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GREEN'S FUNCTIONS, ELECTRIC NETWORKS, AND THE GEOMETRY OF HYPERBOLIC RIEMANN SURFACES

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Abstract. We compare Green's function g on an infinite volume, hyperbolic Riemann surface X with an analogous discrete function g_{disc} on a graphical caricature Γ of X. The main result, modulo technical hypotheses, is that g and g_{disc} differ by at most an additive constant C which depends only on the Euler characteristic of X. In particular, the estimate of g by g_{disc} remains uniform as the geometry (i.e., the conformal structure) of X varies.

1. Introduction

In [DPRS], Dodziuk et al. demonstrate that one can estimate the small eigenvalues of the Laplacian on a Riemann surface by associating an electric network with the surface and studying a discrete problem on the network. Our goal in this paper is to use the same association to the approximate Green's function on a Riemann surface. More specifically, we recall how one can use the collar theorem to define a projection $\Pi : X \to \Gamma$ from a hyperbolic Riemann surface X onto an electric network Γ, referring to Γ and Π as a *circuit diagram* for X. Roughly speaking, Π maps components of the thick part of X to vertices and components of the thin part of X to edges. Each edge is assigned a resistance according to the diameter of the thin component which maps to it. If X has finitely generated fundamental group and infinite volume, Green's function $g(\cdot, q)$ exists on X—i.e., g is harmonic on $X \setminus \{q\}$ with a logarithmic pole at q, and g tends to 0 through any infinite volume end of X. Furthermore, using basic linear algebra, one can define a discrete version of Green's function $g_{\text{disc}}(\cdot, \Pi(q))$ on Γ with pole at $\Pi(q)$—i.e., g_{disc} is the potential function induced on Γ by placing a unit positive charge at $\Pi(q)$

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and a unit negative charge at the vertex corresponding to ground. Our main theorem is as follows:

THEOREM 1.1. *Let X be a connected, hyperbolic Riemann surface with infinite volume and finite Euler characteristic. Let $\Pi : X \to \Gamma$ be a circuit diagram for X. Then for any $\epsilon > 0$, there exists a constant C depending only on ϵ and the Euler characteristic of X such that*

$$|g(p,q) - g_{\mathrm{disc}}(\Pi(p),\Pi(q))| \leq C$$

for all $p, q \in X$ such that

(1) *$d(p,q) \geq \epsilon$;*
(2) *there exists no non-joining component $\mathcal{C}$ of the thin part of X that contains both p and q.*

A non-joining component of the thin part of X is, roughly speaking, one whose boundary is contained in a single connected component of the thick part of X. Heuristically, one expects little variation of Green's function over a non-joining thin component— unless of course it contains the pole q. Both conditions (1) and (2) on p and q in Theorem 1.1 are more cosmetic than necessary. For instance, the first condition on p and q accounts for the fact that $g(p,q)$ has a logarithmic pole at $p = q$, whereas $g_{\mathrm{disc}}(\Pi(p),\Pi(q))$ is continuous across $\Pi(p) = \Pi(q)$. The theorem remains true for $\mathrm{dist}(p,q) < \epsilon$ if we modify the conclusion to read

$$\left| g(p,q) - \log^{+} \frac{\epsilon}{\mathrm{dist}(p,q)} - g_{\mathrm{disc}}(\Pi(p),\Pi(q)) \right| < C,$$

where $\log^{+} t = \max\{0, \log t\}$. Later on (in Remark 5.10) we explain how one can modify the approximation of g by g_{disc} so that it holds when condition (2) is violated.

The utility of Theorem 1.1 is that it provides a relatively simple function g_{disc} that approximates Green's function g with a bound that is insensitive to the geometry of X. One need only consider the case where X is an annulus whose modulus is tending to infinity to realize that Green's function itself depends greatly on the geometry of the surface X. More generally, Green's function (with zero boundary values) will become unboundedly large everywhere as the geometry of X tends to that of a hyperbolic surface with finite volume. So Theorem 1.1 gives a more precise description of how this degeneration takes place.

We now indicate the organization of the rest of this paper. Section 2 begins by reviewing some basic facts about hyperbolic geometry and harmonic functions. Most of the results we state are well-known, though perhaps not exactly in the form that we state them. The main goal of the section is to prove the following result, which to our knowledge is new, and which plays a central role in the proof of Theorem 1.1.

THEOREM 1.2. *Suppose that Green's function for X exists. Given $p \in X$, let ρ be the minimum of* $\operatorname{dist}(p, q)$ *and the injectivity radius of X at p. Then there exists a universal constant C such that*

$$\langle dg_q(p) \rangle \leq \frac{C}{\tanh \rho}.$$

One should compare this theorem with Harnack's inequality (Theorem 2.2, item (3)). While Harnack's inequality applies to any positive harmonic function and allows good control on the derivative where the function is small, Theorem 1.2 holds only for Green's function (in particular, the theorem will not apply to large multiples of Green's function), and it provides good control of the derivative at any point p away from q where the injectivity radius is not too small. It is useful for our purposes because, when combined with Theorem 4.1, it implies that Green's function will not vary much over a component of the thick part of X.

In Section 3 we concern ourselves with providing a good approximation of Green's function when X is doubly connected. The main result (Theorem 3.2) is that Green's function on a round annulus or a punctured disk is well-approximated by a piecewise affine function of $\log |z|$. This special case is important for several reasons. By the collar lemma, thin components of X are isometric to round annuli or punctured disks, so a good understanding of Green's function on annuli is essential for understanding Green's function on more general surfaces. Moreover, it will become clear from the way we define the circuit diagram of X that Theorem 1.1 does not speak directly to the doubly connected case. Section 3 fills that gap. Finally, though the topological details are simpler for doubly connected domains, most of the analytic technicalities that arise in the general case are already present in the doubly connected case. So Section 3 provides a useful "introduction" to some of what occurs later—especially in Section 5.

Section 4 deals only with hyperbolic geometry. After reviewing the collar lemma for hyperbolic surfaces, our main result is a sort of diameter bound (Theorem 4.1) for connected components K of the thick part of X. That such bounds exist might well be expected from heuristic compactness arguments involving the Teichmüller space of K. However, since X is open and K need not be compact, the heuristic arguments would not seem to suffice. For instance, there is no universal bound on the diameter of K, or even on the diameter of K intersected with the convex core of X.

In Section 5, we begin by making precise what we mean by a circuit diagram Γ for X. We then proceed to define an "approximate Green's function" g_{approx} on Γ. This is not the same as the discrete Green's function g_{disc} mentioned above. In fact, we use g_{approx} as an intermediary between g and g_{disc}. There are two main results in Section 5. The first (Theorem 5.4) states that g_{approx} is nearly continuous at vertices of Γ. The second (Theorem 5.9) states that in fact g_{approx} does provide a good approximation of g. The proofs of the results are accomplished more or less simultaneously in a series of analytic lemmas.

In Section 6, we begin by giving a precise definition of the discrete Green's function for the circuit diagram Γ. The main result (Theorem 6.2) of the section states that $|g_{\text{disc}} - g_{\text{approx}}|$ is bounded by a constant depending only on the topology of X. The main difficulty in proving Theorem 6.2 is strictly linear algebraic, and it is surmounted in the proof of Theorem 6.3. Ironically, the proof of Theorem 6.3 remains to us the most mysterious aspect of this paper. We suspect that there exists a more general result with a more straightforward proof, but thus far have been able to find neither.

Theorems 5.9 and 6.2 combine in a straightforward fashion to completely justify Theorem 1.1. However, we conclude the paper in Section 7 by indicating some weaker but still rather useful results about Green's function that one can deduce from this paper. We also indicate what we think is a promising direction for further extending our results.

We have organized this paper with a bias toward explaining the proof of Theorem 1.1 (as opposed to explaining its statement). For this reason, we try to separate the more local and analytic aspects of the argument from the global and combinatorial aspects of the argument. One consequence is that important definitions are delayed until late in the paper. To someone who wishes merely to obtain a good grasp of the statement of Theorem 1.1, we suggest the following approach. Read the first couple of paragraphs of Section 2. If more background on hyperbolic geometry would be helpful, read the following several paragraphs of Section 2 and the part of Section 4 concerned with the collar lemma. Then read Section 5 through Definition 5.1 and Section 6 through the definition of the discrete Green's function. This should be sufficient to convey precise meaning to Theorem 1.1. For a slightly better intuitive understanding, we recommend also reading Section 3 through Theorem 3.2.

Finally, David Barrett has pointed out to us another approach to approximating Green's functions with discrete functions. In [HS], Hardt and Sullivan consider Green's function for a bordered Riemann surface given as a branched cover of the unit disk. Rather than use the thick/thin decomposition of X given by the hyperbolic metric, they use a "swiss cheese" decomposition of $X \setminus \{q\}$ induced by the branched cover. They then compute all of their analytic estimates with respect to a "plumbing metric" defined piecewise on the swiss cheese decomposition. The result is a similar kind of piecewise linear approximation, but one which depends on the (extrinsic) branched cover, rather than on the (intrinsic) hyperbolic metric.

2. Hyperbolic geometry and harmonic functions

In this section we will restate some basic facts about harmonic functions in terms of hyperbolic geometry. Our main goal is to prove Theorem 1.2. Let X denote a Riemann surface. We assume throughout this paper that X is connected and hyperbolic and has finitely generated fundamental group. Hence we can realize X as a subdomain of a compact, connected Riemann surface without boundary $\hat{X}$ such that $\hat{X} \setminus X$ is a finite union of isolated points (which we refer to as *punctures*) and closed, smoothly bounded, topological disks. When we write $\overline{X}$ and $b\overline{X}$ below, we mean closure with respect to $\hat{X}$. Note that $b\overline{X}$ consists of all components of bX which are not punctures.

By assumption, X is uniformized by the unit disk $\Delta = \{|z| < 1\} \subset \mathbb{C}$. Therefore X inherits a *hyperbolic* metric with constant curvature -1 from the Poincaré metric

$$ds = ds_\Delta = \frac{2|dz|}{1-|z|^2} \tag{2–1}$$

on Δ. We will also use ds to denote the hyperbolic length element on X. If η is a section of some tensor bundle on X, we let $\langle \eta(p) \rangle$ denote the hyperbolic length of η at p. Given $K \subset X$ we let $B_K(r)$ denote the set of points whose hyperbolic distance from K is less than r.

In computations below we will need to use the specific form of the hyperbolic metric on the upper half plane $\mathbb{H} = \{\operatorname{Im} z > 0\}$:

$$ds = ds_{\mathbb{H}} = \frac{|dz|}{\operatorname{Im} z}; \tag{2–2}$$

on the annulus $A = A_R = \{e^R > |z| > e^{-R}\}$:

$$ds = ds_R = \frac{\pi |dz|}{2|z| R \cos \frac{\pi \log|z|}{2R}}; \tag{2–3}$$

and on the punctured disk $\Delta^* = \{0 < |z| < 1\}$:

$$ds = ds_* = \frac{-|dz|}{|z| \log|z|}. \tag{2–4}$$

We will make abundant use of the fact that the unit circle is a geodesic of length π^2/R with respect to the hyperbolic metric on A_R.

We let $\operatorname{inj}(p)$ denote the *injectivity radius* of X at p; that is, $\operatorname{inj}(p)$ is the largest number t such that $B_p(t)$ is a topological disk. Equivalently, if $\pi_p : \Delta \to X$ is a uniformizing map sending 0 to p, then one can apply (2–1) to see that $\pi_p : \{|z| < \tanh(t/2)\} \to B_p(t)$ is an injection for $t \le \operatorname{inj}(p)$ but not for $t > \operatorname{inj}(p)$. We refer to the restriction of π_p to the disk $\{|z| < \tanh(\frac{1}{2}\operatorname{inj}(p))\}$ as *standard coordinates about* p.

We now turn to harmonic functions h on X. The first result we state appears in [Ah] (see Chapter 4, Sections 6.1 and 6.2) in the case of plane domains containing round annuli, but the proof generalizes perfectly well to a larger setting. Given a smooth function h on X, we let $*dh$ denote the conjugate differential of h.

Theorem 2.1: Circular Averages. *Let $h : X \to \mathbb{R}$ be a harmonic function. Let γ be a union of smooth, oriented closed curves. Then the quantity*

$$\alpha(\gamma, h) \stackrel{def}{=} \frac{1}{2\pi} \int_\gamma *dh,$$

depends only on h and the homology class of γ in X. If $f : A_R \to X$ is a holomorphic map and $b\Delta \subset A_R$ is given the usual counterclockwise orientation, then there is a constant $\beta \in \mathbb{R}$ such that

$$\frac{1}{2\pi} \int_0^{2\pi} h \circ f(re^{i\theta})\, d\theta = \alpha(f_* b\Delta, h) \log r + \beta.$$

holds for every r between e^{-R} and e^R.

This result plays an absolutely crucial role in the proof of Theorem 1.1. In particular, we will use it in the next section to approximate Green's function on an annulus. On more general Riemann surfaces, we will use it to estimate the amount by which Green's function changes across a tubular neighborhood (the collar) of a short geodesic. Item (1) in the next result is simply a restatement of Harnack's inequality for positive harmonic functions. Item (2) generalizes (1) to the case where h is bounded above *and* below, and item (3) to the case where h is harmonic and positive on only part of X.

Theorem 2.2: Harnack Inequalities. *Let $h : X \to \mathbb{R}$ be harmonic. Then:*

(1) *if $h > 0$, then $\langle dh \rangle \le h$;*
(2) *if $|h| < R$, then $\langle dh \rangle \le \frac{2R}{\pi} \cos \frac{\pi h}{2R}$;*
(3) *if $h > 0$ on some subdomain Ω of X (h needn't be harmonic outside Ω), then $\langle dh(p) \rangle \le \frac{h(p)}{\tanh(1/2) \operatorname{dist}(p, X \setminus \Omega)}$ for all $p \in \Omega$.*

Proof. Let $\pi : \Delta \to X$ be a universal cover. Since π is a local isometry, it is enough to estimate $\langle d(h \circ \pi) \rangle$ where the norm is measured with respect to the Poincaré metric on Δ. If h is harmonic on all of X, there exists a well-defined harmonic conjugate h^* on Δ for $h \circ \pi$. To prove (1), note that $f = -h^* + ih \circ \pi$ is analytic and $f(\Delta) \subset \mathbb{H}$. According to the Schwarz lemma, f shrinks the hyperbolic metric. Therefore,

$$\langle d(h \circ \pi) \rangle = \langle \partial f \rangle \le \operatorname{Im} f = h \circ \pi.$$

The proof of (2) is the same, except that we use the fact that $e^{h \circ \pi + ih^*}$ is an analytic function mapping Δ into A_R.

To prove (3), choose the universal cover π of X so that $\pi(0) = p$. Note that $h \circ \pi$ is a positive harmonic function on $D = \{|z| < \tanh \frac{1}{2} \operatorname{dist}(p, X \setminus \Omega)\}$. Hence, by (1) we have

$$\langle d(h \circ \pi) \rangle_D \leq h \circ \pi$$

on D, where the norm is evaluated with respect to the hyperbolic metric on D. The proof is finished by comparing the Poincaré metric on Δ with the hyperbolic metric on D. □

Corollary 2.3: Bounded Deviation. *Suppose that $h : \Delta \to \mathbb{R}$ is harmonic and $||h - h(0)||_\infty \leq M$. Then*

$$\max_{|z|=r} |h(z) - h(0)| \leq M \left(\frac{2}{\pi} \sin^{-1} \frac{2r}{1+r^2} \right) \leq \frac{2rM}{1+r^2}$$

for all $r < 1$. Suppose that $h : A_R \to \mathbb{R}$ is harmonic. For $e^{-R} < r < e^R$, let $a(r) = \frac{1}{2\pi} \int_0^{2\pi} h(re^{i\theta})\, d\theta$ and $m(r) = \max_\theta |h(re^{i\theta}) - a(r)|$. Suppose that $||m||_\infty \leq M$. Then

$$m(r) \leq \frac{2M}{\pi} \sin^{-1} \tanh \left(\frac{\pi^2}{2R} \sec \frac{\pi \log r}{2R} \right) \leq M \tanh \left(\frac{\pi^2}{2R} \sec \frac{\pi \log r}{2R} \right).$$

Proof. In both parts of the corollary it suffices to assume that h is continuous up to the boundary of its domain. In the first part we can add a constant to h so that

$$0 = h(0) = \int_0^{2\pi} h(re^{i\theta})\, d\theta$$

for all $r \leq 1$. Note that if $|z| = r$, then $\operatorname{dist}(z, 0) = \tanh r/2$. Thus by integrating (2–1) along a radial segment γ from 0 to z, we obtain

$$\log \frac{1+r}{1-r} = \int_\gamma ds \geq \left| \int_{h(0)}^{h(r)} \frac{\pi \sec \frac{\pi h}{2M}}{2M}\, dh \right| = \frac{1}{2} \log \frac{1 + |\sin \frac{\pi h}{2M}|}{1 - |\sin \frac{\pi h}{2M}|}$$

Thus,

$$\frac{h}{M} \leq \sin \frac{\pi h}{2M} \leq \frac{2r}{1+r^2},$$

which proves the first claim.

By Theorem 2.1, the function $a(r)$ in the second claim is an affine function of $\log r$. So after subtracting this from h, we can assume that $a(r) \equiv 0$. By the intermediate value theorem, for each r there exists a point z_0 such that $|z_0| = r$ and $h(z_0) = 0$. Given any other z with $|z| = r$, we proceed as in the proof of the first claim and apply (2–3) to the shortest segment of the circle of radius r joining z to z_0. This gives

$$\frac{h}{M} \leq \sin \frac{\pi h}{2M} \leq \tanh \frac{\pi^2}{2R \cos \frac{\pi \log r}{2R}}$$

as desired. □

Recall that *Green's function* with the pole at q on X is the smallest harmonic function (if it exists) $g_q = g(\cdot, q) : X \setminus \{q\} \to \mathbb{R}^+$ such that $|g(z,q) + \log|z||$ is bounded in some/any local coordinate z about q. If X has infinite volume (i.e., if $b\overline{X}$ is non-empty), then g_q exists regardless of q. Furthermore, g_q extends to a continuous function on all of $\overline{X} \setminus \{q\}$ such that g_q is harmonic in a neighborhood of any puncture and identically 0 on $b\overline{X}$. An elementary fact (see [Ah], Chapter 5, Section 5.2) which turns out to be very important for our purposes (see the proof of Theorem 1.2 at the end of this section) is the following:

Theorem 2.4: Symmetry. *$g(p,q) = g(q,p)$ for all $p, q \in \overline{X}$.*

We employ a slight variant of the notation from Theorem 2.1:

Definition 2.5. Given a union $\gamma \subset X$ of smooth, oriented closed curves and $q \in X \setminus \gamma$, we define the *flux* across γ with respect to q by

$$\alpha(\gamma, q) = \int_\gamma *dg(\cdot, q).$$

Since $\alpha(\gamma, q)$ depends only on the homology class of γ in $X \setminus \{q\}$, we can clearly extend its definition to curves which lie in $\overline{X}$. In particular, we can discuss flux across oriented components of $b\overline{X}$. Suppose that γ_1 and γ_2 are homologous in $\overline{X}$. Then after removing $q \notin \gamma_1, \gamma_2$ from $\overline{X}$, γ_1 is homologous to $\gamma_2 - k\gamma_3$ where k is an integer and γ_3 is any circle about q with small enough radius. Letting the radius of γ_3 tend to 0 and using the fact that $g(p,q) = -\log \operatorname{dist}(p,q) + O(1)$ for p near q, we can compute that

$$\alpha(\gamma_1, q) = \alpha(\gamma_2, q) - k.$$

In particular, if $\Omega \subset X$ is a smoothly bounded subdomain, and $b\Omega \subset \overline{X}$ is oriented with respect to its outward unit normal vector then $\alpha(b\Omega) = 0$ or 1, according to whether $q \notin K$ or $q \in K$, respectively. Thus $\alpha(b\overline{X}, q) = 1$.

The main step in the proof of Theorem 1.2 is the next lemma. The proof of the lemma is essentially the classical "alternating method" for constructing harmonic "dipoles" on a surface (see the introduction to Chapter III in [AS]). We thank David Barrett for calling our attention to the alternating method and for suggesting its usefulness in the present context.

Lemma 2.6. *Given $p_1, p_2 \in X$, suppose that there is a holomorphic injection $\varphi : \Delta \to X$ such that $p_j = \varphi(z_j)$. Let $r = \max_{j=1,2} |z_j|$. Then there exists a harmonic "dipole" $h : X \setminus \{p_1, p_2\} \to \mathbb{R}$ such that*

(1) *$h \circ \varphi - \log|z - z_1|$ extends harmonically past z_1;*
(2) *$h \circ \varphi + \log|z - z_2|$ extends harmonically past z_2;*
(3) *$h \le \frac{Cr}{1-r}$ on $X \setminus \varphi(\Delta)$, where C is a universal constant.*

Proof. We will construct h on the compact surface $\hat{X}$ containing X. By restricting φ to $\{|z| < 1-\delta\}$ and then letting δ tend to 0, we can suppose that φ is actually holomorphic and injective in a neighborhood of $\overline{\Delta}$. In particular $\varphi(b\Delta)$ is a smooth Jordan curve. We can suppose that $z_1 = -r$ and $z_2 = r$, because if this were not the case, then we could replace φ with $\varphi \circ T$ for some automorphism T of Δ and arrange for $z_1 = -x$, $z_2 = x$ for some $0 < x < r$.

Let $\Delta_1 = \varphi(\Delta)$ and $\Delta_2 = \varphi(\{|z| < (1+r)/2\})$. We let $h_0 : \Delta_1 \setminus \{p_1, p_2\} \to \mathbb{R}$ be the harmonic function with zero boundary values, a logarithmic pole at p_1, and a negative logarithmic pole at p_2. That is, $h_0 = H_0 \circ \varphi^{-1}$ where

$$H_0 = \log \frac{|z-r|}{|1-zr|} - \log \frac{|z+r|}{|1+zr|}.$$

Note that $|h_0| < Cr$ on $b\Delta_2$. Then for $j \geq 1$ we define a sequence of harmonic functions $h_j : \Delta_1 \to \mathbb{R}$, $g_j : \hat{X} \setminus \Delta_2 \to \mathbb{R}$ inductively as follows: g_{j+1} extends the boundary values $h_j|_{b\Delta_2}$ harmonically to $\hat{X} \setminus \Delta_2$, and h_j extends the boundary values $g_j|_{b\Delta_1}$ harmonically to Δ_1.

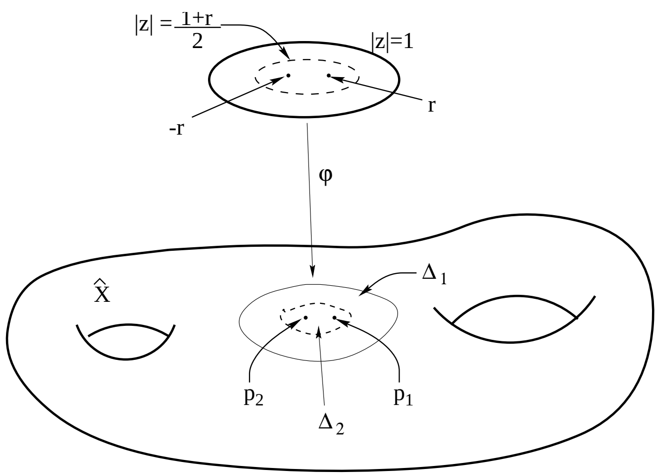

FIGURE 2.1. Setup for the proof of Lemma 2.6

Using the maximum principle, Theorem 2.1, the fact that $b\Delta_2$ is homologous to 0 in $\hat{X} \setminus \Delta_2$, and the fact that $b\Delta_1$ is homologous to 0 in $\overline{\Delta_1}$, we have

$$||g_j|_{b\Delta_1}||_\infty \leq ||g_j|_{b\Delta_2}||_\infty,$$

and

$$\int_0^{2\pi} g_j \circ \varphi(te^{i\theta})\, d\theta = \int_0^{2\pi} h_j \circ \varphi(te^{i\theta})\, d\theta = \int_0^{2\pi} H_0\left(\frac{1+r}{2}e^{i\theta}\right)\, d\theta = 0,$$

for all $j \geq 1$ and all $(1+r)/2 \leq t \leq 1$. Now Corollary 2.3 gives us that

$$||h_j|_{b\Delta_2}||_\infty \leq [1 - C(1-r)]||h_j|_{b\Delta_1}||_\infty = [1 - C(1-r)]||g_j|_{b\Delta_1}||_\infty,$$

for some $C > 0$. So by induction, the sums

$$h = h_0 + \sum_{j=1}^{\infty} h_j \qquad \text{and} \qquad g = \sum_{j=1}^{\infty} g_j$$

converge geometrically to harmonic functions. By construction $g = h$ on $b\Delta_1$ and $b\Delta_2$, so in fact $g = h$ on the region in between; i.e., the two functions patch together to create a well-defined dipole h on $\hat{X}$. Checking the bounds on g_j reveals that

$$||g||_\infty \leq \frac{\max_{b\Delta_2} |h_0|}{1 - [1 - C(1-r)]} \leq \frac{Cr}{1-r},$$

which establishes (3) and finishes the proof of the Lemma. □

Proof of Theorem 1.2. Note that the hypothesis of the theorem implies the existence of a uniformization map $\pi : \Delta \to X$ such that $\pi : \{|z| < \tanh(\rho/2)\} \to B_p(\rho)$ is injective and omits q. Let $\varphi : \Delta \to B_p(\rho)$ be given by $\varphi(z) = \pi(z/\tanh(\rho/2))$; let $p_1 = p = \varphi(0)$ and $p_2 = \varphi(z)$ for some other $z \in \Delta$. Then we can apply Lemma 2.6 to produce a harmonic function $h : \hat{X} \setminus \{p_1, p_2\} \to \mathbb{R}$ such that h has a positive logarithmic pole at p_1 and a negative logarithmic pole at p_2, and $|h| < \frac{C|z|}{1-|z|}$ outside of $\varphi(\Delta)$. In particular, since $b\overline{X} \cap \varphi(\Delta) = \emptyset$, the harmonic extension h_1 to $\overline{X}$ of $h|_{bX}$ is bounded by $\frac{C|z|}{1-|z|}$. If we let $h_2 = h_{2,p_1,p_2} = h - h_1$, then on $\overline{X}$, h_2 has the same properties as h (after possibly doubling the constant C) as well as zero boundary values on $b\overline{X}$. Since $b\overline{X}$ is non-empty (because X supports Green's function), we conclude that h_2 is unique even without the bound outside of $B_p(\rho) = \varphi(\Delta)$. Now it is trivial to compute

$$\begin{aligned}
\langle dg_q(p) \rangle &= \limsup_{p_2 \to p_1} \frac{|g_q(p_1) - g_q(p_2)|}{\operatorname{dist}(p_1, p_2)} \\
&= \limsup_{p_2 \to p_1} \frac{|g_{p_1}(q) - g_{p_2}(q)|}{\operatorname{dist}(p_1, p_2)} \quad \text{(by Theorem 2.4)} \\
&= \limsup_{p_2 \to p_1} \frac{|h_{2,p_1,p_2}(q)|}{\operatorname{dist}(p_1, p_2)} \quad \text{(by uniqueness)} \\
&\leq \limsup_{|z| \to 0} \frac{C|z|}{1-|z|} \frac{1}{\operatorname{dist}(\varphi(z), p)} \\
&\leq \limsup_{|z| \to 0} \frac{C|z|}{|z| \tanh \rho} = \frac{C}{\tanh \rho}.
\end{aligned}$$

To get the last inequality, we used the relationship between φ and π and the fact that π is a local isometry from Δ to X to compute the distance from $\varphi(z)$ to p. □

3. Green's function on annuli: a warm-up case

Green's functions for Δ and Δ^* are identical and can be written down explicitly. The simplest case in which explicit formulas are hard to present and, therefore, in which approximations are useful to have is that of the annulus. In fact, while the combinatorial details of the case $X = A_R$ are simple (see Lemma 3.1), the analytic details are nearly as involved as they are in the general case. We therefore assume for the remainder of this section that $X = A_R$. Our main goal (Theorem 3.2) is to show that Green's function $g(z, w)$ on X is well-approximated by an explicit, piecewise linear function of $|z|$.

Fix a number $\epsilon > 0$. It will be helpful (but not necessary) to assume that ϵ is not too large—e.g., $\epsilon < 1/2$ will do. Constants occurring below will sometimes depend on ϵ, but they will always be independent of X. Let $T(r)$ denote the circle of (Euclidean) radius r oriented counterclockwise about 0, and let $L(r)$ be the hyperbolic length of $T(r)$ in X. Set

$$a(r) = \frac{1}{2\pi}\int_0^{2\pi} g(re^{i\theta}, w)\, d\theta = \frac{1}{L(z)}\int_{z\in T(r)} g(z, w)\, ds.$$

We can compute $a(r)$ exactly.

LEMMA 3.1. *If $X = A_R$, then $a(r)$ is given by*

$$a(r) = \begin{cases} \dfrac{1}{2R}(R - \log|w|)(R + \log r) & \text{if } r \le |w|, \\ \dfrac{1}{2R}(R + \log|w|)(R - \log r) & \text{if } r \ge |w|. \end{cases}$$

Proof. From Theorem 2.1 we know that $a(r) = \alpha(T(e^{-R}), w)\log r + \beta_1$ for $r \le |w|$ and $a(r) = \alpha(T(e^R), w)\log r + \beta_2$ for $r \ge |w|$. Suppose first that $X = A_R$. Then we have

(1) $\alpha(T(e^R), w) = \alpha(T(e^{-R}), w) - 1$ (from the discussion following Definition 2.5);
(2) $a(e^{\pm R}) = 0$ (from the definition of Green's function);
(3) $a(r)$ is continuous across $r = |w|$ (since the logarithmic singularity of g is integrable along a smooth path);

It is easily verified that these conditions determine $\alpha(T(e^{-R}), w)$, $\alpha(T(e^R), w)$, β_1, β_2, and lead to the formula for $a(r)$ in the case $X = A_R$. □

The main result of this section is as follows:

THEOREM 3.2. *Fix $\epsilon > 0$. Then there is a constant $C = C(\epsilon)$ independent of X such that*

$$|g(z,w) - a(|z|)| \leq C$$

for all $z \in X \setminus B_w(\epsilon)$.

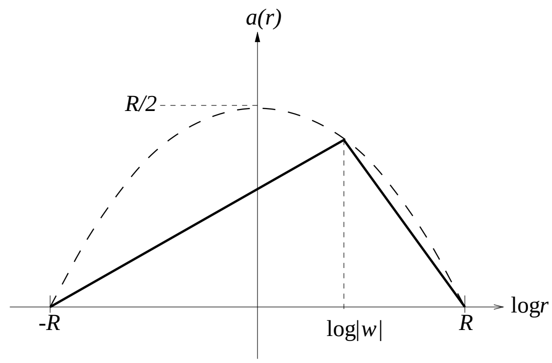

FIGURE 3.1. Circular average $a(r)$ of Green's function $g(z,w)$ on A_R

REMARK 3.3. Suppose that X is embedded in a larger hyperbolic Riemann surface Y and that g is Green's function on X. Since the inclusion map $X \hookrightarrow Y$ is distance decreasing, the conclusion of Theorem 3.2 holds with the same or a smaller constant C if we replace the assumption $\text{dist}_X(z,w) \geq \epsilon$ with $\text{dist}_Y(z,w) \geq \epsilon$.

It is convenient for the present proof and especially in the sections to follow to distinguish between the part of an annulus where the injectivity radius is large and the part where it is small. In order to remain consistent with later sections, we take a slightly indirect route to making this distinction, and we extend our attention momentarily to include the case $X = \Delta^*$ as well as $X = A_R$.

We define the *thin part* X_{thin} of X as follows. If $X = \Delta^*$, $X_{\text{thin}} = \{0 < |z| < e^{-\pi}\}$. We also refer to the thin part of Δ^* as the *cusp* $\mathcal{C}(0)$ about 0. If $X = A_R$ and $\gamma = b\Delta$ is the core geodesic of X, then the *collar* about γ is $\mathcal{C}(\gamma) = B_\gamma(w(\gamma))$, where

$$w(\gamma) = \sinh^{-1}\frac{1}{\sinh(\frac{1}{2}\,\text{Length}(\gamma))}. \tag{3–1}$$

Alternatively, from (2–3) we see that $\mathcal{C}(\gamma) = A_{R'}$ where $R' < R$ satisfies

$$\tan\frac{\pi R'}{2R} = \frac{1}{\sinh(\frac{1}{2}\,\text{Length}(\gamma))}. \tag{3–2}$$

If $\mathrm{Length}(\gamma) \geq 2\sinh^{-1} 1$, then we say that γ is *long* and declare X_{thin} to be empty. Otherwise, we say that γ is *short* and declare X_{thin} to be $\mathcal{C}(\gamma)$. In any cases, we define the *thick part* of X to be the complement of X_{thin}. The rather arbitrary looking values for the constants used in defining X_{thin} are chosen to be consonant with the Collar Lemma (see Section 4 below). The following proposition, which justifies the choice of terminology to some extent, can be proven by a straightforward, if messy, computation using the explicit form of the hyperbolic metric on X.

PROPOSITION 3.4. *If $z \in X_{\text{thick}}$, then $\mathrm{inj}(z) \geq \sinh^{-1} 1$. The area of X_{thin} is universally bounded above as are $L(|z|)$ and $\mathrm{inj}(z)$ for any $z \in X_{\text{thin}}$. Furthermore, there exist universal constants $C_1, C_2 > 0$ such that for all $z, z' \in X_{\text{thin}}$,*

(1) $C_1 L(|z|) \leq \mathrm{inj}(z) \leq L(|z|)$;
(2) $|\log \mathrm{inj}(z) - \log \mathrm{inj}(z')| \leq C_2 \,\mathrm{dist}(z, z')$.

Now we are ready to prove Theorem 3.2 in a series of lemmas.

LEMMA 3.5. *Suppose that $z \in X_{\text{thick}}$ and $d(z, w) > \epsilon$. Then*

$$a(|z|), g(z, w) < C(\epsilon).$$

Proof. From Lemma 3.1 we have that

$$a(|z|) \leq \frac{R}{2}\left(1 - \left(\frac{\log|z|}{R}\right)^2\right).$$

By the definition of X_{thick} and the fact that $\mathrm{Length}(b\Delta) = \pi^2/R$, we have either $R < R_0$ for some fixed R_0, or by (3–2)

$$\tan\frac{\pi\log|z|}{2R} \geq \frac{1}{\sinh \pi^2/R}.$$

In either case, direct estimation gives a uniform upper bound for $a(|z|)$.

To get the upper bound on $g(z, w)$, note that for any $z \in A_R$, with $|z| = r$,

$$\begin{aligned}\int_{z' \in T(r)} g(z', w)\, ds &= a(r)L(r)\\ &\leq \pi^2\left(1 - \left(\frac{\log r}{R}\right)^2\right)\sec\frac{\pi\log r}{2R} \leq \max_{-1<t<1} \pi^2(1-t^2)\sec\frac{\pi t}{2},\end{aligned}$$

which is a finite upper bound independent of w and R. By hypothesis, we have $z \in X_{\text{thick}}$, so we can assume without loss of generality that $\mathrm{inj}(z) > \epsilon$. In particular, we can choose θ_0 so that the segment $\gamma = \{ze^{i\theta} \in T(r) : |\theta| \leq \theta_0\}$ has length equal to ϵ. So we have some point $z'' \in \gamma$ such that

$$g(z'', w)\,\mathrm{Length}(\gamma) \leq \int_{z' \in \gamma} g(z', w)\, ds \leq \int_{z' \in T(z)} g(z', w)\, ds.$$

That is, $g(z'', w) \leq C/\epsilon$.

Let γ' be the subset of γ between z and z''. Note that if $\operatorname{dist}(z, w) \geq \epsilon$, then $\operatorname{dist}(z', w) > \epsilon/2$ for all $z' \in \gamma'$. We apply (3) of Theorem 2.2 to obtain

$$\begin{aligned}\log g(z, w) &= \log g(z'', w) + \int_{\gamma'} d \log g(\cdot, w)\\ &\leq C - \log \epsilon + \int_{\gamma'} \langle d \log g \rangle \, ds\\ &\leq C - \log \epsilon + \frac{C'}{\tanh(\epsilon/4)} \cdot \frac{\epsilon}{2} \leq C(\epsilon). \qquad \square\end{aligned}$$

LEMMA 3.6. *Suppose that* $z \in X_{\text{thin}}$ *and* $\operatorname{dist}(T(|z|), T(|w|)) \geq \epsilon/2$. *Then*

$$|g(z, w) - a(|z|)| \leq C(\epsilon).$$

Proof. Let $|z| = r$. By the intermediate value theorem there exists a point $z' \in T(r)$ such that $g(z', w) = a(r)$. Let γ be the shortest segment of $T(r)$ connecting z to z'. We consider two cases according to whether $L(r) \leq \epsilon$ or $L(r) > \epsilon$. If $L(r) \leq \epsilon$, then by (1) of Proposition 3.4, we have $\operatorname{inj}(z') \geq CL(r)$ for all $z' \in \gamma$. Hence, we can apply Theorem 1.2 along γ with $\rho = CL(r)$. The result is

$$\begin{aligned}|g(z, w) - a(r)| &= |g(z, w) - g(z', w)|\\ &\leq \int_{\gamma} \langle dg(\cdot, w) \rangle \, ds\\ &\leq \frac{C' \operatorname{Length}(\gamma)}{\tanh CL(r)}\\ &\leq \frac{C' L(r)}{\tanh CL(r)} \leq C,\end{aligned}$$

since $L(r) \leq \epsilon$. If $L(r) > \epsilon$, then we have $\operatorname{inj}(z') \geq CL(r) \geq C\epsilon$. Thus we can apply Theorem 1.2 along γ with $\rho = C\epsilon$. Repeating the estimation performed in the first case and using Proposition 3.4 to bound $L(r)$, we arrive at the desired conclusion. $\square$

LEMMA 3.7. *Suppose that* $z \in X_{\text{thin}}$ *and* $\operatorname{dist}(z, w) > \epsilon$, *but* $\operatorname{dist}(T(|z|)$ *and* $T(|w|)) < \epsilon/2$. *Then*

$$|g(z, w) - a(|z|)| \leq C(\epsilon).$$

Proof. Let z' be the point closest to z that satisfies $\operatorname{dist}(T(|z'|), T(|w|)) = \epsilon/2$, and let γ be the (radial) line segment from z to z'. Note that $\operatorname{Length} \gamma < \epsilon/2$. Let $w' \in T(|w|)$ be chosen as close as possible to z. Then

$$\operatorname{dist}(w', w) \geq \operatorname{dist}(z, w) - \operatorname{dist}(z, w') \geq \epsilon/2.$$

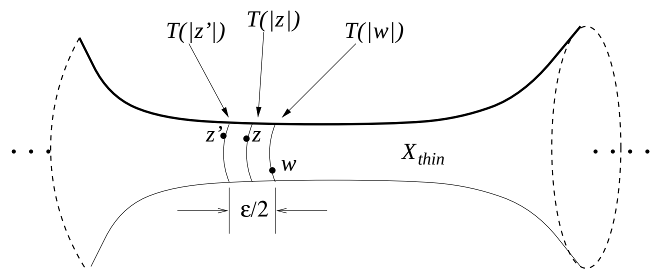

FIGURE 3.2. Setup for Lemma 3.7

In particular, by (1) of Proposition 3.4,

$$\operatorname{inj}(w') \geq C \operatorname{Length}(T(|w|)) \geq C \operatorname{dist}(w, w') \geq C\epsilon.$$

By (2) of Proposition 3.4, we also have

$$\operatorname{inj}(z'') \geq C\epsilon$$

for all $z'' \in \gamma$. Now we estimate

$$|g(z,w) - a(|z|)| \leq |g(z,w) - g(z',w)| + |g(z',w) - a(|z'|)| + |a(|z'|) - a(|z|)|.$$

The second term on the right is dominated by $C(\epsilon)$ according to Lemma 3.6. The first term on the right can be estimated using Theorem 1.2 along γ as we did in the proof of Lemma 3.6. The result is again an upper bound depending only on ϵ. To estimate the third term, we note that for $z'' \notin T(|w|)$, we have from Lemma 3.1 and equation (2–3) that

$$\langle da(|z''|) \rangle \leq \frac{C}{L(|z''|)}.$$

Since $L(|z''|) \geq 2 \operatorname{inj}(z'') \geq C\epsilon$ is bounded below for $z'' \in \gamma$, we have

$$|a(|z'|) - a(|z|)| \leq \frac{C}{\epsilon} \operatorname{Length}(\gamma) \leq C.$$

This finishes the proof. □

Proof of Theorem 3.2. Suppose that $z \in X$ satisfies $\operatorname{dist}(z,w) > \epsilon$. If $z \in X_{\text{thick}}$, then by Lemma 3.5, we have

$$|g(z,w) - a(|z|)| \leq |g(z,w)| + |a(|z|)| \leq C\epsilon.$$

If $z \in X_{\text{thin}}$, then z satisfies the hypotheses of either Lemma 3.6 or Lemma 3.7. □

4. More on hyperbolic geometry

In this section we review the statement of the well-known collar lemma for hyperbolic surfaces. We then use the collar lemma to define the thick and thin parts of a Riemann surface X and to prove the main result of the section—a sort of diameter estimate for components of the thick part of X.

In order to state the collar lemma, we need to extend the notions of cusps and collars from doubly connected domains to general Riemann surfaces X. Let $\gamma \subset X$ be a simple closed geodesic. As in Section 3, we call γ *long* if $\mathrm{Length}(\gamma) \geq 2\sinh^{-1} 1$, and *short* otherwise. We also define the *collar* about γ by $\mathcal{C}(\gamma) = B_\gamma(w(\gamma))$, where $w(\gamma)$ is given by (3–1). There is an annulus A_R and a natural holomorphic covering map $\pi_\gamma : A_R \to X$ such that π_γ injects the core geodesic $b\Delta$ onto γ. Since π_γ is a local isometry, we have $R = \pi^2/\,\mathrm{Length}(\gamma)$ and $\pi_\gamma(\mathcal{C}(b\Delta)) = \mathcal{C}(\gamma)$.

Similarly, if $p \in \overline{X}$ is a puncture, there exists a holomorphic cover $\pi_{\overline{p}} : \Delta^* \to X$ such that $\pi_{\overline{p}}$ extends conformally past 0 when we set $\pi_{\overline{p}}(0) = \overline{p}$. We define the *cusp* about $\overline{p}$ to be the image of the cusp in Δ^* about 0. That is,

$$\mathcal{C}(\overline{p}) = \pi_{\overline{p}}(\{0 < |z| < e^{-\pi}\}).$$

It is a wonderful fact of hyperbolic geometry that the covering maps corresponding to simple closed geodesics and to punctures are injections onto the respective collars and cusps.

Collar Lemma. *The following are true on any hyperbolic Riemann surface X:*

1. *given any simple closed geodesic $\gamma \subset X$, the covering map π_γ is an (isometric) injection of $\mathcal{C}(b\Delta)$ onto $\mathcal{C}(\gamma)$;*
2. *given any puncture $\overline{p} \in \overline{X}$, the covering map $\pi_{\overline{p}}$ is an (isometric) injection of $\mathcal{C}(0)$ onto $\mathcal{C}(\overline{p})$;*
3. *distinct cusps and collars about short geodesics are disjoint;*
4. *a point $p \in X$ not contained in a cusp or in the collar about a short geodesic satisfies* $\mathrm{inj}(p) \geq \sinh^{-1} 1$.

Buser's book ([Bu], Chapter 4) gives a good exposition of the collar theorem and some of its generalizations. Let R' be as in (3–2). Because of (1), we refer to the restriction of π_γ to $A_{R'} = \mathcal{C}(\{|z| = 1\})$ as *standard coordinates about* γ. Likewise, because of (2) we refer to the restriction of $\pi_{\overline{p}}$ to $\{0 < |z| < e^{-\pi}\}$ as *standard coordinates about* $\overline{p}$.

If $b\overline{X}$ is non-empty then one can add slightly to the collar theorem. Namely, given a Jordan curve $\overline{\gamma} \subset b\overline{X}$, there exists a simple closed geodesic γ homotopic to $\overline{\gamma}$ in $X \cup \overline{\gamma}$. Following [DPRS], we refer to the annulus A between γ and $\overline{\gamma}$ as a *funnel*, and we call γ a *peripheral* geodesic. There is a holomorphic covering $\pi_\gamma : A_R \to X$, where $R = \pi^2/\,\mathrm{Length}(\gamma)$, such that π_γ maps the inner half of A_R injectively onto A. Clearly A is disjoint from any cusp and from the collar about any short geodesic different from γ. As above, we refer to $\pi_\gamma|_{A_R\cap\Delta}$ as *standard coordinates on the funnel A*.

In light of the collar theorem, we single out some important subsets of X. We define the *core* X_{core} of X to be the subset of X obtained by removing all funnels. We define the *thin part* X_{thin} of X to be the union of all cusps and all collars of short geodesics, and we define the thick part X_{thick} of X to be $X \setminus X_{\text{thin}}$. Note that while $\text{inj}(p)$ is bounded below universally on X_{thick} and above universally on X_{thin}, it is not the case under our definition that $X_{\text{thick}} = \{p \in X : \text{inj}(p) \geq C\}$ for some $C > 0$. We call a connected component $\mathcal{C}$ of X_{thin} *joining* if its boundary intersects two distinct connected components of X_{thick}, at least one of which is compact in X. Note that all cusps are non-joining; so are collars of short geodesics whose boundary lies either entirely in non-compact components of X_{thick}, or entirely in a single compact connected component of X_{thick}.

Ultimately, we will identify components of X_{thick} with vertices in a graph. Hence it is helpful to know that metrically speaking, these components are not too far from being points—i.e., we would like to control the diameter of each connected component $K \subset X_{\text{thick}}$. This is a reasonable thing to try, but a little finesse is called for when K is not compact in X. Since peripheral geodesics can be arbitrarily long, one does not have a universal bound for the diameter of components of $X_{\text{thick}} \cap X_{\text{core}}$. Nonetheless, we have the following theorem.

THEOREM 4.1. *There are universal constants $C_1, C_2, C_3 > 0$ such that the following are true. Let $K \subset X_{\text{thick}}$ be any connected component, and let $p_1, p_2 \in K$ be any two points.*

1. *If K is compact in X, then there is a path γ connecting p_1 and p_2 such that* $\text{Length}(\gamma) \leq C_1|\chi(K)|$.
2. *If K is not compact in X, then either p_1 lies in a funnel, or $p_1 \in X_{\text{core}}$ and there exists a path γ connecting p_1 to a long peripheral geodesic such that* $\text{Length}(\gamma) \leq C_1|\chi(K)|$.
3. *Given any $q \in X$ the path γ in* (1) *and* (2) *can be chosen so that* $\text{dist}(\gamma, q) \geq \min\{C_2, \text{dist}(p_1, q), \text{dist}(p_2, q)\}$.
4. *The path γ in* (1)–(3) *can be chosen so that* $\text{inj}(p) > C_3$ *for every $p \in \gamma$.*

We remark that the conclusion of this theorem still holds for points in $B_K(d)$ if we allow the constants to depend on d. The proof remains the same as the one we give here. We recall the following lemma (see [Di, Lemma 4.1] for a proof).

LEMMA 4.2. *Let γ be a length minimizing geodesic in X. Let $\tilde{\gamma}$ be a subset of the positive imaginary axis isometric to γ in $\mathbb{H}$. Then there is a universal* *covering map $\pi : \mathbb{H} \to X$ mapping $\tilde{\gamma}$ to γ such that $\pi|_{\tilde{W}}$ is injective, where*

$$\tilde{W} = \bigcup_{z \in \tilde{\gamma}} B_z(s \circ \pi(z))$$

for any function $0 < s(p) \leq \operatorname{inj}(p)/3$ on γ. In particular,

$$W = \pi(\tilde{W}) = \bigcup_{p \in \gamma} B_p(s(p))$$

satisfies

$$\operatorname{Area}(W) \geq \int_\gamma 2 \sinh(s(p))\, ds.$$

Proof of Theorem 4.1. First suppose that K is compact. Let $K_1 = B_K(\sinh^{-1} 1)$ and note that by the collar theorem, $K_1 \setminus K$ lies in X_{thin}. In fact, by (3–1) K_1 cannot even intersect a short geodesic. Furthermore,

(i) $\operatorname{Length}(bK) \leq Cn$ where n is the number of components of bK,
(ii) $\operatorname{inj}(p) > C > 0$ for all $p \in K_1$,
(iii) $\operatorname{Area}(K_1) \leq C|\chi(K)|$,

where the constants in all three statements are universal. The first two statements follow from Proposition 3.4; The third statement is a consequence of the Gauss-Bonet theorem.

Now let γ be a shortest path from p_1 to p_2. Apply Lemma 4.2 to $\gamma \cap K$ with $s \equiv \sinh^{-1} 1$ and note that the resulting set W lies in K_1. Hence, direct comparison of areas reveals that $\operatorname{Length}(\gamma \cap K) \leq C \operatorname{Area}(W) \leq C|\chi(K)|$. By adding segments of bK to $\gamma \cap K$, we obtain a new path γ' from p_1 to p_2 lying entirely in K and satisfying (1). Note that if we take C_2 to be the constant C occurring in (ii) above, then $B_q(C_2)$ does not disconnect K_1 for any $q \in X$. So if $m = \min\{C_2, \operatorname{dist}(p_1, q), \operatorname{dist}(p_2, q)\}$, we can arrange for (3) to be true by replacing $\gamma' \cap B_q(m)$ with a segment of $bB_q(m)$. This adds at most a fixed constant to the length of the path γ', and since the resulting path γ remains in K_1, we are done with the compact case.

If K is not compact in X, then $K \setminus X_{\text{core}}$ consists of funnels bounded by long peripheral geodesics. Let $K_0 = K \cap X_{\text{core}}$ and $K_1 = B_K(\sinh^{-1} 1) \cap X_{\text{core}}$. Then $bK_0 \cap \operatorname{int} X_{\text{core}}$ satisfies (i) above, and K_1 satisfies (ii) and (iii). Suppose that $p_1 \in K_0$ and let γ be a shortest path from p_1 to the nearest point p_2 on a long peripheral geodesic. Let $\gamma_0 = \gamma \cap K_0$ and $\gamma_1 = \gamma_0 \setminus \{p \in \gamma_0 : \operatorname{dist}(p, bX_{\text{core}}) > \sinh^{-1} 1\}$. As above, we apply Lemma 4.2 to γ_1 with $s(p) \equiv \sinh^{-1} 1$, and we obtain a bound $\operatorname{Length}(\gamma_1) \leq C|\chi(K_1)|$. Hence,

$$\operatorname{Length}(\gamma_0) \leq 1 + \operatorname{Length}(\gamma_1) \leq 1 + C|\chi(K_1)| \leq C|\chi(K)|.$$

Now by adding segments of $bK_0 \cap \operatorname{int} X_{\text{core}}$ to γ_0, we obtain a path satisfying (2). Items (3) and (4) are obtained as in the compact case. $\square$

5. Circuit diagrams and approximate Green's functions

In this section we will introduce the notions of a circuit diagram and an approximate Green's function for a surface X. The main goal of the section will be to show that the approximate Green's function does indeed provide a good approximation of Green's function.

First we describe what we mean by an electric network. Let Γ be a finite, non-empty, directed graph with edge set $\mathbf{E}$ and vertex set $\mathbf{V}$. We allow distinct edges to join the same pair of vertices (i.e., Γ is really a *multigraph*), but we require that Γ be connected and that each edge be incident to two distinct vertices. Let δ denote the oriented incidence matrix for Γ. That is, rows of δ are indexed by vertices and columns by edges; and $\delta_{v,e} = 0$ if v and e are not incident, $\delta_{v,e} = 1$ if v and e are incident with e directed toward v, and $\delta_{v,e} = -1$ otherwise. For our purposes, Γ becomes an *electric network* when we make a choice of a distinguished vertex $v_0 \in \mathbf{V}$ (the *ground* vertex) and assign a positive real number $\mathcal{R}(e)$ (the *resistance*) to each edge $e \in \mathbf{E}$. While it is not our goal to study actual electric circuits, it seems good to us to use the terminology of circuits as much as possible for intuition's sake.

It will be convenient for us to identify the edge e with the open line segment of length $\mathcal{R}(e)$; hence we will refer to a point in e as (t, e) where $-(1/2)\mathcal{R}(e) < t < (1/2)\mathcal{R}(e)$. Identifying vertices with points and edges with line segments, we can view Γ as a compact connected metric space by declaring

$$\lim_{t \to (1/2)\mathcal{R}(e)} (t, e) = v^+(e),$$

where $v^+(e)$ is the unique vertex such that $\delta_{v^+(e),e} = 1$, and similarly

$$\lim_{t \to -(1/2)\mathcal{R}(e)} = v^-(e),$$

where $\delta_{v^-(e),e} = -1$.

The following idea was used (albeit without the name) in [DPRS].

Definition 5.1. The *circuit diagram* for X is an electric network Γ together with a continuous, surjective map $\Pi : X \to \Gamma$. Specifically,

(1) Γ contains a vertex v_K for each compact, connected component $K \subset X_{\text{thick}}$, as well as a ground vertex $v_0 = v_K$ for K equal to the union of all non-compact connected components of X_{thick};
(2) Γ contains an edge e_γ for each joining component $\mathcal{C}(\gamma) \subset X_{\text{thin}}$, $\mathcal{R}(\gamma) \stackrel{\text{def}}{=} \mathcal{R}(e_\gamma) = 2R'$ where $A_{R'}$ is the annulus conformally equivalent to $\mathcal{C}(\gamma)$;
(3) $\delta_{K,\gamma} \stackrel{\text{def}}{=} \delta_{v_K, e_\gamma}$ is non-zero if and only if $K \cap b\mathcal{C}(\gamma) \neq \emptyset$.

For each joining component $\mathcal{C}(\gamma) \subset X_{\text{thin}}$ choose standard coordinates $A_{R'}$ on $\mathcal{C}(\gamma)$ so that the inner boundary component of $A_{R'}$ lies in the component $K \subset X_{\text{thick}}$ such that $\delta_{K,\gamma} = -1$. Then Π is given by

(4) $\Pi(p) = v_K$ if $p \in K$, or if p lies in a non-joining component of X_{thin} whose boundary lies in K;
(5) $\Pi(z) = (\log|z|, e_\gamma)$ if $z \in \mathcal{C}(\gamma) \cong A_{R'}$.

Observe that the circuit diagram for X is unique up to the choice of edge orientation. We will assume implicitly throughout the rest of this paper that standard coordinates on joining collars are chosen to be consistent with the edge orientations (i.e., increasing $|z|$ in standard coordinates corresponds to moving in the positive direction along the edge).

Definition 5.2. Let Γ be a circuit diagram for X. Given $q \in X$, we define the *approximate Green's function* $g_{\text{approx}}(\cdot, q)$ on Γ as follows:
(1) given a vertex v_K, we set $g_{\text{approx}}(v_K, q) = \min_{p\in K} g(p,q)$;
(2) given a point (t, e_γ) in an edge e_γ, we set

$$\begin{aligned} g_{\text{approx}}((t,e_\gamma),q) &= \frac{1}{2\pi}\int_0^{2\pi} g(e^{t+i\theta},q)\,d\theta \\ &= \frac{1}{\operatorname{Length}\Pi^{-1}(t,e_\gamma)}\int_{\Pi^{-1}(t,e_\gamma)} g(p,q)\,ds(p), \end{aligned}$$

where the middle integral is performed with respect to standard coordinates on $\mathcal{C}(\gamma)$.

Remark 5.3. According to the definitions we have given, the circuit diagram for a disk, a punctured disk, or an annulus consist of a single (ground) vertex. The thin parts of these surfaces contain at most one non-joining connected component and do not contribute edges to the diagram. Of course, approximate Green's function will be trivial automatically. Especially when X is an annulus, it is probably better to regard the function $a(|z|)$, defined in Section 3, as approximate Green's function.

It is easy to see that g_{approx} is continuous (in fact, from Theorem 2.1 g_{approx} is piecewise affine) along any edge of Γ. Only in trivial situations (i.e., when Γ consists of a single vertex and no edges) will it turn out that g_{approx} is continuous at vertices. However, we have the following result, which we will prove with the next sequence of lemmas.

Theorem 5.4. *The discontinuity of g_{approx} at a vertex v_K is bounded. More specifically, let $p_j \in X$ be a sequence of points whose distance from K tends to 0 as $j \to \infty$. Then*

$$\limsup_{j\to\infty} |g_{\text{approx}}(\Pi(p_j),q) - g_{\text{approx}}(v_K,q)| \le C|\chi(K)|,$$

where C is a universal constant. If $v_K = v_0$ is the ground vertex, then the right hand side can be improved to $C \max|\chi(K')|$ where the maximum is taken over connected components K' of K.

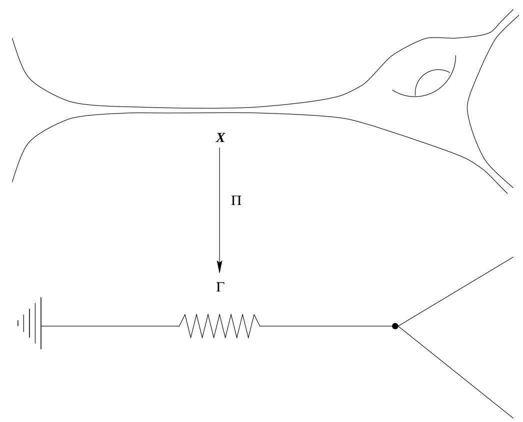

FIGURE 5.1. Projection of a hyperbolic surface onto an electric network

LEMMA 5.5. *Let K be a compact connected component of X_{thick}. Then given $\epsilon > 0$, there exists a constant $C = C(\epsilon)$ such that for any $p_1, p_2 \in K \setminus B_q(\epsilon)$,*

$$|g(p_1, q) - g(p_2, q)| \leq C|\chi(K)|.$$

The estimate continues to hold if we allow p_1 and/or p_2 to lie in $\mathcal{C} \setminus B_q(\epsilon)$, where $\mathcal{C}$ is any non-joining component of X_{thin} such that $b\mathcal{C} \subset K$ and $q \notin \mathcal{C}$.

Proof. Suppose first that $p_1, p_2 \in K$. Choose a path γ from p_1 to p_2 satisfying the conclusions of Theorem 4.1. By Theorem 1.2 and (3) and (4) of Theorem 4.1, we have C such that

$$\langle dg(\cdot, q)\rangle \leq C(\epsilon)$$

at all points in γ. Hence by (1) of Theorem 4.1,

$$|g(p_1, q) - g(p_2, q)| \leq C\,\mathrm{Length}(\gamma) \leq C|\chi(K)|.$$

We remark that if $d > 0$ is a fixed constant, then this estimate continues to hold with a larger constant $C' = C(\epsilon, d)$ if we only assume $p_j \in B_K(d)$. Indeed, as we noted earlier, the conclusion of Theorem 4.1 applies to points in $B_K(d)$ as long as we allow constants to depend on d. Hence, the argument we have given here for K still applies to $B_K(d)$.

Now consider a non-joining component $\mathcal{C}$ such that $b\mathcal{C} \subset K$ and $q \notin \mathcal{C}$. Let $\mathcal{C}(\epsilon) = \{p \in \mathcal{C} : \mathrm{dist}(p, K) > \epsilon\}$. Then $b\mathcal{C}(\epsilon) \subset \overline{B_K(\epsilon)}$ and $b\mathcal{C}(\epsilon) \cap B_q(\epsilon) = \emptyset$. In case $\mathcal{C}$ is a cusp recall that $g(\cdot, q)$ extends harmonically past the puncture in $\mathcal{C}$. Hence we can apply the maximum principle to obtain

$$|\max_{\mathcal{C}(\epsilon)} g - \min_{\mathcal{C}(\epsilon)} g| = |\sup_{B_K(\epsilon)\setminus B_q(\epsilon)} g - \inf_{B_K(\epsilon)\setminus B_q(\epsilon)} g| \leq C|\chi(K)|.$$

Since $\mathcal{C} \subset \mathcal{C}(\epsilon) \cup B_K(\epsilon)$, we are done. □

Lemma 5.6. *Let $\gamma \subset X$ be a simple closed curve homologous to a short and/or peripheral geodesic, and let $q \in X$ be a point not lying on γ. Then the flux α across γ satisfies*

$$|\alpha(\gamma, q)| \leq C$$

for some universal constant C and any point $q \in X$.

Our proof of this lemma does not yield a very explicit value for the constant C. However, we wonder whether an alternative argument might reveal that $C = \alpha(\gamma_q, q) = 1$ works, where γ_q is a small circle about q.

Proof. The hypotheses on γ imply that γ is homologous in $\overline{X}$ to a simple closed curve γ' that is either a component of $b\overline{X}$ or a component of $b\mathcal{C}$ for the collar $\mathcal{C}$ of some short geodesic. In the latter case, since $b\mathcal{C}$ consists of two components, we can assume without loss of generality that $\operatorname{dist}(\gamma', q) > \epsilon_0$ for some fixed constant $\epsilon_0 > 0$. By the discussion following Definition 2.5, $\alpha(\gamma, q)$ and $\alpha(\gamma', q)$ differ by at most 1, so it is enough to prove the lemma for γ'.

Suppose first that $\gamma' \subset b\overline{X}$. We observed after Definition 2.5 that $\alpha(b\overline{X}, q) = 1$. Since $g(\cdot, q)$ is zero on $b\overline{X}$ and positive elsewhere, we see that in fact $1 > \alpha(\gamma'', q) > 0$ for every outward oriented component γ'' of $b\overline{X}$. In particular, $|\alpha(\gamma', q)| < 1$.

Now suppose that γ' is a component of $b\mathcal{C}$, and that γ' avoids q by at least ϵ_0. From Proposition 3.4, we have that $\operatorname{Length}(\gamma')$ is bounded above. Since $\gamma' \subset X_{\text{thick}}$, we see that $\operatorname{inj}(p)$ is bounded below at all $p \in \gamma'$. Thus

$$\begin{aligned}
|\alpha(\gamma', q)| &= \frac{1}{2\pi} \left| \int_{\gamma'} *dg(\cdot, q) \right| \\
&\leq \frac{1}{2\pi} \int_{\gamma'} \langle dg \rangle \, ds \\
&\leq C \operatorname{Length} \gamma' \qquad \text{(by Theorem 1.2)} \\
&\leq C.
\end{aligned}$$

□

Lemma 5.7. *Let $A \subset X$ be a funnel. Then given $\epsilon > 0$, there exists a constant $C(\epsilon)$ such that for all $p \in (A \cap X_{\text{thick}}) \setminus B_q(\epsilon)$,*

$$g(p, q) \leq C.$$

Proof. We work in standard coordinates on $A \cong A_R \cap \Delta$. Let $a(r)$ be as in Section 4 for $e^{-R} < r < 1$. We know $g(re^{i\theta}, q)$ is zero if $r = e^{-R}$. Hence, by Lemma 5.6

$$a(r) \leq C(\log r + R)$$

for some constant C. From here the proof is similar to that of Lemma 3.5. □

LEMMA 5.8. *Let K be a connected component of X_{thick} that is not compact in X. Then there is a constant $C(\epsilon)$ such that for all $p \in K \setminus B_q(\epsilon)$*

$$|g(p,q)| \leq C|\chi|,$$

where $|\chi|$ is the maximum value of $|\chi(K')|$ over all non-compact connected components $K' \subset X_{\text{thick}}$. The estimate continues to hold if $p \in \mathcal{C}$, where $\mathcal{C}$ is a non-joining component of X_{thin} such that $b\mathcal{C} \cap K \neq \emptyset$ and $q \notin \mathcal{C}$.

We could replace χ with $\chi(K)$ in the conclusion of this lemma, except that a non-joining component $\mathcal{C}$ of X_{thin} might have boundary components lying in two distinct unbounded components of X_{thick}.

Proof. If $p \in K \setminus X_{\text{core}}$, then this lemma follows from the previous one. If $p \in K \cap X_{\text{core}}$, let γ be the path from p to some point p' on a long peripheral geodesic guaranteed by Theorem 4.1. Then by Theorem 1.2

$$\langle dg(\cdot, q)\rangle < C$$

for some constant $C(\epsilon)$ and all $q \in \gamma$. We apply this estimate and the previous lemma to conclude

$$g(p,q) = g(p',q) + \int_\gamma dg(p,q) \leq C + C' \operatorname{Length}(\gamma) \leq C|\chi(K)| \leq C|\chi|.$$

This estimate carries over to points in non-joining components of X_{thin} incident to K just as it did in Lemma 5.5. □

Proof of Theorem 5.4. Choose a vertex v_K of Γ, and let e_γ be an edge incident to v_K. Suppose for the sake of convenience that e_γ is oriented away from v_K. Then we have by definition that $\lim\limits_{t \to -(1/2)\mathcal{R}(\gamma)} g_{\text{approx}}((e_\gamma, t), q)$ is the average of $g(p,q)$ over $b\mathcal{C}(\gamma) \cap K$. In particular,

$$g_{\text{approx}}(v_K, q) = \min_K g(\cdot, q) \leq \lim_{t \to -(1/2)\mathcal{R}(\gamma)} g_{\text{approx}}((e_\gamma, t), q) \leq \max_K g(\cdot, q).$$

By Lemmas 5.5 and 5.8, the difference between the left and right hand quantities is at most $C|\chi(K)|$. □

Besides establishing Theorem 5.4, Lemmas 5.5 through 5.8 also suffice to prove the next theorem when $p \in X$ is a vertex.

THEOREM 5.9. *Given $\epsilon > 0$, there exists a constant C depending only on ϵ such that*

$$|g_{\text{approx}}(\Pi(p), q) - g(p,q)| \leq C|\chi|,$$

where χ maximizes $\chi(K)$ over connected components $K \subset X_{\text{thick}}$, and $p, q \in X$ are points such that $\operatorname{dist}(p,q) \geq \epsilon$ and there exists no non-joining component of X_{thin} containing both p and q.

Remark 5.10. If $p, q \in \mathcal{C}$ for some non-joining component $\mathcal{C}$ of X_{thin}, then Theorem 5.9 is still true in the sense that

$$|g_{\text{approx}}(\Pi(p), q) - g(p, q) + g_{\mathcal{C}}(p, q)| \leq C|\chi|,$$

where $g_{\mathcal{C}}$ is Green's function for $\mathcal{C}$. Since $\mathcal{C}$ is isometric to either a punctured disk or a round annulus, we can either write down $g_{\mathcal{C}}$ explicitly in standard coordinates or approximate $g_{\mathcal{C}}$ using Theorem 3.2.

The proof of Theorem 5.9 is completed for general surfaces X by the following lemma. The proof of the lemma also suffices to justify the assertion made in Remark 5.10.

Lemma 5.11. *Let $\mathcal{C} = \mathcal{C}(\gamma) \cong A_{R'}$ be a joining component of X_{thin}. In standard coordinates on $\mathcal{C}$ set*

$$a(z) = \int_0^{2\pi} g(ze^{i\theta}, q)\, d\theta.$$

Then $a(r)$ is a continuous, piecewise affine function of $\log r$, and there exists a constant $C(\epsilon)$ such that

$$|g(z, q) - a(z)| < C$$

for all $z \in \mathcal{C} \setminus B_q(\epsilon)$.

Proof. That $a(z)$ is a continuous piecewise affine function of $\log |z|$ follows from Theorem 2.1; we specify "piecewise" to allow for $q \in \mathcal{C}$.

We divide the proof of the estimate on $|g(z, q) - a(z)|$ into three cases. The proof is more or less the same in each, but the latter cases present more technical difficulties. Fix a small number $\epsilon_0 > 0$. Then we consider in order the cases

(1) $\operatorname{dist}(q, \mathcal{C}) \geq \epsilon_0$;
(2) $q \in \mathcal{C}$, but $\operatorname{dist}(q, b\mathcal{C}) \geq \epsilon_0$;
(3) $\operatorname{dist}(q, b\mathcal{C}) < \epsilon_0$.

In cases (1) and (2), Theorem 1.2 gives us a bound $C_1 \geq \langle dg(\cdot, q) \rangle$ effective at all points in $b\mathcal{C}$. Since we also have $\operatorname{Length}(b\mathcal{C}) \leq C_2$, we conclude that

$$|g(z, q) - a(z)| < C = C(\epsilon_0)$$

for all $z \in b\mathcal{C}$. In case (1), $g(z, q) - a(z)$ is harmonic on all of $\mathcal{C}$, so the estimate extends to $\mathcal{C}$ by the maximum principle.

In case (2), let $g_{\mathcal{C}}(z, q)$ be Green's function on $\mathcal{C}$ with pole at q. Let $a_{\mathcal{C}}(z)$ be the circular averages of $g_{\mathcal{C}}$. Then $h(z) = g(z, q) - g_{\mathcal{C}}(z, q) - a(z) + a_{\mathcal{C}}(z)$ is harmonic on all of $\mathcal{C}$, and h equals $g(z, q) - a(z)$ on the boundary. Hence, by the maximum principle again

$$|g(z, q) - a(z)| \leq C(\epsilon_0) + |g_{\mathcal{C}}(z, q) - a_{\mathcal{C}}(z)|$$

for all $z \in \mathcal{C}$. Now the ball $B_q(\epsilon)$ defined with respect to the hyperbolic metric on X contains the corresponding ball defined with respect to the hyperbolic metric on the annulus $A_{R'} \cong \mathcal{C}$. Therefore, we invoke Theorem 3.2 and Remark 3.3 to conclude

$$|g_{\mathcal{C}}(z,q) - a_{\mathcal{C}}(z)| < C(\epsilon)$$

for all $z \in \mathcal{C} \setminus B_q(\epsilon)$, where the ball is defined with respect to the hyperbolic metric on X. This proves Lemma 5.11 in case (2).

To prove case (3), let $A_{R''} = \{z \in A_{R'} : \mathrm{dist}(z, bA_{R'}) > 2\epsilon_0\}$. Then, as we remarked in the proof of Lemma 5.5, we can obtain the desired estimate at points in $A_{R''}$ just as we did in case (1). Furthermore, by the same remarks, we conclude that

$$|g(z_1,q) - g(z_2,q)| \le C(\epsilon_0,\epsilon)$$

for all $z_1, z_2 \in A_{R'} \setminus (A_{R''} \cup B_q(\epsilon))$. One can easily verify that $\log R' - \log R'' \le C(\epsilon_0)$. Hence, Lemma 5.6 gives us that

$$|a(z_1) - a(z_2)| \le \left|\log \frac{|z_1|}{|z_2|}\right| \max_{R''<r<R'} \alpha(\{|z| = r\}, q) \le C(\epsilon_0)$$

for all $z_1, z_2 \in A_{R'} \setminus A_{R''}$. Since $g(z,q)$ and $a(z)$ agree for some $z \in A_{R'} \setminus (A_{R''} \cup B_q(\epsilon_0))$, we have in fact that

$$|g(z_1,q) - a(z_2)| < C(\epsilon,\epsilon_0)$$

for all $z_1, z_2 \in A_{R'} \setminus (A_{R''} \cup B_q(\epsilon_0))$. For points in $A_{R''} \setminus (A_{R''} \cup B_q(\epsilon_0))$, this is at least as strong as the estimate we are seeking. □

6. Discrete Green's function

Now that we have related Green's function on X to the approximate Green's function on the circuit diagram Γ of X, we proceed in this section to show that one can approximately compute approximate Green's functions using only linear algebra. That is, we define the discrete Green's function on Γ as the solution of a linear algebra problem, and our main goal will be to bound the difference between discrete and approximate Green's functions. The crucial role will be played by the fact (to be proven) that bounded potential drop functions on an electric network induce bounded potentials on the network.

For the time being, let Γ be any (finite) electric network. It is not important until the end of this section that Γ arises as the circuit diagram of a Riemann surface. Let $\mathbf{V}, \mathbf{E}$ denote the vertices and edges of Γ, respectively. Let v_0 denote the ground vertex. Let $\delta : \mathbb{R}^{\mathbf{E}} \to \mathbb{R}^{\mathbf{V}}$ denote the incidence matrix and $\mathcal{R} : \mathbf{E} \to \mathbb{R}$ the resistance (weight) function of Γ. Suppose we are given a "potential" function $U : \mathbf{V} \to \mathbb{R}$. We extend the definition of U to points

(t, e) on edges e by interpolating linearly between vertices. We call the slope $I(e)$ of U along the edge e the *current* through e. That is,

$$I(e) = \frac{U(v^+(e)) - U(v^-(e))}{\mathcal{R}(e)} = \frac{1}{\mathcal{R}(e)} \sum_v \delta_{v,e}\, U(v),$$

We then define the (relative) charge $Q(v)$ at the vertex v by adding up the currents (with orientation) through edges adjacent to v. That is,

$$Q(v) = \sum_e \delta_{v,e} I(e).$$

If we consider U and Q as column vectors indexed by vertices, I as a column vector indexed by edges, and $\mathcal{R}$ as a diagonal square matrix with rows and columns indexed by edges, then we can restate these equations more conveniently:

$$\delta^T U = \mathcal{R} I, \tag{6–1}$$

$$\delta I = Q. \tag{6–2}$$

The reader will note that the first equation is simply Ohm's law, and the second equation is one of Kirchoff's rules for electric circuits. We can combine the equations to a single equation relating charge and potential directly,

$$\mathcal{L} U = Q,$$

where $\mathcal{L} = \delta \mathcal{R}^{-1} \delta^T$ is the *discrete Laplacian* operator for Γ. It is an interesting fact, though not necessary for our purposes, that the definition of $\mathcal{L}$ does not depend on the orientations of the edges of Γ. Facts which are more germane to the present paper are summarized by the following result:

Proposition 6.1. *The kernel of $\mathcal{L}$ is the set of potential functions which are constant. The range of $\mathcal{L}$ consists of those charge functions Q satisfying*

$$\sum_{v \in \mathbf{V}} Q(v) = 0. \tag{6–3}$$

In particular, there is a unique linear operator $\mathcal{L}^{-1}$ acting on the subspace of charge functions determined by (6–3) such that $\mathcal{L}\mathcal{L}^{-1}, Q = Q$, and $[\mathcal{L}^{-1} Q](v_0) = 0$.

Proof. It is clear that $\mathcal{L}U = 0$ if U is constant. If U is not constant, then there are adjacent vertices v_1 and v_2 such that $U(v_1)$ is maximal and $U(v_2) < U(v_1)$. Hence, the sum of all currents flowing into v_1 is strictly positive—that is, $\mathcal{L}U(v_1) > 0$. (Note that by this observation, a "subharmonic" potential U would be one for which $\mathcal{L}U \leq 0$; though $\mathcal{L}$ is convenient for our purposes, convention would argue that $-\mathcal{L}$ is a better candidate for the discrete Laplacian.) We conclude that the kernel of $\mathcal{L}$ is the set of constant functions.

Each column of δ contains two non-zero entries, one of which is 1, the other -1. In particular each column of δ adds up to zero. So by (6–2) any Q in the range of $\mathcal{L}$ satisfies (6–3). On the other hand, the dimensions of the kernel and the range of $\mathcal{L}$ must add up to the number of vertices, so (6–3) must also be sufficient for Q to lie in the range of $\mathcal{L}$.

The set of potential functions U satisfying $U(v_0) = 0$ is a codimension one linear subspace whose intersection with the kernel of $\mathcal{L}$ is trivial. Hence $\mathcal{L}$ maps this set linearly and bijectively onto the set of charge functions satisfying (6–3). We let $\mathcal{L}^{-1}$ be the inverse of the bijection. □

Given any point q in an electric network Γ, we define the *augmented* network Γ_q to be such that

(1) if q is a vertex then $\Gamma = \Gamma_q$;
(2) if $q = (t, e)$ is a point on an edge, we obtain Γ_q from Γ by declaring q to be a vertex and dividing e into two edges e_1, e_2 of resistances $\mathcal{R}(e_1) = t + (1/2)\mathcal{R}(e)$, $\mathcal{R}(e_2) = (1/2)\mathcal{R}(e) - t$ (with the obvious incidence relations).

Note that, while Γ and Γ_q can be different as graphs, there is a natural isometry $\Gamma_q \to \Gamma$ between the associated metric spaces. Let $\mathcal{L}_q$ denote the Laplacian on Γ_q and let Q_q be the vertex function such that $Q_q(q) = 1$, $Q_q(v_0) = -1$, and $Q_q(v) = 0$ for all other vertices (in the case $q = v_0$, we take Q_q to be identically zero). Then the *discrete Green's function* $g_{\text{disc}}(\cdot, q) = \mathcal{L}_q^{-1}(Q_q)$ *with unit charge at* q is the unique solution of $\mathcal{L}_q g_{\text{disc}}(\cdot, q) = Q_q$ satisfying $g_{\text{disc}}(v_0, q) = 0$. As with other potentials, g_{disc} extends linearly to points on the edges of Γ_q. It is this linear extension to which we refer in the statements of Theorem 1.1 and of the main result of this section:

THEOREM 6.2. *There exists a constant C depending only on $\chi(X)$ such that*

$$|g_{\text{approx}}(p, q) - g_{\text{disc}}(p, \Pi(q))| < C$$

for all $p \in \Gamma$, $q \in X$.

Technically speaking, g_{approx} is defined on Γ, whereas $g_{\text{disc}}(\cdot, \Pi(q))$ is defined on the augmented graph $\Gamma_{\Pi(q)}$. However, as we pointed out earlier, the two graphs are canonically equivalent as metric spaces, so there is no ambiguity in the statement of this theorem. Note that Theorem 1.1 follows immediately from this theorem and Theorem 5.9.

In order to prove Theorem 6.2, we need an important preliminary result. Consider an edge function $D : \mathbf{E} \to \mathbb{R}$, which we imagine to be the drop in potential between adjacent vertices. Clearly it is not always possible to find a potential $U : \mathbf{V} \to \mathbb{R}$ which "realizes" D—i.e., which satisfies

$$D(e) = U(v^+(e)) - U(v^-(e)) \tag{6–4}$$

for each edge e. However, even in the absence of U, we can use (6–4) formally in (6–1) and apply (6–2) to obtain a charge function $Q = \delta\mathcal{R}^{-1}D$ associated with D. As we noted in the proof of Proposition 6.1, the entries of each column of δ sum to zero. Hence, by Proposition 6.1, Q lies in the range of $\mathcal{L}$ and we can associate (in a looser sense) a potential $\mathcal{L}^{-1}Q$ with D after all. The next result states that bounded drop functions induce bounded potential functions and that the bound is insensitive to the resistances assigned to various edges. We use the usual Euclidean norm $||\cdot||$ to measure the lengths of vectors in $\mathbb{R}^{\mathbf{E}}$ and $\mathbb{R}^{\mathbf{V}}$.

Theorem 6.3. *There is a constant C depending only on the number of edges in Γ such that*

$$||\mathcal{L}^{-1}(\delta\mathcal{R}^{-1}D)|| \leq C||D||,$$

for all drop functions $D : \mathbf{E} \to \mathbb{R}$.

The proof of this theorem rests entirely on linear algebra, and we abandon the analogy with electrical circuits for the moment to present some linear algebraic lemmas which we will need in the proof. We let $\angle(V, W) = \cos^{-1}\frac{V\cdot W}{||V||\,||W||}$ denote the angle between two vectors in $\mathbb{R}^n$.

Lemma 6.4. *Given vectors $V, W \in \mathbb{R}^n$,*

$$||V||, ||W|| \leq \frac{||V - W||}{\sin\angle(V, W)}.$$

Proof. Apply the law of sines to the triangle with vertices $0, V, W$. □

Definition 6.5. Let $V \in \mathbb{R}^n$ be a non-zero vector. The *quadrant* of V is the set

$$\operatorname{quad}(V) = \{V' \in \mathbb{R}^n : V'_j = t_j V_j \text{ for some } t_j > 0, 1 \leq j \leq n\}.$$

The *grand quadrant* of V is the cone

$$\operatorname{grand}(V) = \{V' \in \mathbb{R}^n : V_j V'_j \geq 0, 1 \leq j \leq n\}.$$

Given a linear subspace $L \subset \mathbb{R}^n$, we set

$$\operatorname{grand}(L) = \bigcup_{V \in L\setminus\{0\}} \operatorname{grand}(V).$$

This definition more or less directly yields the following result:

Proposition 6.6. *Any non-zero vector W lies in the interior of* grand(W). *If V satisfies $V \cdot W = 0$, then* quad(V) *does not intersect the interior of* grand(W).

Corollary 6.7. *Given a linear subspace $L \subset \mathbb{R}^n$ and a non-zero vector V such that $V \cdot W = 0$ for all $W \in L$, there exists a constant $C > 0$ depending only on L such that*

$$\angle(W, V') \geq C$$

for all non-zero $W \in L$ and $V' \in \operatorname{quad}(V)$.

Proof. Since $\operatorname{grand}(L)$ is a cone containing L and each non-zero vector $W \in L$ lies in the interior of $\operatorname{grand}(L)$, we have

$$C(W) = \min_{W' \notin \operatorname{grand}(L)} \angle(W, W') > 0.$$

Clearly, C varies continuously with W and does not change when W is multiplied by a positive number. Since the intersection of L with the unit sphere is compact, we have

$$C(L) = \min_{\substack{W' \notin \operatorname{grand}(L) \\ W \in L \setminus \{0\}}} \angle(W', W) > 0.$$

By hypothesis and the previous lemma, we have in particular that

$$\angle(W, V') \geq C(L)$$

for all $W \in L$ and all $V' \in \operatorname{quad}(V)$. □

Proof of Theorem 6.3. Note that the number of vertices of Γ cannot exceed the number of edges by more than one. Hence, ignoring the resistance matrix $\mathcal{R}^{-1}$, there are only finitely many combinatorial possibilities for an electric network with at most n edges. By this observation, we need only show that a constant C exists for each particular Γ, such that C satisfies the conclusion of the Theorem and is independent of $\mathcal{R}^{-1}$.

Let $A = \delta^T \mathcal{L}^{-1}(\delta \mathcal{R}^{-1} D)$. By definition of $\mathcal{L}$ we have

$$\delta \mathcal{R}^{-1}(A - D) = 0.$$

That is, $D = A - B$ where $A \in \operatorname{ran} \delta^T$ and $B \in \ker \delta \mathcal{R}^{-1}$. Since all diagonal entries of $\mathcal{R}$ are positive, we have in fact that $B \in \operatorname{quad}(V)$ where $V = \mathcal{R}^{-1} B \in \ker \delta$. But $\ker \delta$—in particular, V—is orthogonal to $\operatorname{ran} \delta^T$. So by Corollary 6.7, we see that

$$\angle(A, B) \geq C > 0$$

for some C depending only on $\operatorname{ran} \delta^T$. In particular, C is independent of $\mathcal{R}$. We apply Lemma 6.4 to conclude that

$$||\delta^T A'|| = ||A|| \leq C||D||,$$

where $A' = \mathcal{L}^{-1}(\delta \mathcal{R}^{-1} D)$. All that remains is to show

$$||A'|| \leq C||\delta^T A'|| \tag{6–5}$$

for some other constant C depending only on the number n of edges. Recall that by the construction of $\mathcal{L}^{-1}$, $A'_{v_0} = 0$. Also, if e is any edge of Γ, then

$A'_{v^+(e)} - A'_{v^-(e)} = (\delta^T A')_e$. Since Γ is connected, any vertex v in Γ is joined to v_0 by a chain of no more than n edges. So proceeding by induction, we have $|A'_v| = |A'_v - A'_{v_0}| \leq n||A'||$. (6–5) follows immediately, and we are done. □

Now we assume that Γ is the circuit diagram for the surface X. We can apply Theorem 6.3 to prove this section's main result and thereby conclude the proof of Theorem 1.1.

Proof of Theorem 6.2. It is easy to see that $g_{\text{approx}}(\cdot, q)$ is linear along any edge in the augmented graph Γ_q. Indeed, for $(t, e) \in \Gamma_q$,

$$\begin{aligned} I_{\text{approx}}(e) &\stackrel{\text{def}}{=} \frac{dg_{\text{approx}}((t,e),q)}{dt} = \alpha(\Pi^{-1}(t,e),q) \\ &= \lim_{t\to\mathcal{R}(e)} \frac{g_{\text{approx}}((\frac{1}{2}t,e),q) - g_{\text{approx}}((-\frac{1}{2}t,e),q)}{\mathcal{R}(e)}, \end{aligned}$$

where $\Pi^{-1}(t, e)$ is oriented counterclockwise with respect to standard coordinates. From the observations following Definition 2.5, we have

$$\sum_e \delta^q_{v,e} I_{\text{approx}}(e) = \begin{cases} 1 & \text{if } v = \Pi(q) \neq v_0, \\ -1 & \text{if } v = v_0 \neq \Pi(q), \\ 0 & \text{otherwise .} \end{cases}$$

We also have

$$\begin{aligned} \sum_v \delta^q_{v,e} g_{\text{approx}}(v,q) &= g_{\text{approx}}(v^+(e),q) - g_{\text{approx}}(v^-(e),q) \\ &= D(e) + \lim_{t\to\mathcal{R}(e)} g_{\text{approx}}((\frac{1}{2}t,e),q) - g_{\text{approx}}((-\frac{1}{2}t,e),q) \\ &= \mathcal{R}_q(e) I_{\text{approx}}(e) + D(e), \end{aligned}$$

where $D(e)$ accounts for the discontinuity of g_{approx} at the vertices incident to e. By Theorem 6.3, $|D(e)| \leq C \max_{K\subset X_{\text{thick}}} |\chi(K)|$. Combining these observations, we arrive at

$$\delta^q \mathcal{R}_q^{-1} (\delta^q)^T g_{\text{approx}}(\cdot, q) = Q_q + \delta^q \mathcal{R}_q^{-1} D.$$

In particular, the difference between the approximate and discrete Green's functions satisfies

$$g_{\text{approx}}(\cdot, q) - g_{\text{disc}}(\cdot, \Pi(q)) = \mathcal{L}_q^{-1}(\delta^q \mathcal{R}_q^{-1} D)$$

at each vertex of Γ. We apply Theorem 5.4 to conclude that

$$|g_{\text{approx}}(v, q) - g_{\text{disc}}(v, \Pi(q))| \leq C||D||,$$

where C depends only on the number of edges of Γ_q. Since both $||D||$ and the number of edges are bounded above in terms of $\chi(X)$, we have proved the desired estimate on vertices of Γ_q. The estimate extends to points on edges by linearity and (approximate) continuity. □

7. Concluding remarks

Clearly Theorem 1.1 offers much useful geometric information about the behavior of Green's function on a hyperbolic surface. However, without drawing on the full strength of Theorem 1.1, we can selectively apply parts of its proof to obtain some simpler, but still rather useful bounds. For instance, we have:

THEOREM 7.1. *Let X be a hyperbolic surface with infinite volume and finitely generated fundamental group. Let $\Pi : \Gamma \to X$ be a circuit diagram for X and $\epsilon > 0$ be a real number. Then there are constants C_1, C_2 depending only on ϵ and $\chi(X)$ such that*

$$g(p,q) < C_1 + C_2 \operatorname{dist}(\Pi(p), v_0)$$

for all $p, q \in X$ such that $\operatorname{dist}(p,q) > \epsilon$ and there exists no non-joining component of X_{thin} containing both p and q.

Proof. We first consider g_{approx} rather than g. By Lemma 5.6 there is an upper bound, depending only on $\chi(X)$, on the slope of g_{approx} along any edge. Furthermore, by Theorem 5.4 the variation in g_{approx} across any vertex is also bounded in terms of $\chi(X)$. Hence, if $\gamma \subset \Gamma$ is a shortest path from v_0 to $\Pi(p)$, we have

$$\begin{aligned} g_{\text{approx}}(\Pi(p), q) &= g_{\text{approx}}(\Pi(p), q) - g_{\text{approx}}(v_0, q) \\ &\leq \sum_{v \in \gamma \cap \mathbf{V}} C_1 + \sum_{e \in \mathbf{E}} C_2 \operatorname{Length}(e \cap \gamma) \\ &\leq C_1 + C_2 \operatorname{Length}(\gamma), \end{aligned}$$

since the number of vertices is bounded in terms of $\chi(X)$. Applying Theorem 5.4 and the hypotheses on p and q to the difference $|g_{\text{approx}}(\Pi(p), q) - g(p,q)|$ finishes the proof. □

COROLLARY 7.2. *Let X be a hyperbolic surface with infinite volume and finitely generated fundamental group. Let ℓ be the length of the shortest closed geodesic in X. Then there exists a constant C depending only on the Euler characteristic of X such that*

$$g(p,q) < \frac{C}{\ell} + \log^+ \frac{1}{\operatorname{dist}(p,q)},$$

for all points $p, q \in X$ that do not both lie in the same cusp.

Proof. Any edge $e_\gamma = \Pi(\mathcal{C}(\gamma)) \subset \Gamma$ has length $\mathcal{R}(\gamma) < 2\pi^2/\operatorname{Length}(\gamma)$. Also, the number of edges contained in any simple path $\gamma \subset \Gamma$ is controlled by $\chi(X)$. Hence, if $p, q \in X$ satisfy the hypotheses of Theorem 7.1, we have

$$g(p,q) < C_1 + \frac{C_2}{\ell} < \frac{C}{\ell},$$

since ℓ is bounded above in terms of $\chi(X)$. If $\operatorname{dist}(p,q) \le \epsilon$, then logarithmic growth of $g(\cdot, q)$ near q still implies that

$$g(p,q) < \frac{C}{\ell} + \log^+ \frac{1}{\operatorname{dist}(p,q)}.$$

Finally, suppose that p and q are both contained in a non-joining collar $\mathcal{C}(\gamma)$ of some short geodesic. Then if g_γ is Green's function for $\mathcal{C}(\gamma)$, we have

$$g(p,q) - g_\gamma(p,q) < \frac{C}{\ell}.$$

But Lemma 3.1 and Theorem 3.2 combine to show that

$$g_{\mathcal{C}}(p,q) < \frac{C}{\operatorname{Length}(\gamma)}.$$

By definition $\operatorname{Length}(\gamma) > \ell$, so we are done. □

There are several directions in which one might try to improve Theorem 1.1. It would be particularly interesting to know whether the theorem can be extended in some form to surfaces with infinitely generated fundamental group. This might allow one to "geometrize" some of the older classification theory of Riemann surfaces. In somewhat different contexts, other authors have used graphical caricatures to demonstrate the (non-)existence of Green's functions on surfaces with infinite topology (see, for instance, [MMT]). Of the two main theorems used to prove Theorem 1.1 in this paper, Theorem 5.9 is more or less local in nature. That is, this theorem holds so long as there is a uniform bound on $\chi(K)$ over all connected components K of X_{thick}.

There are, however, two difficulties that one encounters in attempting to generalize Theorem 6.2 to surfaces with infinitely generated fundamental group. First of all, Theorem 6.2 relies on the fact (Theorem 6.3) that bounded drop functions D induce bounded potentials $U = \mathcal{L}^{-1}\delta\mathcal{R}^{-1}D$ on an electric network Γ. The proof we give for Theorem 6.3 does not yield a bound on $||U||$ that is very specific in terms of $||D||$; at any rate, the proof relies heavily on the fact that Γ has finitely many edges and vertices. The second difficulty is that we used Theorem 5.4 (discontinuity of the approximate Green's function at vertices) to obtain a bound on $||D||$. If Γ has infinitely many edges, then Theorem 5.4 will not suffice unchanged. We would need an improved version of the theorem that gave smaller discontinuities on vertices further from the pole of Green's function. Obtaining this improvement would consist, essentially, of obtaining a similar improvement in Theorem 1.2 (a geometric bound on the derivative of Green's function).

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