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UNIQUENESS THEOREMS FOR p-ADIC HOLOMORPHIC CURVES

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1. Introduction

It is well known that two non-constant polynomials f and g over an algebraically closed field of characteristic zero are identical if there exist two distinct values a and b such that $f(x) = a \Leftrightarrow g(x) = a$ and $f(x) = b \Leftrightarrow g(x) = b$. In 1926, R. Nevanlinna [Ne] extended this result to meromorphic functions by showing that two non-constant meromorphic functions of a complex variable which attain five distinct values at the same points must be identical.

It has been observed that p-adic entire functions behave in many ways more like polynomials than like entire functions of a complex variable. Confirming this observation, W.W. Adams and E.G. Straus [AS] proved the following result.

THEOREM A. *Let f and g be two non-constant p-adic entire functions so that for two distinct (finite) values a and b we have $f(x) = a \Leftrightarrow g(x) = a$ and $f(x) = b \Leftrightarrow g(x) = b$. Then $f \equiv g$.*

For p-adic meromorphic functions, Adams and Straus obtained the following result, which is an analog of Nevanlinna's result.

THEOREM B. *Let f and g be two non-constant p-adic meromorphic functions so that there exist four distinct values a_1, a_2, a_3, and a_4, such that $f(x) = a_i \Leftrightarrow g(x) = a_i$ for $i = 1, 2, 3, 4$. Then $f \equiv g$.*

The aim of this paper is to extend Theorem B to p-adic holomorphic curves in projective spaces.

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2. Uniqueness problems without counting multiplicity

Before we state our theorems, we recall some definitions and known results. Let p be a prime number, and let $|\ |_p$ be the standard p-adic valuation on $\mathbb{Q}$ normalized so that $|p|_p = p^{-1}$. Let $\mathbb{Q}_p$ be the completion of $\mathbb{Q}$ with respect to this valuation, and let $\mathbb{C}_p$ be the completion of the algebraic closure of $\mathbb{Q}_p$. As is well known, $\mathbb{C}_p$ is algebraically closed. For simplicity, we denote the p-adic norm $|\ |_p$ on $\mathbb{C}_p$ by $|\ |$. We note that the results of this paper also hold for a general complete, algebraically closed non-Archimedean field of characteristic zero.

It is known that an infinite sum converges in a non-Archimedean norm if and only if its general term approaches zero. Thus a function of the form

$$h(z) = \sum_{n=0}^{\infty} a_n z^n,\ a_n \in \mathbb{C}_p$$

is well defined whenever

$$|a_n z^n| \to 0 \text{ as } n \to \infty.$$

Functions of this type are called *p-adic analytic functions*. If h is analytic on $\mathbb{C}_p$, then h is called a p-adic entire function. Let

$$h(z) = \sum_{n=0}^{\infty} a_n z^n,\ a_n \in \mathbb{C}_p$$

be a p-adic analytic function on $|z| < R$. For $0 < r < R$, define $M_h(r) = \max_{|z|=r} |h(z)|$. We have the following lemma (see [AS]).

Lemma 2.1. *The following statements hold:*

1. *We have $M_h(r) = \max_{n\geq 0} |a_n| r^n$.*
2. *The maximum on the right of (1) is attained for a unique value of n except for a discrete sequence of values $\{r_\nu\}$ in the open interval $(0, R)$.*
3. *If $r \notin \{r_\nu\}$ and $|z| = r < R$, then $|h(z)| = M_h(r)$.*
4. *If h is a non-constant p-adic entire function, then $M_h(r) \to \infty$ as $r \to \infty$.*
5. *We have $M_{h'}(r) \leq M_h(r)/r$ $(r > 0)$.*
6. *We have $M_{fg}(r) = M_f(r) M_g(r)$ for any analytic functions f and g,*

A *p-adic holomorphic curve* f is a map $f = [f_0 : \cdots : f_n] : \mathbb{C}_p \to \mathbb{P}^n(\mathbb{C}_p)$, where $f_0, \ldots, f_n$ are p-adic entire functions without common zeros. The map $\mathbf{f} = (f_0, \cdots, f_n) : \mathbb{C}_p \to \mathbb{C}_p^{n+1} - \{0\}$ is called a reduced representation of f. The p-adic holomorphic curve $f : \mathbb{C}_p \to \mathbb{P}^n(\mathbb{C}_p)$ is said to be linearly non-degenerate if $f(\mathbb{C}_p)$ is not contained in any proper subspace of $\mathbb{P}^n(\mathbb{C}_p)$. Hyperplanes $H_1, \ldots, H_q$ in $\mathbb{P}^n(\mathbb{C}_p)$ are said to be in general position if any

$n+1$ of them are linearly independent. The following theorem generalizes Theorem B.

THEOREM 2.1. *Let $f_1, f_2, \ldots, f_\lambda : \mathbb{C}_p \to \mathbb{P}^n(\mathbb{C}_p)$ be linearly non-degenerate p-adic holomorphic curves. Denote by $\mathbf{f}_i$ a reduced representation of f_i for $1 \le i \le \lambda$. Let $H_1, \ldots, H_q$ be hyperplanes in $\mathbb{P}^n(\mathbb{C}_p)$ located in general position, and assume that $f_1^{-1}(H_j) = \cdots = f_\lambda^{-1}(H_j)$. Let $D_j = f_1^{-1}(H_j)$, $D = \cup_{j=1}^q D_j$, and assume that for $i \ne j$, $D_i \cap D_j = \emptyset$. Let $l \in \{2, 3, \ldots, \lambda\}$ be the minimal index such that for any increasing sequence $1 \le j_1 < j_2 < \cdots < j_l \le \lambda$, we have $\mathbf{f}_{j_1}(z) \wedge \cdots \wedge \mathbf{f}_{j_l}(z) = 0$ for every point $z \in D$, where $\wedge$ is the usual wedge product, and suppose that $q \ge \frac{\lambda n}{\lambda - l + 1} + n + 1$. Then $f_1, \ldots f_\lambda$ are algebraically dependent over $\mathbb{C}_p$, i.e., $\mathbf{f}_1(z) \wedge \cdots \wedge \mathbf{f}_\lambda(z) \equiv 0$ on $\mathbb{C}_p$.*

In the case of $\lambda = 2$, Theorem 2.1 gives the following result:

THEOREM 2.2. *Let $f, g : \mathbb{C}_p \to \mathbb{P}^n(\mathbb{C}_p)$ be two p-adic linearly non-degenerate holomorphic curves. Let $H_1, \ldots, H_{3n+1}$ be hyperplanes in $\mathbb{P}^n(\mathbb{C}_p)$ located in general position. Assume that $f^{-1}(H_j) = g^{-1}(H_j)$ for $1 \le j \le 3n+1$ and that $f^{-1}(H_i) \cap f^{-1}(H_j) = \emptyset$ for $i \ne j$. If $f(z) = g(z)$ for every point $z \in \cup_{j=1}^q f^{-1}(H_j)$, then $f \equiv g$.*

We will first give a proof of Theorem 2.2, and then outline the proof of Theorem 2.1.

Proof of Theorem 2.2. Let $\mathbf{f}$, $\mathbf{g} : \mathbb{C}_p \to \mathbb{C}_p^{n+1} - \{0\}$ be the reduced representations of f and g, and write $\mathbf{f} = (f_0, \ldots, f_n)$, $\mathbf{g} = (g_0, \ldots, g_n)$. Let

$$H_j = \{w = [w_0 : \ldots : w_n] \in \mathbb{P}^n(\mathbb{C}_p) : a_{j0}w_0 + \cdots + a_{jn}w_n = 0\},\ 1 \le j \le q,$$

and set $L_j(X) = a_{j0}x_0 + \cdots + a_{jn}x_n$, where $X = (x_0, \ldots, x_n)$ and L_j is the corresponding linear form of H_j.

Without loss of generality, we can assume that there exists a sequence $z_k \in \mathbb{C}_p$ such that $r_k = |z_k| \to \infty$, $r_k \notin \{r_\nu\}$, where the set $\{r_\nu\}$ is the discrete set appearing in part (2) of Lemma 2.1, $L_j(\mathbf{f})(z_k) \ne 0$ for $1 \le j \le 3n+1$, and

$$|f_0(z_k)| \ge \max_{0 \le i \le n} \{|f_i(z_k)|, |g_i(z_k)|\}. \tag{2.1}$$

Define

$$\Psi = \frac{W(f_0, \ldots, f_n) \cdot (f_0 g_1 - f_1 g_0)^n}{\prod_{j=1}^{3n+1} L_j(\mathbf{f})},$$

where $W(f_0, \ldots, f_n)$ is the Wronskian of $f_0, \ldots, f_n$. Since f is linearly non-degenerate, we have $W(f_0, \ldots, f_n) \not\equiv 0$. We first show that Ψ is p-adic entire. In fact, since the sets $f^{-1}(H_i)$ are disjoint, each point $z \in \cup_{j=1}^{3n+1} f^{-1}(H_j)$ satisfies $z \in f^{-1}(H_{i_0})$ for some i_0 with $1 \le i_0 \le 3n+1$, and $z \notin f^{-1}(H_j)$ for $j \ne i_0$. Hence $L_j(\mathbf{f})(z) \ne 0$ when $j \ne i_0$. Assume that $L_{i_0}(\mathbf{f})$ vanishes at z with vanishing order m. Then, since $W(f_0, \ldots, f_n) = a_{i_0 0}^{-1} W(L_{i_0}(\mathbf{f}), f_1, \ldots, f_n)$ (where we assume, without of generality, that $a_{i_0 0} \neq 0$), $W(f_0, \dots, f_n)$ vanishes at z with order at least $m - n$. On the other hand, by assumption we have $f(z) = g(z)$, so $(\mathbf{f} \wedge \mathbf{g})(z) = 0$. Thus, $(f_0 g_1 - f_1 g_0)^n$ vanishes at z with order at least n. Hence, by the definition of Ψ, Ψ is continuous at z, so Ψ is p-adic entire.

Now, for each fixed z_k, by rearranging the indices we may assume that

$$|L_1(\mathbf{f})(z_k)| \leq |L_2(\mathbf{f})(z_k)| \leq \cdots \leq |L_{3n+1}(\mathbf{f})(z_k)|.$$

Solving the system of linear equations

$$L_j(\mathbf{f})(z_k) = a_{j0} f_0(z_k) + \cdots + a_{jn} f_n(z_k), \quad 1 \leq j \leq n+1,$$

we obtain

$$|f_0(z_k)| \leq B|L_{n+1}(\mathbf{f})(z_k)| \leq \cdots \leq B|L_{3n+1}(\mathbf{f})(z_k)|,$$

where $B > 0$ is a constant independent of z_k. Hence

$$\begin{aligned} |\Psi(z_k)| &= \frac{|W(f_0, \dots, f_n)(z_k)||(f_0 g_1 - f_1 g_0)(z_k)|^n}{|\prod_{j=1}^{3n+1} L_j(\mathbf{f})(z_k)|} \\ &\leq \frac{B^{2n}|W(f_0, \dots, f_n)(z_k)||(f_0 g_1 - f_1 g_0)(z_k)|^n}{|L_1(\mathbf{f})(z_k)| \cdots |L_{n+1}(\mathbf{f})(z_k)||f_0(z_k)|^{2n}}. \end{aligned} \tag{2.2}$$

By Lemma 2.1,

$$M_{\frac{(L_j(\mathbf{f}))'}{L_j(\mathbf{f})}}(r) \leq \frac{1}{r}.$$

Since for $1 \leq i \leq n$,

$$\frac{(L_j(\mathbf{f}))^{(i)}}{L_j(\mathbf{f})} = \frac{(L_j(\mathbf{f}))^{(i)}}{(L_j(\mathbf{f}))^{(i-1)}} \cdots \frac{(L_j(\mathbf{f}))'}{L_j(\mathbf{f})},$$

it follows that

$$M_{(L_j(\mathbf{f}))^{(i)}/L_j(\mathbf{f})}(r) \leq \frac{1}{r^i},$$

and hence

$$\left| \frac{(L_j(\mathbf{f}))^{(i)}}{L_j(\mathbf{f})}(z_k) \right| \leq \frac{1}{|z_k|^i}. \tag{2.3}$$

By the properties of the Wronskian and the assumption that the hyperplanes are in general position, we have

$$\frac{|W(f_0, \dots, f_n)(z_k)|}{|L_1(\mathbf{f})(z_k)| \cdots |L_{n+1}(\mathbf{f})(z_k)|} = \frac{C|W(L_1(\mathbf{f}, \dots, L_{n+1}(\mathbf{f}))(z_k)|}{|L_1(\mathbf{f})(z_k)| \cdots |L_{n+1}(\mathbf{f})(z_k)|}, \tag{2.4}$$

where $C > 0$ is a constant. By the properties of the p-adic norm and (2.3), we have

$$\begin{aligned} &\frac{|W(L_1(\mathbf{f})(z_k),\dots,L_{n+1}(\mathbf{f})(z_k)|}{|L_1(\mathbf{f})(z_k)|\cdots|L_{n+1}(\mathbf{f})(z_k)|} \\ &\qquad \le \max_{i_1+\cdots+i_{n+1}=n}\left|\frac{(L_1(\mathbf{f}))^{(i_1)}}{L_1(\mathbf{f})}(z_k)\right|\cdots\left|\frac{(L_{n+1}(\mathbf{f}))^{(i_{n+1})}}{L_{n+1}(\mathbf{f})}(z_k)\right| \\ &\qquad \le \frac{1}{|z_k|^n}. \end{aligned} \tag{2.5}$$

On the other hand, by (2.1) and the properties of the p-adic norm, we also have

$$|(f_0g_1 - f_1g_0)^n(z_k)| \le |f_0(z_k)|^{2n}. \tag{2.6}$$

Combining (2.2), (2.4), (2.5) and (2.6) yields

$$|\Psi(z_k)| \le \frac{B^{2n}C}{|z_k|^n} \to 0 \text{ as } k \to \infty,$$

where $B > 0$ and $C > 0$ are two constants which depend only on the hyperplanes. This implies that $\Psi \equiv 0$. Hence

$$\frac{g_1}{g_0} \equiv \frac{f_1}{f_0}.$$

Similarly, we can prove that, for $1 \le i \le n$,

$$\frac{g_i}{g_0} \equiv \frac{f_i}{f_0}.$$

So $f \equiv g$. This completes the proof of Theorem 2.2. □

Proof of Theorem 2.1. Let $\mathbf{f}_\lambda = (f_{\lambda,0},\dots,f_{\lambda,n})$ be the reduced representation of f_λ. Without loss of generality, we can assume that there exists a sequence $z_k \in \mathbb{C}_p$ such that $r_k = |z_k| \to \infty$, $L_j(\mathbf{f}_1)(z_k) \ne 0$ for $1 \le j \le 3n+1$ and

$$|f_{1,0}(z_k)| \ge \max_{0\le i\le n, 1\le t\le\lambda}\{|f_{t,i}(z_k)|\}.$$

Assume that $f_1,\dots f_\lambda$ are not algebraically dependent over $\mathbb{C}_p$, i.e., $\mathbf{f}_1 \wedge\cdots\wedge \mathbf{f}_\lambda \not\equiv 0$. Take a non-trivial component $h(z)$ of $\mathbf{f}_1 \wedge\cdots\wedge \mathbf{f}_\lambda$ and set

$$\Phi = \frac{W(f_{1,0},\dots,f_{1,n})\cdot h(z)^{\frac{n}{\lambda-l+1}}}{\prod_{j=1}^q L_j(\mathbf{f}_1)},$$

where $q \ge \frac{n\lambda}{\lambda-l+1} + n + 1$. Let $\Psi = \Phi^{\lambda-l+1}$. We now show that Ψ is p-adic entire. In fact, since $D_i \cap D_j = \emptyset$ for $i \ne j$, each point $z \in D = \cup_{j=1}^q D_j$ satisfies $z \in f_1^{-1}(H_{i_0})$ for some i_0 with $1 \le i_0 \le q$, and $z \notin f_1^{-1}(H_j)$ for $j \ne i_0$. Thus, $L_j(\mathbf{f}_1)(z) \ne 0$ when $j \ne i_0$. Assume that $L_{i_0}(\mathbf{f})$ vanishes at z with vanishing order m. Then $W(f_{1,0},\dots,f_{1,n})$ vanishes at z with order at least $m-n$. On the other hand, it is easy to verify, using the assumptions of Theorem 2.1, that for any $z \in D$, $|h(z)|^{\frac{n}{\lambda-l+1}}$ vanishes at z with vanishing order at least n. Therefore Ψ is continuous at z, and hence Ψ is p-adic entire. The rest of proof follows that of Theorem 2.2. □

3. Uniqueness problems counting multiplicity

The results in Section 2 are concerned with uniqueness problems without counting multiplicity. In this section we consider the uniqueness problem counting multiplicity. In this case, the result is simple and elegant:

THEOREM 3.1. *Let $f,\ g : \mathbb{C}_p \to \mathbb{P}^n(\mathbb{C}_p)$ be two p-adic holomorphic curves, at least one of which is linearly non-degenerate. Let $H_1, \ldots, H_{n+2}$ be hyperplanes in $\mathbb{P}^n(\mathbb{C}_p)$ located in general position such that $f(\mathbb{C}_p) \not\subset H_j$ and $g(\mathbb{C}_p) \not\subset H_j$ for $1 \le j \le n+2$. Denote by L_j the linear form associated with H_j, and assume that $L_j(f)/L_j(g)$, $1 \le j \le n+2$, is non-vanishing on $\mathbb{C}_p$ (i.e., that $L_j(f)$ and $L_j(g)$ vanish at the same points with the same vanishing order). Then $f \equiv g$.*

Proof. Without loss of generality, we can assume that g is linearly non-degenerate. We recall the fact that any non-vanishing p-adic entire function must be constant (see [R1]). Consider the functions

$$h_j = \frac{L_j(f)}{L_j(g)},\ 1 \le j \le n+2.$$

Each h_j is a non-vanishing p-adic entire function, so $h_j = c_j$, where c_j is constant. Without loss of generality, we may assume that the hyperplanes H_j are represented by

$$H_j = \{w = [w_0 : \cdots : w_n] \in \mathbb{P}^n(\mathbb{C}_p) \mid w_{j-1} = 0\},\ 1 \le j \le n+1$$

and

$$H_{n+2} = \{w = [w_0 : \cdots : w_n] \in \mathbb{P}^n(\mathbb{C}_p) \mid w_0 + \cdots + w_n = 0\}.$$

Thus we can write $c_{n+1}(g_0 + \cdots + g_n) = f_0 + \cdots + f_n$, and hence

$$(c_{n+1} - c_0)g_0 + \cdots + (c_{n+1} - c_n)g_n = 0.$$

By the linear-nondegeneracy condition, this implies $c_0 = c_1 = \cdots = c_{n+1}$. Hence $f \equiv g$.

□

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