transversally elliptic differential operator D', and an invertible map $\Theta : \Gamma\left(E'\right)^{O(n)} \to \Gamma\left(E\right)^{G}$ on invariant sections such that $D\Theta = \Theta D'$. There are obvious consequences for spectral theory and index theory.

There are many known results concerning the topology and geometry of such group actions. We remark that if G acts smoothly on a smooth, compact manifold, we may choose a Riemannian metric so that the action is isometric.

By [6], if G acts smoothly and properly, then there is a real analytic structure on M compatible with the given smooth structure such that the action becomes real analytic. It is known that the orbit space is triangulable (see [7], [10], [9]). Also, it is well-known that the orbit space of every G-manifold may be locally modelled as the space of orbits of a subgroup of the orthogonal group acting by isometries on a metric ball. In [16], R. Palais showed among other results that any G-manifold may be imbedded in Euclidean space such that the group G acts by orthogonal transformations. (G. D. Mostow proved similar results in [12].) Thus, the orbit space of every G-manifold M is diffeomorphic to the space of orbits of a subgroup H of an orthogonal group acting on M. In the foliation setting, the theory of P. Molino [13] gives a homeomorphism between the leaf closure space of a Riemannian foliation and the space of orbits of $O(q)$ acting on the basic manifold, a compact manifold associated to the foliation. The results of this paper show that the metric on the basic manifold may be chosen so that the homeomorphism preserves the transverse geometry and transfers the basic analysis to invariant analysis.

2. Constructing Riemannian foliations from G-manifolds

Let G be a compact Lie group that acts on the right by isometries on an n-dimensional Riemannian manifold M. We now construct a Riemannian foliation $\left(\widetilde{M}, \mathcal{F}\right)$ such that the leaf closure space $\widetilde{M}\diagup\overline{\mathcal{F}}$ is isometric to the orbit space $M\diagup G$. This foliation is a suspension of a dense subgroup of G. First, let T_1, T_2, ..., T_ℓ be a collection of maximal tori of G such that their Lie algebras span the Lie algebra of G. For each i, choose $g_i \subset T_i$ such that the cyclic group generated by g_i is dense in T_i. The subgroup Γ_0 generated by $\{g_1, g_2, ..., g_\ell\}$ is dense in the connected component of the identity in G. By adding a finite set $\{g_{\ell+1}, g_{\ell+2}, ..., g_k\}$ of group elements to the list, we may assume that the subgroup Γ generated by $\{g_1, g_2, ..., g_k\}$ is dense in G. Next, let X be any compact, connected Riemannian manifold with volume 1 such that there is a surjective homomorphism $\mu : \pi_1(X) \to \Gamma$. For example, X could be homeomorphic to the connected sum of k copies of $S^1 \times S^2$, since its fundamental group is the free group on k generators. We now form the Riemannian foliation defined by suspension. Let $\widetilde{X}$ be the universal cover of X with the induced metric, and let $\pi_1(X)$ act isometrically on $\widetilde{X}$ on the left by deck transformations. Let $\widetilde{M} = \widetilde{X} \times M\diagup \sim$, where the metric on $\widetilde{M}$ is locally the product metric and where $(x, y) \sim \left([\gamma^{-1}]x, y\,\mu([\gamma])\right)$ for every $x \in \widetilde{X}$, $y \in M$, and $[\gamma] \in \pi_1(X)$. The codimension n foliation $\mathcal{F}$ on $\widetilde{M}$ is defined by letting the leaves be sets of the form $L_{y_0} = \left\{[(x, y_0)]_\sim \mid x \in \widetilde{X}\right\}$ (note that $L_{y_0} = L_{y_1}$ does not necessarily imply that $y_1 = y_0$). One can check that the chosen metric is bundlelike for this foliation. We have the following:

LEMMA 2.1. *The leaf closures of* $\left(\widetilde{M},\mathcal{F}\right)$ *are sets of the form* $L_C = \left\{[(x,y)]_\sim \mid x\in\widetilde{X},\, y\in C\right\}$, *where* C *is an orbit of* G *in* M. *Given such a leaf closure, the orbit* C *is uniquely determined; the closure of the leaf* L_{y_0} *is* L_{C_0}, *where* C_0 *is the orbit of* G *in* M *containing* y_0.

Proof. Given a leaf L_{y_0} and any $[(x,y_0)]_\sim\in L_{y_0}$, we also have $[([\gamma]x,y_0)]_\sim \in L_{y_0}$ for any $[\gamma]\in\pi_1(X)$. This implies that $[(x,y_0\,\mu([\gamma]))]_\sim\in L_{y_0}$ for every $[\gamma]\in\pi_1(X)$, so that $[(x,y_0\,g)]_\sim\in L_{y_0}$ for every $g\in\Gamma$. Thus, $[(x,y_0\,g)]_\sim\in \overline{L_{y_0}}$ for every $g\in G$, so that $L_{y_0}\subseteq L_{C_0}\subseteq\overline{L_{y_0}}$. Next, suppose that $S=\{[(x_i,y_0g_i)]_\sim\}$ is a sequence of points in L_{C_0}. Then, if $p:\widetilde{X}\to X$ is the covering map, the sequence $\{p(x_i)\}$ has a subsequence which converges to $\overline{x}\in X$ since X is compact. By trivializing the cover near $\overline{x}$ and choosing different representatives (x_i,y_0g_i) if necessary, we may assume that we have a subsequence $S_2=\{[(x_i,y_0g_i)]_\sim\}$ of S such that $x_i\to x$ for some $x\in\widetilde{X}$. Since G is compact, we may assume that there is another subsequence $S_3=\{[(x_i,y_0g_i)]_\sim\}$ of S such that $x_i\to x$ for some $x\in\widetilde{X}$ and $g_i\to g$ for some $g\in G$. Thus, S_3 converges to $[(x,y_0g)]_\sim\in L_{C_0}$. We have shown that L_{C_0} is compact, so that $L_{C_0}=\overline{L_{y_0}}$. □

Observe that the distance between leaf closures, respectively orbits, makes $\widetilde{M}\diagup\overline{\mathcal{F}}$, respectively $M\diagup G$, into a metric space. We will now show that the leaf closure space $\widetilde{M}\diagup\overline{\mathcal{F}}$ is isometric to the orbit space $M\diagup G$. Let $\Phi: M\diagup G\to\widetilde{M}\diagup\overline{\mathcal{F}}$ be defined by $\Phi(C)=L_C$. By the previous lemma, this map is a bijection. Let C_1 and C_2 be two orbits on M, and let α be a minimal geodesic connecting them, so that the length of α is the distance between them. Then α is perpendicular to each of the orbits. For any $x\in\widetilde{X}$, the curve $\widetilde{\alpha}=[(x,\alpha)]_\sim\subset\widetilde{M}$ connects L_{C_1} to L_{C_2} and is a geodesic because the metric on $\widetilde{M}$ is locally the product metric. We claim that the length of $\widetilde{\alpha}$, which is the same as the length of α, is the distance between the leaf closures L_{C_1} and L_{C_2}. Suppose that the claim is false; then there exists a smooth curve $\beta(t)=[(x(t),y(t))]_\sim$, $0\le t\le 1$, that is shorter than $\widetilde{\alpha}$ and such that $y(0)\in C_1$ and $y(1)\in C_2$. By choosing representatives in the equivalence class $[(x(t),y(t))]_\sim$ carefully, we may assume that $y(t)$ is a smooth curve in M connecting the leaf closures C_1 and C_2. Since we are using the product metric, the length of $\beta(t)=[(x(t),y(t))]_\sim$ is greater than or equal to the length of $y(t)$, so that the length of $y(t)$ is less than the length of α. This contradicts the fact that α realizes the distance between C_1 and C_2. We have the following result.

THEOREM 2.2. *The function* $\Phi: M\diagup G\to\widetilde{M}\diagup\overline{\mathcal{F}}$ *defined above is an isometry of metric spaces. In addition, for any orbit* C, *the volume of* $\Phi(C)$ *has the same volume as* C.

Proof. We have already shown that the continuous bijection Φ preserves distances and is therefore an isometry. The last statement follows from the fact that the volume of X is one. $\square$

REMARK 2.1. Observe that if G acts by orientation-preserving isometries, then $\left(\widetilde{M}, \mathcal{F}\right)$ is transversally orientable.

REMARK 2.2. It is possible to modify the arguments above to prove that the pseudogroups of local isometries generated by (M, G) and by the singular, orbit-like Riemannian foliation $\left(\widetilde{M}, \overline{\mathcal{F}}\right)$ are isometrically equivalent (see [19] for a discussion of these definitions).

3. Constructing G-manifolds from Riemannian foliations

Let $(\widetilde{M}, \mathcal{F})$ be a transversally oriented n-codimensional foliation. We now construct the basic manifold W, a compact $SO(n)$-manifold associated to the Riemannian foliation $\widetilde{M} \diagup \mathcal{F}$. We will show that with a naturally chosen metric on W , the orbit space $W \diagup SO(n)$ is isometric to the leaf closure space $\widetilde{M} \diagup \overline{\mathcal{F}}$. We remark that if $(\widetilde{M}, \mathcal{F})$ is not transversally orientable, we replace $SO(n)$ with $O(n)$ in the following discussion.

Let $\widehat{M}$ be the oriented transverse orthonormal frame bundle of $(\widetilde{M}, \mathcal{F})$, and let π be the natural projection $\pi : \widehat{M} \to \widetilde{M}$. The manifold $\widehat{M}$ is a principal $SO(n)$-bundle over $\widetilde{M}$. Associated to $\mathcal{F}$ is the lifted foliation $\widehat{\mathcal{F}}$ on $\widehat{M}$. The lifted foliation is transversely parallelizable, and the closures of the leaves are fibers of a fiber bundle $\rho : \widehat{M} \to W$. The manifold W is smooth and is called the *basic manifold* (see [13]). The right action of $SO(q)$ on $\widehat{M}$ naturally descends to an isometric right action of $SO(n)$ on W. We remark that on an open, dense subset of W, the orbits are principal; these orbits correspond to the leaf closures of maximal dimension on $\widetilde{M}$. Let $\overline{\widehat{\mathcal{F}}}$ denote the foliation of $\widehat{M}$ by leaf closures of $\widehat{\mathcal{F}}$.

Endow $\widehat{M}$ with the metric $g^{\widetilde{M}} + g^{SO(n)}$, where $g^{\widetilde{M}}$ is the pullback of the metric on $\widetilde{M}$, and $g^{SO(n)}$ is the standard, normalized, biinvariant metric on the fibers. By this, we mean that we are using the transverse Levi–Civita connection to do the following. We calculate the inner product of two horizontal vectors in $T_{\hat{x}}\widehat{M}$ by using $g^{\widetilde{M}}$, and we calculate the inner product of two vertical vectors using $g^{SO(n)}$. We require that vertical vectors are orthogonal to horizontal vectors. This metric is bundlelike for both $(\widehat{M}, \widehat{\mathcal{F}})$ and $(\widehat{M}, \overline{\widehat{\mathcal{F}}})$. We remark that the tangent bundle to the foliation $\widehat{\mathcal{F}}$ is the intersection of the tangent bundle of $\pi^{-1}\mathcal{F}$ with the horizontal subbundle of $T\widehat{M}$ coming from the transverse Levi-Civita connection. The transverse metric on $(\widehat{M}, \overline{\widehat{\mathcal{F}}})$ induces a well-defined Riemannian metric on W. Note that the leaf closures of $(\widehat{M},\widehat{\mathcal{F}})$ cover the leaf closures of $\widetilde{M}$, so that for every orbit $wSO(n)$ on W, $\rho^{-1}(wSO(n)) = \pi^{-1}L_C$ for some leaf closure L_C of $(\widetilde{M},\mathcal{F})$. By the way the metrics have been defined, the transverse metric to the orbit $wSO(n)$ is the same as the transverse metric to the leaf closure L_C. That is, given two vectors $X, Y \in N_x\mathcal{F}$, the unique horizontal lifts $\widehat{X}$, $\widehat{Y}$ as $SO(n)$–invariant vector fields on $\pi^{-1}(x)$ get mapped by ρ_* in a well-defined way to vector fields X_W, Y_W that are normal to the orbit $\rho\left(\pi^{-1}(x)\right)$. The inner product of X and Y in $N_x\mathcal{F}$ is the same as the inner product of X_W and Y_W at any $w \in \rho\left(\pi^{-1}(x)\right)$. In fact, even more is true.

Theorem 3.1. *The map $\Theta : \widetilde{M}\diagup\overline{\mathcal{F}} \longrightarrow W\diagup SO(n)$ defined by $\Theta(\overline{L}) = \rho\left(\pi^{-1}(\overline{L})\right)$ is a metric space isometry. Moreover, the volume $\mathrm{Vol}\left(\overline{L}\right)$ of the leaf closure $\overline{L}$ is related to the volume $\mathrm{Vol}\left(\Theta\left(\overline{L}\right)\right)$ of the orbit $\Theta\left(\overline{L}\right)$ by the formula $\mathrm{Vol}\left(\Theta\left(\overline{L}\right)\right)\cdot\psi(\overline{L}) = \mathrm{Vol}\left(\overline{L}\right)$. Here, $\psi : \widetilde{M}\diagup\overline{\mathcal{F}} \longrightarrow \mathbb{R}$ is the continuous function defined uniquely by $\psi(\overline{L}) = \mathrm{Vol}\left(\widehat{\overline{L}}\right)$, and $\widehat{\overline{L}}$ is any leaf closure of $(\widehat{M},\widehat{\mathcal{F}})$ that projects to $\overline{L}$.*

Proof. The map Θ is clearly one-to-one and continuous. Suppose that the leaf closures $\overline{L_1}$ and $\overline{L_2}$ are separated by a distance D, so that there exists a minimal geodesic $\alpha : [0,1] \to \widetilde{M}$ of length D such that $\alpha(0) \in \overline{L_1}$ and $\alpha(1) \in \overline{L_2}$. Let the geodesic $\hat{\alpha} : [0,1] \to \widehat{M}$ be any horizontal lift of α. By the choice of the metric on $\widehat{M}$, the length of $\hat{\alpha}$ is D as well. The curve $\rho\circ\widehat{\alpha} : [0,1] \to W$ has the property that $\rho\circ\widehat{\alpha}\,(0) \in \Theta\left(\overline{L_1}\right)$ and $\rho\circ\widehat{\alpha}\,(1) \in \Theta\left(\overline{L_2}\right)$. Therefore, $\mathrm{dist}\left(\Theta\left(\overline{L_1}\right),\Theta\left(\overline{L_2}\right)\right) \leq \mathrm{length}\,(\rho\circ\widehat{\alpha}) \leq D$ since ρ is a Riemannian submersion. Next, let $\beta : [0,1] \to W$ be a geodesic such that $\beta\,(0) \in \Theta\left(\overline{L_1}\right)$, $\beta\,(1) \in \Theta\left(\overline{L_2}\right)$, β' is orthogonal to the orbits, and $\mathrm{length}\,(\beta) = \mathrm{dist}\left(\Theta\left(\overline{L_1}\right),\Theta\left(\overline{L_2}\right)\right)$. Choosing any ρ-horizontal lift $\widehat{\beta}$ of β to $\widehat{M}$, we find that $\widehat{\beta}$ is a geodesic of length $\mathrm{dist}\left(\Theta\left(\overline{L_1}\right),\Theta\left(\overline{L_2}\right)\right)$ connecting a leaf closure $\widehat{\overline{L_1}}$ to a leaf closure $\widehat{\overline{L_2}}$, where $\pi\left(\widehat{\overline{L_i}}\right) = \overline{L_i}$. Then $\pi\circ\widehat{\beta}$ is a curve in $\widetilde{M}$ connecting $\overline{L_1}$ and $\overline{L_2}$ such that $D \leq \mathrm{length}\left(\pi\circ\widehat{\beta}\right) \leq \mathrm{length}\left(\widehat{\beta}\right) = \mathrm{dist}\left(\Theta\left(\overline{L_1}\right),\Theta\left(\overline{L_2}\right)\right)$. We have shown that Θ preserves distance and is therefore an isometry.

Next, observe that for any leaf closure $\overline{L} \subset \widetilde{M}$, the volume of $\pi^{-1}\overline{L} \subset \widehat{M}$ is the same as the volume of $\overline{L}$, by the way the metrics have been defined. The last statement of the theorem follows from the definition of the metric on W and the fact that $SO(n)$ acts by foliation-preserving isometries on $\widehat{M}$. □

We will now show that by choosing a different metric on W, we can show that there exists a Riemannian $SO(n)$-manifold that is isometric to $\widetilde{M}\diagup\overline{\mathcal{F}}$ as before and such that the isometry preserves volumes. We have the following result concerning G -manifolds.

THEOREM 3.2. *Let (W, g) be a compact, connected, m-dimensional, Riemannian manifold endowed with an isometric right action by a compact Lie group G. Let $h : W \to \mathbb{R}$ be a positive, smooth, G-invariant function. Let W_0 be the union of principal orbits in W. Given $w \in W_0$, consider the orbit wG. Let g_w^{T} denote the restriction of g to $T_w(wG)$, and let g_w^{N} denote the restriction of g to the normal space $N_w(wG)$. Define the metric g_w^h on T_wW by $g_w^h = h(w)\, g_w^{\mathrm{T}} + g_w^{\mathrm{N}}$. Then g^h extends to a smooth metric on W, and the extended g^h is a smooth function of the parameter $h \in C^\infty(W)$. The extended metric has the additional property that given any $w \in W \setminus W_0$, the metric on $T_w(wG)$ is $h(w)\, g_w^{\mathrm{T}}$.*

Proof. We prove this result by induction on m. If $\dim W = 1$, then $W = S^1$, and the orbits are either unions of isolated points or the entire manifold; the result follows trivially. We now assume that the result is true if the manifold has dimension less than m. Observe that since W_0 is open and dense, the metric g^h is clearly smooth and depends smoothly on h on any open subset whose closure is contained in W_0. Choose a biinvariant metric on G. Given any $w \in W \setminus W_0$ with isotropy H, let $B_\varepsilon \subset N_w(wG)$ be the ball of radius ε in the normal space, and let D_ε be a metric ball of radius ε centered at 0 in the $N_eH \subset \mathfrak{g}$. Choose coordinates $\eta : D_\varepsilon \times B_\varepsilon \to W$ for a neighborhood of $w \in W$ by $\eta(x, y) = \exp_w^W(y)\, \exp_e^G(x)$. Using geodesic polar coordinates $y = r\sigma$ with $0 \le r < \varepsilon$ and $\sigma \in S^{k-1}$ with $k = \dim N_w(wG)$, the original metric g has the form

$$g\,(x, r, \sigma) = g_1\,(x, r\sigma) + dr^2 + r^2 g_2\,(x, r, \sigma)\,,$$

where $g_1\,(x, r\sigma)$ is the inner product on the normal bundle $N(x, r\sigma)$ of the submanifold $V(x) = \eta\,(x, B_\varepsilon) \subset W$ and $dr^2 + r^2 g_2\,(x, r, \sigma)$ is the metric on the tangent bundle of $V(x)$. Observe that $g_2\,(x, r, \sigma)$ is a family of metrics on S^{k-1} depending smoothly on r and x and converging to a smooth metric on S^{k-1} as $r \to 0$ and $x \to 0$. Because the local foliation $\{V(x) \,|\, -\varepsilon \le x \le \varepsilon\}$ is smooth, its normal and tangent bundles are smooth subbundles. Also, observe that the local $\exp^G(D_\varepsilon)$-orbits $W(r\sigma) = \eta\,(D_\varepsilon, r\sigma)$ form a smooth foliation of the image of η as well. Since $T_wW(0) = N(0,0) = T_w(wG)$, the inner product $g_1\,(x, r\sigma)$ on $N(x, r\sigma)$ and the inner product on $T_{\eta(x,r\sigma)}W(r\sigma)$ both converge smoothly to the inner product on $T_w(wG)$ as $(x, r\sigma) \to (0,0)$. Using the same coordinates, we now write

$$g^h\,(x, r, \sigma) = g_1^h\,(x, r\sigma) + dr^2 + r^2 g_2^h\,(x, r, \sigma)\,,$$

where $g_1^h\,(x, r\sigma)$ is the metric on $N(x, r\sigma)$ and $dr^2 + r^2 g_2^h\,(x, r, \sigma)$ is the metric on the tangent bundle of $V(x)$ in polar coordinates, as before. Since the original inner product on $T_{\eta(x,r\sigma)}W(r\sigma)$ is multiplied by the smooth factor $h(x, r\sigma)$, the comments above imply that $g_1^h\,(x, r\sigma)$ converges smoothly to an inner product $g_1^h\,(0,0) = h(0,0) g_1\,(0,0) = h(w) g_1\,(w)$ on $T_w(wG)$ as $(x, r\sigma) \to (0,0)$. Observe that $g_1\,(0,0) = g_1(w)$ is the metric g_w^{T}. Next, the family of metrics $g_2^h(x,r,\sigma)$ on S^{k-1} is obtained by multiplying the metric $g_2(x,r,\sigma)$ by $h(x,r\sigma)$ along the orbits of the H-action given by $\sigma' = \sigma t$ if and only if $\eta(x,r\sigma') = \eta(x,r\sigma)\cdot\left(\exp^G x\right)^{-1} t \left(\exp^G x\right)$ for any $t\in H$. By the induction hypothesis, the resulting metric $g_2^h(x,r,\sigma)$ is smooth and depends smoothly on r and x and converges smoothly to a smooth metric on S^{k-1} as $(x,r)\to(0,0)$. Therefore, the metric g^h satisfies the conclusion of the theorem. □

Remark 3.1. In the theorem above, the G-action on $\left(W,g^h\right)$ is isometric, because the operation that is applied to the original metric commutes with the induced G-action on $(0,2)$-tensors.

Let $\phi : W \to \mathbb{R}$ be the function defined by $\phi(w) = \mathrm{Vol}\left(\rho^{-1}(w)\right)$; ϕ is positive and smooth because ρ is a smooth fibration.

Theorem 3.3. *There exists a (smooth) metric g' on the basic manifold W such that*

(1) *the $SO(n)$-action on (W,g') is isometric,*
(2) *the map $\Theta : \widetilde{M}\diagup\overline{\mathcal{F}} \longrightarrow (W,g')\diagup SO(n)$ defined by $\Theta(\overline{L}) = \rho\left(\pi^{-1}(\overline{L})\right)$ is a metric space isometry, and*
(3) $\mathrm{Vol}_{g'}\left(\Theta\left(\overline{L}\right)\right) = \mathrm{Vol}\left(\overline{L}\right)$ *for every leaf closure $\overline{L}\in\widetilde{M}\diagup\overline{\mathcal{F}}$ of minimum codimension.*

Proof. Let g be the original metric defined naturally on W. Let $wSO(n)$ denote the orbit of $w\in W$, and let $\overline{L}_w$ denote the corresponding leaf closure $\Theta^{-1}(wSO(n)) = \pi\left(\rho^{-1}(wSO(n))\right)$. By Theorem 3.1, we have the equation

$$\mathrm{Vol}_g(wSO(n))\cdot\phi(w) = \mathrm{Vol}\left(\overline{L}_w\right).$$

Let s be the dimension of the principal orbits. Let $g' = g^h$ as in the last theorem, where $h(w) = (\phi(w))^{-2/s}$. Then the volumes of the s-dimensional orbits in W will be multiplied by a factor of $\frac{1}{\phi(w)}$, so that

$$\mathrm{Vol}_{g'}(wSO(n)) = \mathrm{Vol}\left(\overline{L}_w\right)$$

if the dimension of $wSO(n)$ is maximal, or equivalently if $\overline{L}_w$ has minimum codimension. The transverse metrics corresponding to g and g' are the same, so all of the conditions are satisfied. □

Remark 3.2. In the context of the theorem above, if $\overline{L}_w$ is a leaf closure whose codimension is not minimal, then the dimension a of the orbit $wSO(n)$ will be less than s. Thus

$$\mathrm{Vol}_{g'}(wSO(n))\cdot(\phi(w))^{a/s} = \mathrm{Vol}\left(\overline{L}_w\right),$$

so that the third statement of the theorem does not necessarily hold for more general orbits.

REMARK 3.3. It is possible to show that the pseudogroups of local isometries generated by (W, G, g') and by the singular, orbit-like Riemannian foliation $\left(\widetilde{M}, \overline{\mathcal{F}}\right)$ are isometrically equivalent (again, see [19] for a discussion).

4. Transverse classification of G-manifolds and Riemannian foliations

The following results are immediate consequences of the main theorems.

COROLLARY 4.1. *Let G be any compact Lie group that acts by isometries on a compact, n-dimensional Riemannian manifold M. Then there exists a Riemannian $O(n)$ -manifold W such that $W \diagup O(n)$ is isometric to $M \diagup G$ via an isometry that preserves the volumes of orbits of maximal dimension. If G acts by orientation-preserving isometries, then the group $O(n)$ may be replaced by $SO(n)$ in the statement above.*

Proof. Let $\left(\widetilde{M}, \mathcal{F}\right)$ be a Riemannian foliation with a bundle-like metric constructed as in Section 2. Next, we use Theorem 3.3 to construct a metric on the basic manifold W associated to $\left(\widetilde{M}, \mathcal{F}\right)$. The result follows from Theorem 2.2 and Theorem 3.3. □

REMARK 4.1. Note that orthogonal group actions are very special cases of compact group actions, so that it is surprising that the orbit spaces of arbitrary compact group actions are classified up to isometry by those corresponding to orthogonal group actions.

COROLLARY 4.2. *Let $(M, \mathcal{F})$ be a codimension-q Riemannian foliation on a compact manifold endowed with a bundlelike metric, and let $\overline{q}$ be the minimal codimension of the leaf closures. Let a be the dimension of the principal isotropy groups corresponding to the orthogonal group action on the basic manifold associated to this foliation. Then there exists a Riemannian manifold $\widetilde{M}$ along with a Riemannian foliation $\widetilde{\mathcal{F}}$ on $\widetilde{M}$ of codimension $\overline{q} + \frac{q(q-1)}{2} - a$ that is constructed by suspending an action of a subgroup of $O(q)$, such that the leaf closure spaces $M \diagup \overline{\mathcal{F}}$ and $\widetilde{M} \diagup \overline{\widetilde{\mathcal{F}}}$ are isometric via an isometry that preserves volumes of the leaf closures of maximal dimension. If $(M, \mathcal{F})$ is transversally orientable, the group $O(q)$may be replaced by $SO(q)$ in the statement above.*

Proof. We use Theorem 3.3 to construct a metric on the basic manifold W associated to $(M, \mathcal{F})$. Next, let $\left(\widetilde{M}, \widetilde{\mathcal{F}}\right)$ be a Riemannian foliation with a bundle-like metric constructed as in Section 2. The result follows from Theorem 3.3 and Theorem 2.2. □

REMARK 4.2. Note that these suspension foliations are very special cases of Riemannian foliations; they are totally geodesic and have an involutive normal bundle that is also totally geodesic and Riemannian. It is surprising that the leaf closure spaces of arbitrary Riemannian foliations are classified by those constructed by suspending subgroups of orthogonal groups.

Therefore, problems concerning the geometry and topology of orbit spaces of general G-manifolds can be reduced to problems concerning orbit spaces of $SO(n)$-manifolds and $O(n)$-manifolds. Similarly, problems concerning the transverse geometry and topology of general Riemannian foliations can be reduced to problems concerning Riemannian foliations obtained by suspending orthogonal group actions.

5. Laplacians on Riemannian foliations and G-manifolds

Let $(M,\mathcal{F})$ be a transversally-oriented, codimension-q Riemannian foliation on a compact Riemannian manifold. We now consider a generalization of the Laplacian that reflects the Riemannian foliation structure—the *basic Laplacian*. Let the set of basic forms $\Omega_B(M)$ be the space of smooth forms ω such that $i\,(X)\,\omega = i\,(X)\,d\omega = 0$ for any $X \in T\mathcal{F}$, where $i\,(X)$ denotes interior product with X. For example, the basic functions are the functions that are constant on the leaves of $\mathcal{F}$. The exterior derivative d maps $\Omega_B(M)$ to itself. The basic Laplacian is defined by $\Delta_B = \delta_B d + d\delta_B$,where δ_B is the adjoint of d on $L^2\left(\Omega_B^*\right)$. If $\mathcal{F}$ is the foliation by points of M, the basic Laplacian is the ordinary Laplacian. In the more general case, analysis of the basic Laplacian provides information about the transverse geometry of $(M,\mathcal{F})$. We remark that this operator is the restriction of the ordinary Laplacian only in special cases, but it is always the restriction of an elliptic operator on the space of all forms (see [15]). Since this operator is not a differential operator on the space of all sections of a vector bundle, many of the standard facts about such operators do not easily follow for the basic Laplacian. We refer the reader to [1], [4], [8], [11], [14], [15], [17], [18], [20], and [21] for some results concerning the basic Laplacian.

On the other hand, suppose that G is a compact Lie group that acts on a compact, connected, oriented Riemannian manifold M' by orientation-preserving isometries. Associated to such a G-manifold are various spaces of differential forms. Let $\Omega^G\left(M'\right)$ denote the space of invariant differential forms, and let Δ^G be the *invariant Laplacian*, which is the restriction of the ordinary Laplacian to $\Omega^G\left(M'\right)$. A *G-basic form* associated to the G-action is a form ω on M' that is G-invariant and satisfies $i\,(X)\,\omega = 0$ for every vector field X that is tangent to the orbits. Let $\Omega_B^G\left(M'\right)$ denote the space of smooth G-basic forms on M', and let Δ_B^G denote the Laplacian on G-basic forms (a restriction of the ordinary Laplacian). Another space of forms associated to a group action is the space $\Omega_{\mathfrak{g}}\left(M'\right)$ of *equivariant forms* (see [2], [3, Chapter 16]). There are many relationships between these spaces of forms;

for example, the spaces of 0-forms and their differentials coincide in the three associated differential complexes.

Let the $(M, \mathcal{F})$ be as above, and let W be the basic manifold associated to this Riemannian foliation, with $G = SO(q)$. We endow W with the metric g' used in Theorem 3.3 . Let ω be a G-basic form on W. If $\rho : \widehat{M} \to W$ is the basic fibration, then $\rho^* \omega$ is a form on the orthonormal transverse frame bundle $\widehat{M}$ that is basic for the pullback foliation $\pi^{-1}\mathcal{F}$. Therefore, $\rho^*\omega = \pi^*\beta_\omega$ for some basic form β_ω on M. Since β_ω is uniquely determined by ω, we have a function $S : \Omega_B^G(W) \to \Omega_B(M, \mathcal{F})$ defined by $S(\omega) = \beta_\omega$. Since pullbacks commute with the exterior derivative, we have that $dS = Sd$. Also, observe that S is invertible on $\left(\Omega_B^G\right)^0(W)$, and $dS^{-1} = S^{-1}d$ on $\Omega_B^0(M, \mathcal{F})$. Observe that if $f \in \Omega_B^0(M, \mathcal{F})$ and $\overline{L} \in M \diagup \overline{\mathcal{F}}$, then $S^{-1}f(w) = f(x)$ for any $x \in \overline{L}$ and $w \in \Theta\left(\overline{L}\right)$, using the notation from Theorem 3.3.

We consider the L^2 inner product on basic functions. On an open, dense subset W_0 of W, the orbits are principal so that $W_0 \to W_0 \diagup G$ is a Riemannian fiber bundle. Similarly, on an open, dense subset M_0 of M, the leaf closures have maximal dimension so that $M_0 \to M_0 \diagup \overline{\mathcal{F}}$ is a Riemannian fiber bundle. By Theorem 3.3, $W_0 \diagup G$ is isometric to $M_0 \diagup \overline{\mathcal{F}}$, and the fibers of these fibrations are mapped to ones with equal volumes under the isometry Θ. Therefore, for any basic function $f \in \Omega_B^0(M,\ \mathcal{F})$ (or,equivalently, any basic function $g = S^{-1}f \in \left(\Omega_B^G\right)^0(W)$), $\int_W \left(S^{-1}f\right)(w)\ dV_W = \int_M f(x)\ dV_M$. Moreover, we have that $\langle g_1, g_2\rangle_W = \langle Sg_1, Sg_2\rangle_M$ for any $g_1, g_2 \in \left(\Omega_B^G\right)^0(W)$.

We now examine the properties of L^2 inner products on basic one-forms. Let f and g be G-basic functions on W. The inner product on one-forms satisfies $\langle df, dg\rangle_W = \langle V_f, V_g\rangle_W$, where V_f and V_g are the gradient vector fields, the vector fields dual to the differentials of the functions. Observe that $V_f(w)$ and $V_g(w)$ are normal to the orbit through any $w \in W$. Let the vector fields $\widehat{V_f}$, $\widehat{V_g}$ on $\rho^{-1}(w)$ be their ρ-horizontal lifts; these vectors are normal to the pullback foliation $\pi^{-1}\mathcal{F}$ on $\widehat{M}$. Then $\pi_*\widehat{V_f}$, $\pi_*\widehat{V_g}$ are well-defined basic vector fields normal to the leaf closure $\pi\left(\rho^{-1}(w)\right)$. By construction and choice of metrics on $\widehat{M}$ and W, we have

$$\langle V_f(w), V_g(w)\rangle_W = \left\langle \pi_*\widehat{V_f}(x), \pi_*\widehat{V_g}(x)\right\rangle_M,$$

where $\hat{x} \in \rho^{-1}(w)$, $\pi(\hat{x}) = x$. Next, observe that $\widehat{V_f}$ is a restriction of the gradient of $\rho^* f$, and using the same reasoning, we have that $\pi_*\widehat{V_f}$ is a restriction of the gradient of Sf. Since similar statements are true for g, we have that for any G-basic functions f and g,

$$\langle df, dg\rangle_W = \langle dSf, dSg\rangle_M = \langle Sdf, Sdg\rangle_M .$$

From the above we have that $\langle df, dg\rangle_W = \langle dSf, dSg\rangle_M = \left\langle f, S^{-1}\delta_B S dg\right\rangle_W$, which implies that $\delta_W = S^{-1}\delta_B S$ on differentials of G-basic functions, where

δ_W denotes the adjoint of d on G-basic differential forms on W. Thus, for G -basic functions f,

$$S\Delta_B^G f = S\delta_W df = \delta_B S df = \delta_B d S f = \Delta_B S f.$$

Similarly, $S^{-1}\Delta_B h = \Delta_B^G S^{-1} h$ for $h \in \Omega_B^0 (M, \mathcal{F})$. We have shown the following:

THEOREM 5.1. *Let $\mathcal{F}$ be a transversally-oriented Riemannian foliation on a compact, connected Riemannian manifold M. Let W be the basic manifold associated to this foliation, endowed with the metric g' used in Theorem 3.3. Then the spectrum of the G -basic (or G-invariant, or G-equivariant) Laplacian Δ_B^G on functions on W is the same as the spectrum of the basic Laplacian Δ_B on $\Omega_B^0 (M, \mathcal{F})$. Moreover, $f \in \left(\Omega_B^G\right)^0 (W)$ is a G-basic eigenfunction with eigenvalue λ if and only if $Sf \in \Omega_B^0 (M, \mathcal{F})$ is a basic eigenfunction with eigenvalue λ.*

COROLLARY 5.2. *With $(M, \mathcal{F})$ and (W, g') as above, the G-invariant (or G-basic, or G -equivariant) heat kernel K^G on $\left(\Omega^G\right)^0 (W)$ is related to the basic heat kernel K_B on $\Omega_B^0 (M, \mathcal{F})$ by the formula*

$$K_B (t, x_1, x_2) = K^G (t, w_1, w_2)$$

if $x_i \in \pi\left(\rho^{-1}(w_i)\right)$ for $i = 1, 2$.

REMARK 5.1. The above results are false for higher degree forms; the problem is that not every basic form $\alpha \in \Omega_B^* (M, \mathcal{F})$ can be written as $\alpha = S\beta$ for some $\beta \in \left(\Omega_B^G\right)^* (W)$.

REMARK 5.2. The corollary above could be taken as the definition of K_B. The G -invariant heat kernel is easily calculated from the ordinary heat kernel K on W:

$$K_G (t, w_1, w_2) = \int_G K (t, w_1, w_2 g)\, dV (g),$$

where $dV(g)$ is the volume form corresponding to the normalized biinvariant metric. The properties of the basic heat kernel on functions follow from the equation in Corollary 5.2, so that we could use the ideas of this section to give a new proof of the existence and asymptotics of the basic heat kernel $K_B (t, x_1, x_2)$. See [8], [14], [15], [17], [18] for previous work on this problem.

Next, let G' be any compact Lie Group, and let M' be any compact, Riemannian manifold on which G' acts isometrically. As in Section 2, we construct a Riemannian foliation $\left(\widetilde{M}, \widetilde{\mathcal{F}}\right)$ and an isometry $\Phi : M' \diagup G' \longrightarrow \widetilde{M} \diagup \widetilde{\mathcal{F}}$ that preserves the volumes of the orbits. Let Γ' be the discrete, dense subgroup used in the construction of $\left(\widetilde{M}, \widetilde{\mathcal{F}}\right)$. The basic forms of the suspension foliation correspond exactly to the Γ' -invariant forms of the suspended manifold

M', which therefore correspond exactly to the G'-invariant forms of M', by the smoothness of the map $G' \to \mathrm{Isom}\,(M')$. Thus, we have an isomorphism $\overline{S} : \Omega^j_B\left(\widetilde{M}, \widetilde{\mathcal{F}}\right) \to \left(\Omega^{G'}\right)^j(M')$, defined in the obvious way by restricting to the suspension fibers. More specifically, given $\omega \in \Omega_B\left(\widetilde{M}, \widetilde{\mathcal{F}}\right)$, we extend it to be an element of $\Omega^\Gamma_B\left(\widetilde{X} \times M'\right)$, the Γ-invariant basic forms on the product foliation $\widetilde{X} \times M'$, where $\gamma \in \Gamma'$ acts on $\widetilde{X} \times M'$ by $(x, y)\,\gamma = \left([\gamma^{-1}]x, y\,\mu\left([\gamma]\right)\right)$. Because the basic forms are simply forms on M', we may define $\overline{S}\omega$ to be the restriction of ω to the M' factor.

We now consider the L^2 inner product on basic forms on M' and on $\widetilde{M}$. By the way the metrics are defined, it is clear that the pointwise inner product of any two basic forms $\alpha, \beta \in \Omega_B\left(\widetilde{M}, \widetilde{\mathcal{F}}\right)$ satisfies $(\alpha, \beta)_{\widetilde{M}} = \left(\overline{S}\alpha, \overline{S}\beta\right)_{M'}$. Because the isometry $\Phi : M' \diagup G'. \to \widetilde{M} \diagup \overline{\widetilde{\mathcal{F}}}$ maps orbits to leaf closures with the same volume, it follows that

$$\langle \alpha, \beta \rangle_{\widetilde{M}} = \int_{\widetilde{M}} (\alpha, \beta)_{\widetilde{M}}\,(x)\; dV_{\widetilde{M}} = \int_{M'} \left(\overline{S}\alpha, \overline{S}\beta\right)_{M'}(y)\; dV_{M'} = \left\langle \overline{S}\alpha, \overline{S}\beta \right\rangle_{M'}.$$

Using the product coordinate description of the map $\overline{S}$, it is clear that $d\overline{S}\alpha = \overline{S}d\alpha$ for all $\alpha \in \Omega_B\left(\widetilde{M}, \widetilde{\mathcal{F}}\right)$. The above implies that $\delta_{M'}\overline{S}\alpha = \overline{S}\delta_B\alpha$, where $\delta_{M'}$ is the adjoint of d on M' and δ_B is the adjoint of d restricted to basic forms on $\left(\widetilde{M}, \widetilde{\mathcal{F}}\right)$. Therefore, $\Delta^G\overline{S} = \overline{S}\Delta_B$.

We now have the following:

THEOREM 5.3. *Let G' be any compact Lie Group, and let M' be any compact, Riemannian manifold on which G' acts isometrically. Let $\left(\widetilde{M}, \widetilde{\mathcal{F}}\right)$ be a Riemannian foliation constructed from this group action as in Section 2. Then the spectrum of the G'-invariant Laplacian $\Delta^{G'}$ on j-forms on M' is the same as the spectrum of the basic Laplacian Δ_B on $\Omega^j_B\left(\widetilde{M}, \widetilde{\mathcal{F}}\right)$. Moreover, $\alpha \in \Omega^j_B\left(\widetilde{M}, \widetilde{\mathcal{F}}\right)$ is a basic eigenform with eigenvalue λ if and only if $\overline{S}\alpha \in \left(\Omega^{G'}\right)^j(M')$ is a G'-invariant eigenform with eigenvalue λ.*

COROLLARY 5.4. *With $\left(\widetilde{M}, \widetilde{\mathcal{F}}\right)$ and (M', G') as above, the G'-invariant heat kernel $K^{G'}$ on $\left(\Omega^G\right)^j(W)$ is related to the basic heat kernel K_B on $\Omega^j_B\left(\widetilde{M}, \widetilde{\mathcal{F}}\right)$ by the formula*

$$\left(K^G\right)^j(t, w_1, w_2) = K^j_B\,(t, x_1, x_2)$$

if $x_i \in \Phi(\mathcal{O}_{w_i})$ for $i = 1, 2$. Here, $\mathcal{O}_w$ denotes the orbit of $w \in M'$, and Φ is the isometry $\Phi : M' \diagup G' \to \widetilde{M} \diagup \widetilde{\mathcal{F}}$.

By constructing in sequence $(M, \mathcal{F}) \dashrightarrow (W, O(q) \text{ or } SO(q)) \dashrightarrow \left(\widetilde{M}, \widetilde{\mathcal{F}}\right)$ and $(M, G) \dashrightarrow \left(\widetilde{M}, \mathcal{F}\right) \dashrightarrow (W, O(q) \text{ or } SO(q))$, we obtain the following corollaries:

Corollary 5.5. *Given any Riemannian foliation $\mathcal{F}$ of codimension q on a compact manifold M, the spectrum of its basic Laplacian on functions is identical to the spectrum of the basic Laplacian on functions of a Riemannian foliation $\left(\widetilde{M}, \widetilde{\mathcal{F}}\right)$ that is constructed by suspending a Γ-action, where Γ is a dense subgroup of $SO(q)$ (for the transversally oriented case) or $O(q)$ (for foliations $(M, \mathcal{F})$ that are not transversally orientable).*

Corollary 5.6. *Given any compact Lie Group G that acts by orientation-preserving isometries on a compact, Riemannian manifold M, the spectrum of its G -invariant Laplacian on functions is identical to the spectrum of the $SO(q)$-invariant Laplacian on a compact, Riemannian manifold W , which is constructed from the original group action. The analogous statement is true for actions that do not necessarily preserve orientation with $SO(q)$ replaced by $O(q)$.*

6. Differential Operators on Sections of Vector bundles

We begin by describing the holonomy groupoid of a Riemannian foliation. Let $\mathcal{F}$ be a p-dimensional Riemannian foliation on a compact, n-dimensional Riemannian manifold N, and let $G_{\mathcal{F}}$ denote the holonomy groupoid of $(N, \mathcal{F})$ (see [22]). An element of $G_{\mathcal{F}}$ is an ordered triple $(x_1, x_2, [\gamma])$, where x_1 and x_2 are points of a leaf L and $[\gamma]$ is an equivalence class of piecewise smooth paths in L starting at x_1 and ending at x_2; two such paths α and β are equivalent if and only if $\beta^{-1}\alpha$ has trivial holonomy. Multiplication is defined by $(x_1, x_2, [\alpha])(x_2, x_3, [\omega]) = (x_1, x_3, [\alpha\omega])$, where $\alpha\omega$ is the obvious concatenation. Because $(N, \mathcal{F})$ is Riemannian, $G_{\mathcal{F}}$ is endowed with the structure of a smooth $n + p$-dimensional manifold (see [22]).

Given a G-manifold M and G-equivariant vector bundle $E \longrightarrow M$, we construct a foliation $\left(\widetilde{M}, \quad \mathcal{F}\right)$ by suspending a dense subgroup Γ as in Section 2. We have that $\widetilde{M} = \widetilde{X} \times M \diagup \pi_1(X)$, where $\pi_1(X)$ acts by $(x, y)[\gamma] = \left(\left[\gamma^{-1}\right] x, y\mu([\gamma])\right)$. We define the space $\widetilde{E} = \widetilde{X} \times E \diagup \pi_1(X)$, where $\pi_1(X)$ acts by $(x, v_p)[\gamma] = \left(\left[\gamma^{-1}\right] x, R_{\mu([\gamma])} v_p\right)$. It is easy to see that $\widetilde{E}$ is a vector bundle over $\widetilde{M}$, with the equivalence class of $(x, v_p) \in \widetilde{X} \times E \diagup \pi_1(X)$ mapping to the equivalence class of $(x, p) \in \widetilde{X} \times M \diagup \pi_1(X)$. Also, rank $\left(\widetilde{E}\right) =$

rank (E). Next, let $a_1 = [(x_1, p_1)]$ and $a_2 = [(x_2, p_2)]$ be points of a leaf of $\widetilde{X} \times M / \pi_1(X)$, and let $g = (a_1, a_2, [\gamma]) \in G_{\mathcal{F}}$, the holonomy groupoid of $(M, \mathcal{F})$. Since a_1 and a_2 are points of the same leaf, we may choose the element (x_2, p_2) of the equivalence class a_2 such that $p_2 = p_1$. We define $\widetilde{S}_g : \widetilde{E}_{a_1} \longrightarrow \widetilde{E}_{a_2}$ by $\widetilde{S}_g [(x_1, v_{p_1})] = [(x_2, v_{p_1})]$; simple calculations show that this map is well-defined and that $\widetilde{S}_g \widetilde{S}_h = \widetilde{S}_{hg}$. Therefore, the action of G on the vector bundle $E \longrightarrow M$ induces an action of $G_{\mathcal{F}}$ on the vector bundle $\widetilde{E} \longrightarrow \widetilde{M}$.

Suppose $s : M \longrightarrow E$ is any G-invariant section. Then, we define $Ks : \widetilde{M} \longrightarrow \widetilde{E}$ by $Ks[(x, p)] = [(x, s(p))]$. Note that

$$\left[\left(\left[\gamma^{-1}\right] x, s\left(y\mu\left([\gamma]\right)\right)\right)\right] = \left[\left(\left[\gamma^{-1}\right] x, R_{\mu([\gamma])} s\left(y\right)\right)\right],$$

so the map is well-defined. For any $g = \left([(x_1, p)], [(x_2, p)], [\gamma]\right) \in G_{\mathcal{F}}$,

$$\widetilde{S}_g Ks[(x_1, p)] = \widetilde{S}_g [(x_1, s(p))] = [(x_2, s(p))] = Ks[(x_2, p)].$$

Thus, $K : \Gamma^G(M, E) \longrightarrow \Gamma_B\left(\widetilde{M}, \widetilde{E}\right)$. Similarly, let $\widetilde{s} : \widetilde{M} \longrightarrow \widetilde{E}$ be a basic section. Fixing $x \in \widetilde{X}$ and $p \in M$, the equation $\widetilde{s}[(x, p)] = [(x, v_p)]$ uniquely defines the vector $v_p \in E_p$. Because $\widetilde{s}$ is basic, the vector v_p is independent of the fixed x chosen; it defines a section v of $E \longrightarrow M$. Since $\widetilde{s}$ is well-defined on equivalence classes, the section v is Γ-invariant and thus G-invariant. It is clear that $Kv = \widetilde{s}$, so the map $K : \Gamma^G(M, E) \longrightarrow \Gamma_B\left(\widetilde{M}, \widetilde{E}\right)$ is a vector space isomorphism with $K^{-1}\widetilde{s} = v$, as defined above.

Now, suppose that $D : \Gamma(M, E) \longrightarrow \Gamma(M, E)$ is a G-equivariant differential operator, so that it maps $\Gamma^G(M, E)$ to itself. Then, we observe that $KDK^{-1} : \Gamma_B\left(\widetilde{M}, \widetilde{E}\right) \longrightarrow \Gamma_B\left(\widetilde{M}, \widetilde{E}\right)$ is also the restriction of a differential operator by examining the map in a local trivialization, and it is $G_{\mathcal{F}}$ -equivariant. We may choose a G-invariant metric on E, and the resulting L^2 metric will allow us to complete $\Gamma(M, E)$ to a Hilbert space. This metric on E induces a metric on $\widetilde{E}$ by the equation $\langle Ks, Kt\rangle_{\widetilde{E}} := \langle s, t\rangle_E$ that is $G_{\mathcal{F}}$-invariant; the analogous equation of L^2-metrics holds. By defining the metric in this way, we force K to be a unitary equivalence between $L^2\left(\Gamma^G(M, E)\right)$ and $L^2\left(\Gamma_B\left(\widetilde{M}, \widetilde{E}\right)\right)$.

Using the observations above and the results of the previous sections, we have the following:

Theorem 6.1. *Let $E \longrightarrow M$ be a G-equivariant vector bundle, endowed with a metric that is invariant under the G-action. Suppose that $D : \Gamma(M, E) \longrightarrow \Gamma(M, E)$ is a G-equivariant differential operator. Then the bundle $\widetilde{E}$ over the suspension foliation $\left(\widetilde{M}, \mathcal{F}\right)$, which is constructed above, is $G_{\mathcal{F}}$-equivariant and inherits a natural $G_{\mathcal{F}}$-invariant metric from the metric on E. The operator $KDK^{-1} : \Gamma_B\left(\widetilde{M}, \widetilde{E}\right) \longrightarrow \Gamma_B\left(\widetilde{M}, \widetilde{E}\right)$ is the restriction of* *a differential operator and is $G_{\mathcal{F}}$ -equivariant. The operators $D|_{\Gamma^G(M,E)}$ and $KDK^{-1}|_{\Gamma_B\left(\widetilde{M},\widetilde{E}\right)}$ have the same spectrum and are unitarily equivalent.*

Remark 6.1. Observe that if $D:\Gamma(M,E)\longrightarrow\Gamma(M,E)$ is an elliptic operator, then KDK^{-1} is the restriction of a transversally elliptic operator on $\Gamma\left(\widetilde{M},\widetilde{E}\right)$.

Next, let $(Y,\mathcal{F})$ be any transversally oriented Riemannian foliation of codimension q, and let $E\longrightarrow Y$ be a $G_{\mathcal{F}}$-equivariant vector bundle of rank k. Assume that we have chosen a $G_{\mathcal{F}}$-invariant metric for E. Let $\pi:\widehat{Y}\to Y$ be the orthonormal transverse frame bundle of $(Y,\mathcal{F})$. Associated to $\mathcal{F}$ is the canonical lifted foliation $\widehat{\mathcal{F}}$ on $\widehat{Y}$. Let $\rho:\widehat{Y}\to W$ be the projection onto the basic manifold $W=\widehat{Y}\diagup\overline{\widehat{\mathcal{F}}}$, the leaf closure space of $\left(\widehat{Y},\widehat{\mathcal{F}}\right)$. An element of $G_{\widehat{\mathcal{F}}}$ is a triple of the form $(\widehat{x},\widehat{y},[\cdot])$, where $[\cdot]$ is the set of all piecewise smooth curves starting at $\widehat{x}$ and ending at $\widehat{y}$, since the holonomy is trivial on $\widehat{Y}$. The $G_{\mathcal{F}}$-action on E induces a $G_{\widehat{\mathcal{F}}}$-action on π^*E, defined as follows.Given a vector $(\widehat{x},v)\in(\pi^*E)_{\widehat{x}}$ so that $\widehat{x}\in\widehat{Y}$, $v\in E_{\pi(\widehat{x})}$, we define the action of $\widehat{g}=(\widehat{x},\widehat{y},[\cdot])$ by $S_{\widehat{g}}(\widehat{x},v)=(\widehat{y},S_g v)$, where $g=(\pi(\widehat{x}),\pi(\widehat{y}),[\gamma])\in G_{\mathcal{F}}$ and $[\gamma]$ is the unique equivalence class of piecewise smooth curves from $\pi(\widehat{x})$ to $\pi(\widehat{y})$ in the leaf containing $\pi(\widehat{x})$ that lift to leafwise curves in $\widehat{Y}$ from $\widehat{x}$ to $\widehat{y}$. It is easy to check that this action makes π^*E into a $G_{\widehat{\mathcal{F}}}$ -equivariant vector bundle.The pullback π^* maps basic sections of E to basic sections of π^*E. Also, the $SO(q)$-action on $\left(\widehat{Y},\widehat{\mathcal{F}}\right)$ induces an action of $SO(q)$ on π^*E that is trivial on the fibers and preserves the basic sections.

In the case of a foliation that is not transversally orientable, replace $SO(q)$ with $O(q)$ in the discussion above.

Observe that if $s\in\Gamma_B(Y,E)$, then π^*s is a basic section of π^*E that is G-invariant. Conversely, if $\widehat{s}$ is G-invariant, then $\widehat{s}=\pi^*s$ for some $s\in\Gamma(Y,E)$. If $\widehat{s}$ is also basic, then for any $g=(\pi(\widehat{x}),\pi(\widehat{y}),[\gamma])\in G_{\mathcal{F}}$ as above, $S_g(s(\pi(\widehat{x})))=S_{(\widehat{x},\widehat{y},[\cdot])}(\widehat{s}(\widehat{x}))=\widehat{s}(\widehat{y})=s(\pi(\widehat{y}))$. Thus, $\pi^*:\Gamma_B(Y,E)\to\left(\Gamma_B\left(\widehat{Y},\pi^*E\right)\right)^G$ is an isomorphism.

We now construct a vector bundle $\widetilde{E}\longrightarrow W$. Given $w\in W$, $\rho^{-1}(w)$ is a leaf closure in $\widehat{Y}$. Consider a section $s\in\Gamma_B(\pi^*E)$ restricted to $\rho^{-1}(w)$. Given any $\widehat{x}\in\rho^{-1}(w)$, the vector $s(\widehat{x})$ determines s on this leaf closure, by the action of $G_{\widehat{\mathcal{F}}}$ on π^*E. Thus, the space $\widetilde{E}_w:=\Gamma_B\left(\widehat{Y},\pi^*E\right)\diagup\tilde{}_w$ is a finite-dimensional vector space, where two basic sections $s,s':\widehat{Y}\longrightarrow\pi^*E$ are equivalent $(s\tilde{}_w s')$ if $s(\widehat{x})=s'(\widehat{x})$ for all $\widehat{x}\in\rho^{-1}(w)$. We let $[s]_w$ denote the equivalence class of $s\in\Gamma_B\left(\widehat{Y},\pi^*E\right)$. The dimension of this vector space is less than or equal to the rank of π^*E ($=$ rank of E), and the union of the vector spaces $\left\{\widetilde{E}_w \mid w \in W\right\}$ forms a smooth vector bundle, as shown in [5, Proposition 2.7.2].

We let $\Phi : \Gamma\left(W, \widetilde{E}\right) \to \Gamma_B\left(\widehat{Y}, \pi^* E\right)$be the almost tautological map defined as follows. Given a section $\widetilde{s}$ of $\widetilde{E}$, its value at each $w \in W$ is an equivalence class $[\widehat{s}]_w$ of basic sections. We define for any $\widehat{x} \in \rho^{-1}(w)$, $\Phi\left(\widetilde{s}\right)\left(\widehat{x}\right) = \widehat{s}\left(\widehat{x}\right)$. It is easily seen that this map is well-defined and maps sections of $\widetilde{E}$ to basic sections of $\pi^* E$. Moreover, Φ is invertible with $\Phi^{-1}\left(\widehat{s}\right)(w) = [\widehat{s}]_w$ for every $\widehat{s} \in \Gamma_B\left(\widehat{Y}, \pi^* E\right)$. Note that the $SO\left(q\right)$-actions on $\left(\widehat{Y}, \widehat{\mathcal{F}}\right)$ and on $\pi^* E$ induce well-defined actions of $SO(q)$ on W and $\widetilde{E}$. We choose the metric on W as in Theorem 3.3. Suppose that we have chosen a $G_{\mathcal{F}}$-invariant metric on E; then this metric may be lifted to a $G_{\widehat{\mathcal{F}}}$-invariant, $SO(q)$-invariant metric on $\pi^* E$. Thus, the metric descends to a natural $SO(q)$-invariant metric on $\widetilde{E}$. Similar to arguments in the proof of Theorem 5.1, the resulting L^2 inner products of sections are preserved under the map Φ. Note that the restricted map $\Phi : \Gamma^G\left(W, \widetilde{E}\right) \to \left(\Gamma_B\left(\widehat{Y}, \pi^* E\right)\right)^{SO(q)}$ also extends to an L^2 isometry. Note that since the fibers of the submersion $\widehat{Y} \xrightarrow{\pi} Y$ have volume one, $\pi^* : \Gamma_B(Y, E) \longrightarrow \left(\Gamma_B\left(\widehat{Y}, \pi^* E\right)\right)^{SO(q)}$ extends to an L^2 isometry as well. We now have the following:

THEOREM 6.2. *Let $(Y, \mathcal{F})$ be any transversally oriented Riemannian foliation, and let $E \longrightarrow Y$ be a $G_{\mathcal{F}}$-equivariant vector bundle, endowed with a metric that is $G_{\mathcal{F}}$-invariant. Suppose that $D : \Gamma_B(Y, E) \longrightarrow \Gamma_B(Y, E)$ is a $G_{\mathcal{F}}$-equivariant differential operator. Then the bundle $\widetilde{E}$ over the basic manifold W, which is constructed above, is $SO(q)$-equivariant, and it inherits a natural $SO(q)$-invariant metric from the metric on E. The operator $\Phi^{-1}\pi^* D\left(\pi^*\right)^{-1}\Phi : \Gamma^{SO(q)}\left(W, \widetilde{E}\right) \longrightarrow \Gamma^{SO(q)}\left(W, \widetilde{E}\right)$ is the restriction of a differential operator and is $SO(q)$-equivariant. The operators $D|_{\Gamma_B(Y,E)}$ and $\Phi^{-1}\pi^* D\left(\pi^*\right)^{-1}\Phi\Big|_{\Gamma^{SO(q)}\left(W, \widetilde{E}\right)}$ have the same spectrum and are unitarily equivalent.*

Proof. All that remains to be shown is that $\Phi^{-1}\pi^* D\left(\pi^*\right)^{-1}\Phi\Big|_{\Gamma^{SO(q)}\left(W, \widetilde{E}\right)}$ is the restriction of a differential operator. Since $D : \Gamma_B(Y, E) \longrightarrow \Gamma_B(Y, E)$ is $G_{\mathcal{F}}$-equivariant, it has a local expression over a neighborhood U of Y in terms of matrices of basic differential operators. Given a small neighborhood V in $\pi^{-1}\left(U\right) \subset \widehat{Y}$, the differential operator $\pi^* D\left(\pi^*\right)^{-1}$ acting on $\left(\Gamma_B\left(\widehat{Y}, \pi^* E\right)\right)^{SO(q)}$ has a local expression in terms of matrices of differential operators that are basic with respect to the pullback foliation $\pi^{-1}\mathcal{F}$. Choosing coordinates adapted to this foliation, we see that this differential operator on V descends to a differential operator on the quotient $\rho(V)$ since the operator is certainly basic with respect to the fibers of ρ. The new operator on $\Gamma\left(\rho(V), \widetilde{E}\Big|_{\rho(V)}\right)$ gives a local expression for $\Phi^{-1}\pi^* D(\pi^*)^{-1}\Phi\Big|_{\Gamma^{SO(q)}\left(W,\widetilde{E}\right)}$. □

REMARK 6.2. Observe that if $D : \Gamma_B(Y,E) \longrightarrow \Gamma_B(Y,E)$ is the restriction of a transversally elliptic operator $\bar{D} : \Gamma(Y,E) \longrightarrow \Gamma(Y,E)$, then $\Phi^{-1}\pi^* D(\pi^*)^{-1}\Phi\Big|_{\Gamma^{SO(q)}\left(W,\widetilde{E}\right)}$ is the restriction of a transversally elliptic operator on $\Gamma\left(W,\widetilde{E}\right)$.

COROLLARY 6.3. *Let G be a compact Lie group that acts on a Riemannian n-manifold M be orientation-preserving isometries. Let $E \longrightarrow M$ be a G-equivariant vector bundle, endowed with a metric that is invariant under the G-action. Suppose that $D : \Gamma(M,E) \longrightarrow \Gamma(M,E)$ is a G-equivariant, elliptic differential operator. Then there exists*

1. *a vector bundle $\bar{E}$ over an $SO(n)$-manifold W that is $SO(n)$-equivariant and inherits a natural $SO(n)$-invariant metric from the metric on E, and*
2. *a transversally elliptic, $SO(n)$-equivariant differential operator $\bar{D} : \Gamma(W,\bar{E}) \longrightarrow \Gamma(W,\bar{E})$,*

such that the operators $D|_{\Gamma^G(M,E)}$ and $\bar{D}\big|_{\Gamma^G(W,\bar{E})}$ are unitarily equivalent with respect to the L^2 inner products.

Proof. Combine Theorem 6.1 and Theorem 6.2. □

As an example of an application of the corollary above, observe that the index of any elliptic operator restricted to sections of an equivariant vector bundle that are invariant under a compact Lie group action is the same as the index of a transversally elliptic operator acting on $O(n)$-invariant sections over an equivariant vector bundle over another manifold (the group $O(n)$ is replaced by $SO(n)$ if the group action preserves orientation). Of course, this corollary contains geometric information as well.

REFERENCES

[1] J. A. Alvarez López, *The basic component of the mean curvature of Riemannian foliations*, Ann. Global Anal. Geom. **10** (1992), 179–194.

[2] M. F. Atiyah and R. Bott, *The moment map and equivariant cohomology*, Topology **23** (1984), 1–28.

[3] J. J. Duistermaat, *The heat kernel Lefschetz fixed point formula for the spin-c Dirac operator*, Progress in Nonlinear Differential Equations and Their Applications, vol. 18, Birkhäuser, Boston, MA, 1996.

[4] A. El Kacimi-Alaoui, *Opérateurs transversalement elliptiques sur un feuilletage riemannien et applications*, Compositio Math. **73** (1990), 57–106.

[5] A. El Kacimi-Alaoui and G. Hector, *Décomposition de Hodge basique pour un feuilletage riemannien*, Ann. Inst. Fourier (Grenoble) **36** (1986), 207–227.

[6] S. Illman, *Every proper smooth action of a Lie group is equivalent to a real analytic action: a contribution to Hilbert's fifth problem*, Prospects in topology (Princeton, NJ, 1994), Annals of Math. Studies, vol. 138, Princeton Univ. Press, Princeton, NJ, 1995, pp. 189–220.

[7] F. E. A. Johnson, *On the triangulation of stratified sets and singular varieties*, Trans. Amer. Math. Soc. **275** (1983), 333–343.

[8] F. W. Kamber and P. Tondeur, *De Rham-Hodge theory for Riemannian foliations*, Math. Ann. **277** (1987), 415–431.

[9] M. Kankaanrinta, *Proper real analytic actions of Lie groups on manifolds*, Ann. Acad. Sci. Fenn. Ser. A I Math. Dissertationes, no. 83 (1991), 41 pp.

[10] S. Matumoto, *Transformation Groups (Posnan, 1985)*, Lecture Notes in Math., vol. 1217, Springer-Verlag, Berlin, 1986.

[11] M. Min-Oo, E. A. Ruh, and P. Tondeur, *Vanishing theorems for the basic cohomology of Riemannian foliations*, J. Reine Angew. Math. **415** (1991), 167–174.

[12] P. Molino, *Riemannian foliations*, Progress in Math., vol. 73, Birkhäuser Boston, MA, 1988.

[13] G. D. Mostow, *Equivariant embeddings in Euclidean space*, Ann. of Math. (2) **65** (1957), 432–446.

[14] S. Nishikawa, M. Ramachandran, and P. Tondeur, *The heat equation for Riemannian foliations*, Trans. Amer. Math. Soc. **319** (1990), 619–630.

[15] R. S. Palais, *Imbedding of compact, differentiable transformation groups in orthogonal representations*, J. Math. Mech. **6** (1957), 673–678.

[16] E. Park and K. Richardson, *The basic Laplacian of a Riemannian foliation*, Amer. J. Math. **118** (1996), 1249–1275.

[17] K. Richardson, *The asymptotics of heat kernels on Riemannian foliations*, Geom. Funct. Anal. **8** (1998), 356–401.

[18] ———, *Traces of heat operators on Riemannian foliations*, preprint.

[19] E. Salem, *Riemannian foliations and pseudogroups of isometries*, in: P. Molino, *Riemannian foliations*, Birkhäuser, Boston, MA, 1988, pp. 265–296.

[20] P. Tondeur, *Foliations on Riemannian manifolds*, Springer-Verlag, New York, 1988.

[21] ———, *Geometry of foliations*, Monographs in Math., vol. 90, Birkhäuser, Basel, 1997.

[22] H. E. Winkelnkemper, *The graph of a foliation*, Ann. Global Anal. Geom. **1** (1983), 51–75.

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