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TRACES OF MONOTONE FUNCTIONS IN WEIGHTED SOBOLEV SPACES

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ABSTRACT. Consider monotone functions $u\colon \mathbb{B}^n \to \mathbb{R}$ in the weighted Sobolev space $W^{1,p}(\mathbb{B}^n; w)$, where $n-1 < p \le n$ and w is a weight in the class A_q for some $1 \le q < p/(n-1)$ which has a certain symmetry property with respect to $\partial\mathbb{B}^n$. We prove that u has nontangential limits at all points of $\partial\mathbb{B}^n$ except possibly those on a set E of weighted (p,w)-capacity zero. The proof is based on a new weighted oscillation estimate (Theorem 1) that may be of independent interest. In the special case $w(x) = |1-|x||^{\alpha}$, the weighted (p,w)-capacity of a ball can be easily estimated to conclude that the Hausdorff dimension of the set E is smaller than or equal to $\alpha + n - p$, where $0 \le \alpha < (p-(n-1))/(n-1)$.

1. Introduction

Traces of monotone Sobolev functions were considered in [MV], where a classical theorem of Lindelöf was extended to monotone functions in the (unweighted) Sobolev space $W^{1,p}(\mathbb{B}^n)$. The case of weak solutions of elliptic partial differential equations

$$\operatorname{div}(\mathcal{A}(x,\nabla u)) = 0,$$

where $\alpha|\xi|^{p-1} \le \langle \mathcal{A}(x,\xi),\xi\rangle \le \beta\,|\xi|^{p-1}$ for some fixed $p \in (1,\infty)$, $\xi \in \mathbb{R}^n$ and $0 < \alpha < \beta$, was considered in [KMV], where the analog of Lindelöf's theorem was established by analytical methods. We refer to these two articles for background and historical references.

Let us start by recalling the definition of monotone functions and of Muckenhoupt A_p weights.

DEFINITION 1.1. Let $\Omega \subset \mathbb{R}^n$ be an open set. A continuous function $u\colon \Omega \to \mathbb{R}$ is monotone, in the sense of Lebesgue, if

$$\max_{\overline{D}} u(x) = \max_{\partial D} u(x)$$

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and

$$\min_{\overline{D}} u(x) = \min_{\partial D} u(x)$$

hold whenever D is a domain with compact closure $\overline{D} \subset \Omega$.

Definition 1.2. Let $q > 1$ and $w \in L^1_{\text{loc}}(\mathbb{R}^n)$. We say that $w \in A_q$, if there exists a constant C such that

$$\sup_B \left(\fint_B w(y)\, dy \right) \left(\fint_B w(y)^{\frac{1}{1-q}}\, dy \right)^{q-1} < C,$$

where the supremum is taken over all balls $B \subset \mathbb{R}^n$.

The Sobolev space $W^{1,p}(\mathbb{B}^n; w)$ (see [HKM, Chapter 1]) consists of functions $u\colon \mathbb{B}^n \to \mathbb{R}$ that have first distributional derivatives ∇u such that

$$\int_{\mathbb{B}^n} \left(|u(x)|^p + |\nabla u(x)|^p \right) \; w(x)\; dx < \infty.$$

The weighted p-capacity, which we will be using throughout this paper, is the relative first order variational (p, w)-capacity (see [HKM, Chapter 2]). We will occasionally need the weighted Sobolev class

$$\mathrm{ACL}^p_w(\mathbb{B}^n) = \left\{ u \in \mathrm{ACL}(\mathbb{B}^n) \text{ such that } \int_{\mathbb{B}^n} |\nabla u(x)|^p\, w(x)\; dx < \infty \right\},$$

where $\mathrm{ACL}(\mathbb{B}^n)$ is the class of functions that are absolutely continuous on almost every line. The gradients of these functions are Borel functions (see, for example, [Va, §26]). When $w \in A_p$, smooth functions are dense in $W^{1,p}(\mathbb{B}^n; w)$ (see [K]). In particular, $\mathrm{ACL}^p_w(\mathbb{B}^n)$ is dense in $W^{1,p}(\mathbb{B}^n; w)$.

Our first result is a weighted version of the Gehring oscillation inequality for monotone functions.

Theorem 1. *Let u be a continuous monotone function in $W^{1,p}(\Omega; w)$, where w is in the class A_q for some q in the range $1 \le q < p/(n-1)$. Suppose that $n - 1 < p \le n$. Then we have*

$$\left(\operatorname{osc}\bigl(u, \mathbb{B}^n(x, r)\bigr) \right)^p \le c(n, p, q, w)\; \frac{r^p}{w(\mathbb{B}^n(x, 2r))} \; \int_{\mathbb{B}^n(x,2r)} |\nabla u(y)|^p\, w(y)\, dy,$$

whenever $\mathbb{B}^n(x, 2r) \subset \Omega$.

This oscillation estimate is one the key ingredients in the proof of the following extension of the classical Lindelöf theorem.

Theorem 2. *Let u be a monotone function in the space $W^{1,p}(\mathbb{B}^n; w)$. Suppose that $n - 1 < p \le n$ and that w is Borel weight in the class A_q for some q in the range $1 \le q < p/(n-1)$. Then, for any $\epsilon > 0$, there exists an* *open set U in $\mathbb{R}^n$ satisfying* $\mathrm{cap}_{p,w}(U) < \epsilon$ *such that, for any $x_0 \in \partial\mathbb{B}^n \setminus U$, if we have a curve γ ending at x_0 in $\mathbb{B}^n$ with*

$$\lim_{x \to x_0,\ x \in \gamma} u(x) = \alpha.$$

It follows that $u(x)$ has nontangential limit α at x_0.

Remark. We do not identify w with the equivalence class of measurable functions which agree with w a. e., but rather work with a fixed representative of w that we assume is a Borel function. The reason for this is that we will need to restrict w to $(n-1)$-dimensional sets to define the weighted (p,w)-modulus relative to a hypersurface.

The limitations $p > n-1$ and w in A_q, for some $1 \le q < p/(n-1)$, appear in a module estimate on $(n-1)$-dimensional spheres (see Lemma 2.3 below). As in the classical case, one can ask if the Lindelöf type Theorem 2 is enough to guarantee that monotone functions in $W^{1,p}(\mathbb{B}^n; w)$ have non-tangential limits except in a subset of $\partial\mathbb{B}^n$ of (p,w)-capacity zero. This is indeed the case if the weight w satisfies a symmetry condition with respect to $\partial\mathbb{B}^n$.

Definition 1.3. We say that a weight w is symmetric with respect to $\partial\mathbb{B}^n$ if we can find a bi-Lipschitz homeomorphism

$$H\colon \left\{\frac{1}{2} < |x| < 2\right\} \mapsto \left\{\frac{1}{2} < |x| < 2\right\}.$$

which is the identity on $\partial(\mathbb{B}^n)$ and sends $\{1 < |x| < 2\}$ onto $\{1/2 < |x| < 1\}$, and a positive constant $\gamma > 0$ such that

$$w(H(x)) \le \gamma\ w(x) \tag{1.4}$$

for all $1/2 \le |x| \le 1$.

There is nothing special about the width of the crown about $|x| = 1$. The important facts are that the derivatives of H and H^{-1} are bounded, and that the inequality (1.4) holds. An example of a symmetric weight is $w(x) = |1 - |x||^\alpha$. In this case, if we write $x = t\omega$ for $1/2 < t < 2$ and $\omega \in \partial\mathbb{B}^n$ we can take $H(t\omega) = h(t)\omega$, where

$$h(t) = \begin{cases} 3 - 2t & \text{for} \quad 1/2 < t \le 1, \\ \frac{1}{2}(3-t) & \text{for} \quad 1 \le t < 2. \end{cases}$$

In fact, for $1/2 \le |x| \le 1$ we have

$$w(H(x)) = |1 - H(x)|^\alpha = 2^\alpha w(x).$$

Our main result is the following:

THEOREM 3. *Let u be a monotone function in $W^{1,p}(\mathbb{B}^n;w)$. Suppose that $n-1<p\le n$ and that w is a symmetric weight with respect to $\partial\mathbb{B}^n$ in the class A_q for some q in the range $1\le q<p/(n-1)$. Let E be the subset of $\partial\mathbb{B}^n$ where the nontangential limit of u does not exist. Then E has (p,w)-capacity zero.*

Note that in the conclusion of Theorem 3 the value q does not appear. This raises the question whether the theorem holds for a larger range of q.

The plan of this paper is as follows. In Section 2 we define the weighted p-modulus relative to a hypersurface and prove Theorem 1. Section 3 contains some facts about weighted Sobolev spaces for which we could not find an explicit reference. We present the proofs of Theorems 2 and 3 in Section 4. Finally, in Section 5 we consider radial weights of the form $|1-|x||^\alpha$.

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2. Preliminaries and an oscillation estimate

The open ball centered at x_0 with radius r is denoted by $\mathbb{B}^n(x_0,r)$. Its boundary is the $(n-1)$-dimensional sphere $S^{n-1}(x_0,r)$. By a cap of a sphere $S^{n-1}(x_0,r)$ we mean a set $H\cap S^{n-1}(x_0,r)$, where H is an open half space in $\mathbb{R}^n$. The spherical distance between two points in $\bar{\mathbb{R}}^n$ is denoted by $q(x,y)$. Given a point $x\in\partial\mathbb{B}^n$, we write $C(x)$ for the Stolz cone at x with a given fixed aperture. There exists a constant $c_n\ge 1$, depending only on the aperture and n, such that if $y\in C(x)$ then

$$|y-x|\le c_n\ (1-|y|). \tag{2.1}$$

By $c(\alpha,\beta,\ldots)$ we denote a constant depending only on the parameters $\alpha,\beta,\ldots$, not necessarily the same at each occurrence.

Let Γ be a family of curves in $\mathbb{R}^n$. Denote by $\mathcal{F}(\Gamma)$ the collection of admissible metrics for Γ. These are nonnegative Borel measurable functions $\rho\colon\mathbb{R}^n\to\mathbb{R}\cup\{\infty\}$ such that

$$\int_\gamma \rho\,ds\ge 1$$

for each locally rectifiable curve $\gamma\in\Gamma$. For $p\ge1$ the weighted (p,w)-module of Γ is defined by

$$M_p^w(\Gamma)=\inf_{\rho\in\mathcal{F}(\Gamma)}\int_{\mathbb{R}^n}\rho^p\,w\,dx.$$

If $\mathcal{F}(\Gamma)=\emptyset$, we set $M_p^w(\Gamma)=\infty$. The same definition applies to families of curves that lie in an $(n-1)$-dimensional submanifold S of $\mathbb{R}^n$, replacing the measure $w\,dx$ by $w\,dS$, where dS is the surface measure in the submanifold.

(Note that nothing prevents w from being identically equal to ∞ on the submanifold.) The surface module is denoted by $M_p^{w,S}(\Gamma)$. Upper bounds for moduli are obtained by testing with a particular admissible metric.

LEMMA 2.2. *Let Ω be a domain in $\mathbb{R}^n$ and $\omega \in A_p$, $p \geq 1$, and let $u : \Omega \to \mathbb{R}$ be a continuous function in $W^{1,p}(\Omega; w)$. Let $-\infty < a < b < \infty$, and let A, B be nonempty subsets of a ball $\mathbb{B}^n(x_0, r) \subset \Omega$ such that $u(x) \leq a$ for any $x \in A$ and $u(x) \geq b$ for any $x \in \mathbb{B}^n(x_0, r)$. Then we have*

$$M_p^w\Big(\Delta\big(A, B; \mathbb{B}^n(x_0, r)\big)\Big) \leq \frac{1}{(b-a)^p} \int_{\mathbb{B}^n(x_0,r)} |\nabla u(x)|^p \, w(x) \; dx,$$

where $\Delta\big(A, B; \mathbb{B}^n(x_0, r)\big)$ is the family of all curves in $\mathbb{B}^n(x_0, r)$ joining A and B.

Proof. Let us temporarily assume that u is in the class $\mathrm{ACL}^p_w(\Omega)$. We can then adapt the proof in [Va] as follows. We start by observing that the family of curves Γ_u consisting of those paths containing a subpath on which u is not absolutely continuous satisfies $M_p^w(\Gamma_u) = 0$.

Set $v(x) = (u(x) - a)/(b - a)$ and write $\Delta = \Delta\big(A, B; \mathbb{B}^n(x_0, r)\big)$. As before, Δ_u denotes the family of paths in Δ containing a subpath on which u is not absolutely continuous. Then we have $M_p^w(\Delta_u) = 0$. Since ∇u is a Borel function, we have

$$1 \leq \int_\gamma |\nabla v| \; ds \text{ for } \gamma \in \Delta \setminus \Delta_u.$$

Thus $|\nabla v|$ is an admissible function for Δ. Therefore we have

$$M_p^w(\Delta) \leq \int_{\mathbb{B}^n(x_0,r)} |\nabla v(x)|^p \, w(x) \, dx = \frac{1}{(b-a)^p} \int_{\mathbb{B}^n(x_0,r)} |\nabla u(x)|^p \, w(x) \, dx.$$

For $u \in W^{1,p}(\Omega; w)$ we can assume without loss of generality that $\mathbb{B}^n(x_0, 2r) \subset \Omega$ (see [K, §3]), by extending u if necessary. Let u_ϵ be a smooth approximation of u in the space $W^{1,p}(\Omega; w)$ (see [K, Theorem 2.5]). Select $\delta < (b - a)/4$. We can always find $\epsilon > 0$ such that $u_\epsilon(x) \leq a + \delta$ if $x \in A$ and $u_\epsilon(x) \geq b - \delta$ if $x \in B$. Since u_ϵ is certainly in $\mathrm{ACL}^p_w(\Omega)$, we have

$$M_p^w\Big(\Delta\big(A, B; \mathbb{B}^n(x_0, r)\big)\Big) \leq \frac{1}{(b-a-2\delta)^p} \int_{\mathbb{B}^n(x_0,r)} |\nabla u_\epsilon(x)|^p \, w(x) \; dx.$$

Since δ is arbitrary and

$$\int_{\mathbb{B}^n(x_0,r)} |\nabla u_\epsilon(x)|^p \, w(x) \; dx \to \int_{\mathbb{B}^n(x_0,r)} |\nabla u(x)|^p \, w(x) \; dx$$

as $\epsilon \to 0$, the lemma follows. $\square$

Lower bounds for modules are harder to obtain. The next lemma is a weighted version of Väisälä's Lemma. Note that we are not assuming that w is an A_q weight.

Lemma 2.3. *Let $n \geq 2$, and let K be a cap of the sphere $S = S^{n-1}(x_0, r)$. Suppose that E and F are disjoint nonempty sets of $\bar{K}$. Let $\Delta(E, F; K)$ be the family of all curves in K joining E and F. Suppose that $n-1 < p \leq n$, $1 < q < p/(n-1)$ and w is a Borel function such that $w > 0$ almost everywhere. Set*

$$\beta = \frac{n-p-1}{q-1} + (n-1).$$

There exists a constant c depending only on n, p and q such that

$$c \leq \frac{1}{r^\beta} \left(M_p^{w,S}\big(\Delta(E,F;K)\big) \right)^{\frac{1}{q-1}} \cdot \left(\int_S w(y)^{\frac{1}{1-q}} \, dS(y) \right). \tag{2.4}$$

Proof. Consider first the case $n = 2$. Then K is an arc of $S^1(x_0, r)$. There is a subarc γ of K such that $\gamma \in \Delta(E, F; K)$. The length of γ is smaller than $2\pi r$. For any admissible metric ρ and $\gamma \in \Delta$ we have

$$1 \leq \int_\gamma \rho \, ds = \int_\gamma \rho \, w^{1/p} \, w^{-1/p} \, ds.$$

Hölder's inequality applied twice gives

$$1 \leq \left(\int_\gamma \rho^p \, w \right) \left(\int_\gamma w^{\frac{1}{1-q}} \right)^{q-1} (2\,\pi\, r)^{p-q},$$

and taking the $q-1$ root on both sides we obtain the desired inequality.

Consider now the case $n \geq 3$. By the change of variables formula for integrals we see that if $f : \mathbb{R}^n \to \mathbb{R}^n$ is defined by $f(x) = k\,x$, where k is a constant, then for the family $\Gamma = \Delta(E, F; K)$ we have

$$M_p^{w \circ f^{-1}, S^{n-1}(f(x_0), kr)}\big(f(\Gamma)\big) = k^{n-p-1} \, M_p^{w, S^{n-1}(x_0, r)}(\Gamma). \tag{2.5}$$

Set $k = 1/2r$. Then $S^{n-1}(x_0, r)$ is mapped to $S^{n-1}(x_0, 1/2)$. By translating if necessary, we may suppose that $x_0 = e_n/2$ and $e_n = (0, \dots, 0, 1) \in E$.

Therefore, it is enough to show that

$$1 \leq c \left(M_p^{w, S^{n-1}(e_n/2, 1/2)}(\Gamma) \right)^{1/q-1} \left(\int_{S^{n-1}(e_n/2,1/2)} w(y)^{\frac{1}{1-q}} \, dS(y) \right), \tag{2.6}$$

where c depends only on n, p and q.

Let ρ be an admissible metric for the module problem

$$M_p^{w, S^{n-1}(e_n/2, 1/2)}(\Gamma).$$

Let $h \colon \overline{\mathbb{R}}^n \to \overline{\mathbb{R}}^n$ be the inversion in the sphere $S^{n-1}(e_n, 1)$ given by

$$h(x) = e_n + \frac{x - e_n}{|x - e_n|^2}.$$

Then h is conformal, $h \circ h = \mathrm{Id}$, and h maps $S = S^{n-1}(e_n/2, 1/2)$ stereographically onto $\overline{\mathbb{R}}^{n-1}$. The image of $S \setminus K$ is either empty or a closed ball, or a closed half-space in $\overline{\mathbb{R}}^{n-1}$. Choose $a \in h(F)$. We may assume that $a = \alpha\, e_1,\ \alpha \geq 0$. Since $h(S \setminus K)$ is convex, there is an open hemisphere G of S^{n-1} such that $a + t\,y \in h(K)$ for every y in G, and all positive t. We define $\gamma_y \colon [0,\infty) \to S$ by

$$\gamma_y(t) = h\big(a + ty\big).$$

Then $\gamma_y \in \Gamma$ for any y in G. Thus,

$$1 \leq \int_{\gamma_y} \rho \, d\sigma = \int_0^\infty \frac{\rho\Big(h\big(a+ty\big)\Big)}{\big(1+|a+ty|^2\big)}\, dt.$$

Integrating over $y \in G$ yields

$$c \leq \int_H \frac{\rho\big(h(y)\big)}{(1+|y|^2)\ |y-a|^{n-2}}\ dS(y),$$

where H is the half space in $\mathbb{R}^{n-1}$ consisting of all the points $a + ty$, where $t > 0$, and $y \in G$. By Hölder's inequality this implies that

$$c \leq \left(\int_H \frac{\Big(\rho\big(h(y)\big)\Big)^p w(h(y))}{(1+|y|^2)^{n-1}}\ dS(y)\right)^{1/p}$$
$$\times \left(\int_H \frac{w(h(y))^{\frac{1}{1-p}}}{\big(1+|y|^2\big)^{\frac{p-n+1}{p-1}}\,|y-a|^{\frac{(n-2)p}{p-1}}}\ dS(y)\right)^{(p-1)/p}.$$

Thus, we have

$$c \leq \left(\int_S \rho^p(x)\, w(x)\, dS(x)\right)^{1/p} \cdot \mathrm{II}\,,$$

where

$$\mathrm{II} = \left(\int_H \frac{w(h(y))^{\frac{1}{1-p}}}{\big(1+|y|^2\big)^{\frac{p-n+1}{p-1}}\,|y-a|^{\frac{(n-2)p}{p-1}}}\ dS(y)\right)^{(p-1)/p}.$$

Since this inequality holds for any admissible metric ρ, we conclude

$$c \leq \left(M_p^{w,\,S}(\Gamma)\right)^{1/p}\ \mathrm{II}\,.$$

Applying Hölder inequality for $1 < q < p$ we have

$$\begin{aligned}
\mathrm{II} \leq & \left(\int_H \frac{w(h(y))^{\frac{1}{1-q}}}{\left(1+|y|^2\right)^{n-1}} \, dS(y) \right)^{\frac{q-1}{p}} \\
& \qquad \times \left(\int_H \frac{1}{\left(1+|y|^2\right)^{\frac{p-q(n-1)}{p-q}} |y-a|^{\frac{(n-2)p}{p-q}}} \, dS(y) \right)^{\frac{p-q}{p}} \\
= & \left(\int_S w(x)^{\frac{1}{1-q}} \, dS(x) \right)^{\frac{q-1}{p}} \\
& \qquad \times \left(\int_H \frac{1}{\left(1+|y|^2\right)^{\frac{p-q(n-1)}{p-q}} |y-a|^{\frac{(n-2)p}{p-q}}} \, dS(y) \right)^{\frac{p-q}{p}} .
\end{aligned}$$

If we choose $p > q\,(n-1)$ as in the statement of the lemma, the last integral above is finite. $\square$

We are now ready for the main oscillation estimate for monotone functions.

Proof of Theorem 1. Select a point $x \in \Omega$ and a positive r such that

$$\mathbb{B}^n(x, 2r) \subset \Omega.$$

Let y be an arbitrary point in $\mathbb{B}^n(x, r)$. Without loss of generality we can assume that $u(x) < u(y)$; the case $u(x) > u(y)$ is handled by a symmetric argument. Set

$$A = \{z \in \mathbb{B}^n(x, 2r) \colon u(z) \leq u(x)\}$$

and

$$B = \{z \in \mathbb{B}^n(x, 2r) \colon u(z) \geq u(y)\}.$$

Since u is monotone we know that

$$A \cap S^{n-1}(x, t) \neq \emptyset$$

and

$$B \cap S^{n-1}(x, t) \neq \emptyset$$

for $r < t \leq 2r$. From now on let us denote

$$M_p^{w, S^{n-1}(x,t)}\big(\Delta(A \cap S^{n-1}(x,t), B \cap S^{n-1}(x,t); S^{n-1}(x,t))\big)$$

by $M_p^{w, S^{n-1}(x,t)}$. Applying Lemma 2.3 with $K = S^{n-1}(x,t)$, we obtain

$$c(n,p,q) \leq \frac{1}{t^{\frac{n-p-1}{q-1}+(n-1)}} \left(M_p^{w, S^{n-1}(x,t)}\right)^{\frac{1}{q-1}} \left(\int_{S^{n-1}(x,t)} w(y)^{\left(\frac{1}{1-q}\right)} \, dS(y) \right)$$

for any t with $r < t \leq 2r$. Using the fact that $(n-p-1)/(q-1) + (n-1)$ is negative and that w is a positive weight, we get
(2.7)
$$\int_r^{2r} \left(\frac{1}{M_p^{w,S^{n-1}(x,t)}} \right)^{\frac{1}{q-1}} \leq c(n,p,q)\, r^{\frac{p+1-n}{q-1}-(n-1)} \left(\int_{\mathbb{B}^n(x,2r)} w(x)^{\frac{1}{1-q}}\, dx \right).$$
By Hölder's inequality we have
$$\begin{aligned} r = \int_r^{2r} dt &= \int_r^{2r} \left(M_p^{w,S^{n-1}(x,t)} \right)^{\frac{1}{q}} \left(\frac{1}{M_p^{w,S^{n-1}(x,t)}} \right)^{\frac{1}{q}} dt \\ &\leq \left(\int_r^{2r} \left(\frac{1}{M_p^{w,S^{n-1}(x,t)}} \right)^{\frac{1}{q-1}} dt \right)^{\frac{q-1}{q}} \left(\int_r^{2r} M_p^{w,S^{n-1}(x,t)}\, dt \right)^{\frac{1}{q}}. \end{aligned}$$
It is immediate from the definition of the module that the second integral above is at most $M_p^w\Big(\Delta\big(A,B;\mathbb{B}^n(x,2r)\backslash\mathbb{B}^n(x,r)\big)\Big)$. Hence, taking the above inequality to the power $q/(q-1)$, we obtain
$$\begin{aligned} r^{\frac{q}{q-1}} \leq &\left(M_p^w\Big(\Delta\big(A,B;\mathbb{B}^n(x,2r)\setminus\mathbb{B}^n(x,r)\big)\Big)\right)^{\frac{1}{q-1}} \\ &\times \left(\int_r^{2r} \left(\frac{1}{M_p^{w,S^{n-1}(x,t)}} \right)^{\frac{1}{q-1}} dt \right). \end{aligned}$$
Using the above inequality and the monotonicity property of the module in (2.7) we obtain
$$\frac{r^{\frac{q}{q-1}}}{\big(M_p^w\left(\Delta(A,B;\mathbb{B}^n(x,2r))\right)\big)^{\frac{1}{q-1}}} \leq c\, r^{\frac{p+1-n}{q-1}-(n-1)} \left(\int_{\mathbb{B}^n(x,2r)} w(x)^{\frac{1}{1-q}}\, dx \right).$$
Since, by assumption, $w \in A_q(\mathbb{R}^n)$, using the A_q-condition we have that
$$\frac{r^{\frac{q}{q-1}}}{\Big(M_p^w\left(\Delta\big(A,B;\mathbb{B}^n(x,2r)\big)\right)\Big)^{\frac{1}{q-1}}} \leq c\; r^{\frac{p+1-n}{q-1}-(n-1)} \; \frac{r^{\frac{nq}{q-1}}}{\left(\int_{\mathbb{B}^n(x,2r)} w(x)\; dx \right)^{\frac{1}{q-1}}},$$
where c is a constant depending on n, p, q and w. Collecting the powers of r yields
$$M_p^w\Big(\Delta\big(A,B;\mathbb{B}^n(x,2r)\big)\Big) \geq c\, r^{-p} \int_{\mathbb{B}^n(x,2r)} w(x)\; dx. \tag{2.8}$$
Using now Lemma 2.2, we obtain
$$\frac{1}{|u(x)-u(y)|^p} \int_{\mathbb{B}^n(x,2r)} |\nabla u(x)|^p\, w(x)\, dx \geq c\, r^{-p} \int_{\mathbb{B}^n(x,2r)} w(y)\, dy, \tag{2.9}$$

which is the desired inequality. □

3. Pointwise behavior of weighted Sobolev functions

In this section we assume that w is a p-admissible weight as defined in [HKM]. We will use the fact that $\mathrm{cap}_{p,w}$ is a Choquet capacity and that

$$\mathrm{cap}_{p,w}(B_r) \approx r^{-p}\, w(B_r).$$

For functions $f \in L^p_w(\mathbb{R}^n) = \{f\colon \int f^p\, w < \infty\}$ consider the fractional maximal function

$$M^w_{s,p} f(y) = \sup_{r>0} \left(\frac{r^{-s}}{w(\mathbb{B}^n(y,r))} \int_{\mathbb{B}^n(y,r)} |f(x)|^p\, w(x)\, dx \right)^{1/p},$$

where $w(\mathbb{B}^n(x,r)) = \int_{\mathbb{B}^n(x,r)} w(x)\, dx$. We will need the following estimate for $f = |\nabla u|$, which is probably known, but which we have been unable to locate in the literature. (For related estimates see [K].)

Lemma 3.1. *Let $f \in L^p_w(\mathbb{R}^n)$ and $1 < p \le n$. Then we have*

$$\mathrm{cap}_{p,w}\left(\left\{x \in \mathbb{R}^n\colon M^w_{-p,p} f(x) > t\right\}\right) \le \frac{c(p,n,w)}{t^p} \int_{\mathbb{R}^n} |f(y)|^p\, w(y)\, dy.$$

Proof. For $t > 0$ let

$$E_t = \{y \in \mathbb{R}^n\colon M^w_{-p,p} f(y) > t\}.$$

Note that E_t is open. Let K be a compact subset of E_t. We claim that

$$\Lambda^\infty_{-p,w}(K) \le \frac{c_n}{t^p} \int_{\mathbb{R}^n} |f(y)|^p\, w(y)\, dy, \tag{3.2}$$

where

$$\Lambda^\infty_{-p,w}(E) = \inf\left\{ \sum r_i^{-p}\, w(\mathbb{B}^n(z_i, r_i))\colon E \subset \bigcup \mathbb{B}^n(z_i, r_i) \right\}$$

is the $\{-p, w\}$-Hausdorff content of E.

If $y \in K$, there exists a ball $\mathbb{B}^n(y, r_y)$ such that

$$w(\mathbb{B}^n(y, r_y))\, r_y^{-p} \le \frac{1}{t^p} \int_{\mathbb{B}^n(y,r_y)} |f(x)|^p\, w(x)\, dx.$$

The compact set K is contained in $\bigcup \mathbb{B}^n(y, r_y/5)$. By the Besicovitch covering lemma there exists a subfamily of balls $\{B_1, B_2, \dots, B_N\}$ such that

$$K \subset \bigcup_{i=1}^N B_i$$

and no point of $\mathbb{R}^n$ is in more than a fixed number c_n of balls. We have

$$\Lambda^\infty_{-p,w}(K) \le \sum_{i=1}^N (r_i)^{-p}\, w(B_i) \le \sum_{i=1}^N \frac{1}{t^p} \int_{B_i} |f(x)|^p\, w(x)\, dx.$$

Since each point is in at most c_n balls, we deduce (3.2).

The open set E_t is capacitable. Therefore $\mathrm{cap}_{p,w}(E_t) = \sup \mathrm{cap}_{p,w}(K)$, where the supremum is taken over all the compact subsets K of E_t. Fix one such K and a covering $K \subset \bigcup B_i$. We have

$$\mathrm{cap}_{p,w}(K) \leq \sum_i \mathrm{cap}_{p,w}(B_i) \leq C \sum_i r_i^{-p}\, w(B_i).$$

Since the covering is arbitrary, we get

$$\mathrm{cap}_{p,w}(K) \leq C\Lambda^{\infty}_{-p,w}(K) \leq \frac{C}{t^p} \int_{\mathbb{R}^n} |f(y)|^p\, w(y)\, dy. \qquad \square$$

LEMMA 3.2. *Let $u \in W^{1,p}(\mathbb{R}^n; w)$, where $1 < p \leq n$ and w is an A_p weight. Then for every $\epsilon > 0$ there exists an open set $U \subset \mathbb{R}^n$ with $\mathrm{cap}_{p,w}(U) < \epsilon$ such that*

$$\lim_{r\to 0} \frac{r^p}{w(\mathbb{B}^n(x,r))} \int_{\mathbb{B}^n(x,r)} |\nabla u(y)|^p\, w(y)\, dy = 0$$

uniformly on $\mathbb{R}^n \setminus U$.

Proof. The proof of this lemma is a straightforward adaptation of the proof of Lemma 3.2 in [MV], using Lemma 3.1 and the fact that smooth functions with compact support are dense in $W^{1,p}(\mathbb{R}^n; w)$ whenever the weight w is in the class A_p (see [K]). $\square$

4. Boundary behavior of monotone functions

We start with a lemma needed for the proof of Theorem 2.

LEMMA 4.1. *Let $u : \mathbb{B}^n \mapsto \mathbb{R}$ be a continuous monotone function in the weighted Sobolev space $W^{1,p}(\mathbb{B}^n; w)$, where $n-1 < p \leq n$ and w is an A_q weight for some q in the range $1 < q < p/(n-1)$. Then, for any $\epsilon > 0$, there exists an open set U in $\mathbb{R}^n$ satisfying $\mathrm{cap}_{p,w}(U) < \epsilon$, with the following property: If $x_0 \in \partial\mathbb{B}^n \setminus U$ and $\{b_k\}_{k=1}^{\infty}$ is a sequence contained in the Stolz cone $C(x_0)$ such that $\lim_{k\to\infty} b_k = x_0$ and $\lim_{k\to\infty} u(b_k) = \beta$, then for any $\eta > 0$ there exists an integer $k_0 \geq 1$ such that the spherical distance between $u(x)$ and β satisfies*

$$q(u(x), \beta) < \eta$$

for any x in the set

$$E = \bigcup_{k\geq k_0} \mathbb{B}^n(b_k, \frac{1}{4}(1 - |b_k|)).$$

Proof. It is well known that if $w \in A_q$ and $q < p$ then $w \in A_p$. By Theorem D in [Ch] we can extend u to a function f in $W^{1,p}(\mathbb{R}^n; w)$ such that

$$\int_{\mathbb{R}^n} |\nabla f(x)|^p \, w(x) \, dx \leq c \int_{\mathbb{B}^n(0,1)} |\nabla u(x)| \, w(x) \, dx,$$

for some constant c depending on n, p and the A_p constant of the weight w. We continue to denote this extension by u. Fix $\epsilon > 0$ and choose U according to Lemma 3.2. For a constant δ_0 to be determined later, choose $r_0 > 0$ such that for $0 < r < r_0$ and $x_0 \in \overline{\mathbb{B}}^n(0,1) \setminus U$,

$$\int_{\mathbb{B}^n(x_0,r)} |\nabla u(y)|^p \, w(y) dy \leq \delta_0 \; r^{-p} \; w(\mathbb{B}^n(x_0, r)). \tag{4.2}$$

Select a point $y \in \mathbb{B}^n(b_k, \frac{1-|b_k|}{4})$. We may assume that $u(b_k) < u(y)$; the case $u(b_k) > u(y)$ is handled by a symmetric argument. Set

$$A = \{z \in \mathbb{B}^n(b_k, \frac{1-|b_k|}{2}) \colon u(z) \leq u(b_k)\}$$

and

$$B = \{z \in \mathbb{B}^n(b_k, \frac{1-|b_k|}{2}) \colon u(z) \geq u(y)\}.$$

Since u is monotone we have

$$A \cap S^{n-1}(b_k, t) \neq \emptyset$$

and

$$B \cap S^{n-1}(b_k, t) \neq \emptyset$$

for $|b_k - y| < t \leq (1 - |b_k|)/2$. Applying Theorem 1 to this setting we obtain

$$\begin{aligned} \frac{1}{|u(b_k) - u(y)|^p} & \int_{\mathbb{B}^n(b_k, \frac{1-|b_k|}{2})} |\nabla u(x)|^p \, w(x) \, dx \\ & \geq \frac{c}{(1 - |b_k|)^p} \int_{\mathbb{B}^n(b_k, \frac{1-|b_k|}{2})} w(y) \, dy. \end{aligned}$$

Since $b_k \in C(x_0)$ one easily checks, using (2.1), that

$$\mathbb{B}^n\left(b_k, \frac{1-|b_k|}{2}\right) \subset \mathbb{B}^n\big(x_0, d_n(1 - |b_k|)\big),$$

where $d_n = c_n + 1/2$. Therefore we have

$$\begin{aligned} & \int_{\mathbb{B}^n(x_0, d_n(1-|b_k|))} |\nabla u(x)|^p \, w(x) \, dx \\ & \qquad \geq c \; |u(b_k) - u(y)|^p \; (1 - |b_k|)^{-p} \int_{\mathbb{B}^n(b_k, \frac{1-|b_k|}{2})} w(y) \, dy \\ & \qquad \geq c \; |u(b_k) - u(y)|^p \; (1 - |b_k|)^{-p} \int_{\mathbb{B}^n\left(b_k, d_n \, (1-|b_k|)\right)} w(y) \, dy. \end{aligned}$$

Choose k_1 so that for $k > k_1$ we have $d_n(1-|b_k|) < r_0$. It follows from (4.2) that

$$|u(b_k) - u(y)|^p \le c\ \delta_0\ d_n^p$$

for $y \in \mathbb{B}^n\big(b_k, \frac{1}{4}(1-|b_k|)\big)$. Here c denotes (different) constants depending only on p, q, n and the A_p constant of the weight w.

Given any positive η choose $k_0 > k_1$ such that $|u(b_k)-\beta| < \eta/2$ for $k > k_0$, and let δ_0 be such that $(c\ \delta_0\ d_n^p)^{1/p} < \eta/2$. Then, for $x \in E$ we have

$$|u(x) - \beta| \le |u(x) - u(b_{k'})| + |u(b_{k'}) - \beta| < \eta$$

for some $k' \ge k_0$. $\square$

Proof of Theorem 2. Given $\epsilon > 0$, choose U according to Lemma 3.2. Note that after extending u to $\mathbb{R}^n$ as in Theorem D of [Ch], the conclusion of Lemma 3.2 applies in $\overline{\mathbb{B}}^n(0,1) \setminus U$. Without loss of generality we may assume that $x_0 = \mathbf{1} = (1, 0, \ldots, 0)$. The proof proceeds by contradiction. Suppose that we have a sequence $\{b_k\}$ in a Stolz cone $C(\mathbf{1})$ such that $\lim_{k\to\infty} b_k = \mathbf{1}$ and

$$\lim_{k\to\infty} u(b_k) = \beta \ne \alpha.$$

Assume that $-\infty < |\alpha| < |\beta| < \infty$. (The other cases are similar or easier.) Let $\eta = (|\beta| - |\alpha|)/6$ and choose k_1 such that for $k > k_1$ we have

$$|u(x)| < |\alpha| + \eta \quad \text{for } x \in \gamma \cap \mathbb{B}^n(\mathbf{1}, |1 - b_k|)$$

and

$$|u(b_k)| > |\beta| - \eta.$$

Choose k_2 according to Lemma 4.1, so that for $k > k_0 = \max(k_1, k_2)$

$$|u(x)| > |\beta| - 2\eta, \text{ for } x \in \mathbb{B}^n\big(b_k, \frac{1}{4}(1 - |b_k|)\big).$$

Consider now the following module problem. Set

$$H_k = \mathbb{B}^n \cap \left(\mathbb{B}^n\big(\mathbf{1}, |1 - b_k|\big) \setminus \bar{\mathbb{B}}^n\big(\mathbf{1}, |1 - b_k| - \frac{1}{8}(1 - |b_k|)\big)\right)$$

and let $E = \gamma \cap H_k$ and $F = \mathbb{B}^n\big(b_k, \frac{1}{4}(1-|b_k|)\big) \cap H_k$.

To get un upper bound for $M_p^w\big(\Delta(E, F; H_k)\big)$, observe that for any locally rectifiable curve l joining E and F in H_k we have

$$\int_l |\nabla u(x)|\, ds \ge \frac{|\beta| - |\alpha|}{2}.$$

Thus, the metric $\rho = 2/(|\beta| - |\alpha|)\,|\nabla u|$ is admissible. From the definition of module we obtain

$$M_p^w\big(\Delta(E, F; H_k)\big) \le \frac{c(p)}{(|\beta| - |\alpha|)^p} \int_{H_k} |\nabla u(x)|^p\, w(x)\, dx. \tag{4.3}$$

On the other hand, considering the spherical caps

$$K_t = \mathbb{B}^n \cap S^{n-1}(1,t) \text{ for } t \in \big(|1-b_k| - \frac{1}{8}(1-|b_k|), |1-b_k|\big),$$

and the nonempty disjoint sets in K_t,

$$E_t = S^{n-1}(1,t) \cap E$$

and

$$F_t = S^{n-1}(1,t) \cap F,$$

we have, by (2.8),

$$M_p^w\big(\Delta(E,F;H_k)\big) \geq c\ (1-|b_k|)^{-p} \int_{\mathbb{B}^n(1,|1-b_k|)} w(x)\,d\,x.$$

Since the sequence $\{b_k\}$ is in the Stolz cone $C(\mathbf{1})$, we have by (2.1)

$$M_p^w\big(\Delta(E,F;H_k)\big) \geq c\ (|1-b_k|)^{-p} \int_{\mathbb{B}^n(1,|1-b_k|)} w(x)\,dx. \tag{4.4}$$

Combining (4.3) and (4.4) gives

$$\begin{aligned} c\ (|1-b_k|)^{-p} \int_{\mathbb{B}^n(1,|1-b_k|)} w(x)\,dx &\leq \frac{c(p)}{(|\beta|-|\alpha|)^p} \int_{H_k} |\nabla u(x)|^p\, w(x)\,dx \\ &\leq \frac{c(p)}{(|\beta|-|\alpha|)^p} \int_{\mathbb{B}^n(\mathbf{1},|1-b_k|)} |\nabla u(x)|^p\, w(x)\,dx. \end{aligned}$$

Therefore we have the inequality

$$0 < c \leq \frac{c(p)}{(|\beta|-|\alpha|)^p}\ \frac{(|1-b_k|)^p}{\int_{\mathbb{B}^n(1,|1-b_k|)} w(x)\,dx}\ \int_{\mathbb{B}^n(\mathbf{1},|1-b_k|)} |\nabla u(x)|^p\, w(x)\,dx,$$

where again c denotes constants depending only on n, p, q and the constant A_p of the weight w. Since, by assumption, $\mathbf{1} \notin U$, the right hand side of the above inequality tends to zero as k goes to ∞. This gives a contradiction. $\square$

Proof of Theorem 3. An elementary argument shows that the exceptional set E is a Borel set. Thus, to show that E has variational (p,w)-capacity zero it is enough to show that any compact subset K of E has variational (p,w)-capacity zero. Fix such a set K. Given an arbitrary $\epsilon >$, let U be an open set in $\mathbb{R}^n$ given by Theorem 2. Let γ be any rectifiable curve in $\mathbb{B}^n$ ending at a point in $(\partial\mathbb{B}^n \setminus U) \cap K$. Since the nontangential limit does not exist at any point in K, Theorem 2 implies that the limit through γ does not exist either. Therefore we have

$$\int_\gamma |\nabla u(x)|\,ds = \infty. \tag{4.5}$$

Let $\Gamma_{\mathbb{B}^n}((\partial\mathbb{B}^n \setminus U) \cap K)$ be the family of all the rectifiable curves in $\mathbb{B}^n$ ending at $(\partial\mathbb{B}^n \setminus U) \cap K$. We claim that

$$M_p^w(\Gamma_{\mathbb{B}^n}((\partial\mathbb{B}^n \setminus U) \cap K)) = 0. \tag{4.6}$$

To see this, we choose the metric in $\mathbb{B}^n$ as $\rho_\eta(x) = \eta|\nabla u(x)|$. It follows from (4.5) that, for any positive η, the metric ρ_η is admissible for our module problem. Hence we have

$$M_p^w\Big(\Gamma_{\mathbb{B}^n}\big((\partial\mathbb{B}^n \setminus U) \cap K\big)\Big) \le \eta^p \int_{\mathbb{B}^n} |\nabla u(x)|^p\, w(x)\, dx < \eta^p (\|u\|^w_{,p})^p.$$

Letting $\eta \to 0$ we obtain

$$M_p^w\Big(\Gamma_{\mathbb{B}^n}\big((\partial\mathbb{B}^n \setminus U) \cap K\big)\Big) = 0.$$

Next, we use a weighted version of a symmetrization result of Vuorinen for weighted modules with symmetric weights (Lemma 4.7 below) to show that if

$$M_p^w\Big(\Gamma_{\mathbb{B}^n}\big((\partial\mathbb{B}^n \setminus U) \cap K\big)\Big) = 0,$$

then, in fact,

$$M_p^w\Big(\Gamma_{\mathbb{R}^n}\big((\partial\mathbb{B}^n \setminus U) \cap K\big)\Big) = 0,$$

where $\Gamma_{\mathbb{R}^n}\big((\partial\mathbb{B}^n \setminus U) \cap K\big)$ is the family of rectifiable curves in $\mathbb{R}^n$ that intersect $(\partial\mathbb{B}^n \setminus U) \cap K$. By the weighted version of Ziemer's theorem (see [HK]), it follows that $\operatorname{cap}_{p,w}\big((\partial\mathbb{B}^n \setminus U) \cap K\big) = 0$. Finally, by the subadditivity of the weighted variational capacities we have

$$\operatorname{cap}_{p,w}(K) \le \operatorname{cap}_{p,w}\big((\partial\mathbb{B}^n \setminus U) \cap K\big) + \operatorname{cap}_{p,w}(U) < \epsilon.$$

Letting $\epsilon \to 0$, we obtain that $\operatorname{cap}_{p,w}(K) = 0$. $\square$

Next we state and prove a weighted version of the symmetrization lemmas 4.2 and 4.3 in [V]. Let E be a subset of $\partial\mathbb{B}^n$. Let $\Gamma_{\mathbb{R}^n}$ be the family of rectifiable curves in $\mathbb{R}^n$ which intersect E, $\Gamma_{\mathbb{B}^n}$ the family of rectifiable curves in $\mathbb{B}^n$ ending in E, and $\Gamma_{\overline{\mathbb{B}}^n}$ the family of rectifiable curves in $\overline{\mathbb{B}}^n$ that intersect E.

Lemma 4.7. *Let w be a symmetric weight with respect to $\partial\mathbb{B}^n$ in the sense of Definition 1.3. If*

$$M_p^w\left(\Gamma_{\mathbb{B}^n}\right) = 0,$$

then

$$M_p^w\left(\Gamma_{\mathbb{R}^n}\right) = 0.$$

Proof. We proceed in several steps.

Step 1. $M_p^w\left(\Gamma_{\mathbb{B}^n}\right) = 0$ *implies* $M_p^w\left(\Gamma_{\overline{\mathbb{B}}^n}\right) = 0$.

The proof of this step follows the exact same argument as the proof of Lemma 4.3 in [V], since the result of Ziemer, [Z, Lemma 2.3], also holds for the weighted moduli of increasing path families as it can be easily seen.

Step 2. Let R be the annulus $\{1 < |x| < 2\}$ and Γ_R the family of rectifiable curves in R that intersect E. Then $M_p^w(\Gamma_R) = 0$ implies $M_p^w(\Gamma_{\mathbb{R}^n}) = 0$.

Let γ be an arbitrary curve in $\Gamma_{\mathbb{R}^n}$ and let ρ be an admissible metric for the module problem $M_p^w(\Gamma_R)$. Extend the metric ρ to $\mathbb{R}^n$ by setting it equal to zero in $\mathbb{R}^n \setminus R$. It is clear that γ either belongs to Γ_R or contains a subcurve $\tilde{\gamma}$ which is in Γ_R. In any case, we have

$$\int_\gamma \rho\, ds \geq \int_{\tilde{\gamma}} \rho\, ds \geq 1.$$

This shows that the metric ρ is also admissible for the module problem $M_p^w(\Gamma_{\mathbb{R}^n})$, and thus

$$M_p^w(\Gamma_{\mathbb{R}^n}) \leq \int_R \rho(x)^p\ w(x)\ dx$$

for any admissible ρ. Taking the infimum and using the hypothesis $M_p^w(\Gamma_R) = 0$, it follows that $M_p^w(\Gamma_{\mathbb{R}^n}) = 0$.

Step 3. $M_p^w\left(\Gamma_{\overline{\mathbb{B}}^n}\right) = 0$ implies $M_p^w\left(\Gamma_{R\cap\overline{\mathbb{B}}^n}\right) = 0$.

As usual, we have denoted by $\Gamma_{R\cap\overline{\mathbb{B}}^n}$ the family of all rectifiable curves in $R \cap \overline{\mathbb{B}}^n$ which intersect E. The assertion follows immediately from the monotonicity of the modulus since $\Gamma_{R\cap\overline{\mathbb{B}}^n} \subset \Gamma_{\overline{\mathbb{B}}^n}$.

Step 4. $M_p^w(\Gamma_{R\cap\overline{\mathbb{B}}^n}) = 0$ implies $M_p^w(\Gamma_R) = 0$.

Let ρ be an admissible metric for the module problem $M_p^w\left(\Gamma_{R\cap\overline{\mathbb{B}}^n}\right)$. Define

$$\tilde{\rho}(x) = \begin{cases} \rho(x) & \text{for} \quad |x| \leq 1, \\ \rho(H^{-1}(x)) & \text{for} \quad |x| > 1, \end{cases}$$

where H is the bi-Lipschitz homeomorphism given in Definition 1.3. We will show the existence of a constant δ depending only on the mapping H and the constant in Definition 1.3 such that $\delta\,\tilde{\rho}$ is admissible for the module problem $M_p^w(\Gamma_R)$. To see this, decompose γ in Γ_R as $\gamma = \gamma_{R\cap\overline{\mathbb{B}}^n} \cup \gamma_{R\setminus\overline{\mathbb{B}}^n}$, where $\gamma_{R\cap\overline{\mathbb{B}}^n} \in R \cap \overline{\mathbb{B}}^n$ and $\gamma_{R\setminus\overline{\mathbb{B}}^n} \in R \setminus \overline{\mathbb{B}}^n$. We have

$$\int_\gamma \tilde{\rho}\, ds = \int_{\gamma_{R\cap\overline{\mathbb{B}}^n}} \rho\, ds + \int_{\gamma_{R\setminus\overline{\mathbb{B}}^n}} \tilde{\rho}\, ds = \mathrm{I} + \mathrm{II}\,.$$

Changing variables in the second line integral II, we obtain

$$\mathrm{II} \geq \beta_H \int_{H^{-1}\circ\gamma_{R\setminus\overline{\mathbb{B}}^n}} \rho\, ds,$$

where β_H is a positive constant depending on H. Observe that $\gamma_{R\cap\overline{\mathbb{B}}^n} \cup H^{-1} \circ \gamma_{R\setminus\overline{\mathbb{B}}^n}$ is a curve in $\Gamma_{R\cap\overline{\mathbb{B}}^n}$. Thus, we have the inequality

$$\int_\gamma \tilde{\rho}\, ds \geq \min\{1, \beta_H\} \int_{\gamma_{R\cap\overline{\mathbb{B}^n}} \cup H^{-1}\circ\gamma_{R\setminus\overline{\mathbb{B}}^n}} \rho\, ds.$$

Hence, by taking $\delta^{-1} = \min\{1, \beta_H\}$, we see that the metric $\delta\tilde{\rho}$ is admissible for the module problem $M_p^w(\Gamma_R)$. We now compute

$$\begin{aligned} M_p^w(\Gamma_R) \leq & \delta^p \int_R \tilde{\rho}^p(x) w(x)\, dx \\ & = \delta^p \left(\int_{R\cap\overline{\mathbb{B}}^n} \rho^p(x) w(x)\, dx + \int_{R\setminus\overline{\mathbb{B}}^n} \rho^p(H^{-1}(x)) w(x)\, dx \right) \\ & \leq \delta^p \left(\int_{R\cap\overline{\mathbb{B}}^n} \rho^p(x) w(x)\, dx + c_H \int_{R\cap\overline{\mathbb{B}^n}} \rho^p(y) w(H(y))\, dy \right) \\ & \leq \delta^p (1 + c_H\gamma) \left(\int_{R\cap\overline{\mathbb{B}}^n} \rho^p(x) w(x)\, dx \right), \end{aligned}$$

where we have changed variables $H^{-1}(x) = y$ on the second integral above.

Since the admissible metric ρ is arbitrary, we obtain

$$M_p^w(\Gamma_R) \leq \delta^p\ (1 + c_H\gamma) M_p^w\left(\Gamma_{R\cap\overline{\mathbb{B}}^n}\right),$$

which gives the desired result. □

5. On a theorem of Carleson

In this section we consider the case of power radial weights w that vanish on $\partial\mathbb{B}^n$. These weights are interesting since $w(\partial\mathbb{B}^n) = 0$, but $\mathrm{cap}_{p,w}(\partial\mathbb{B}^n) > 0$ for $p > \alpha + 1$.

Lemma 5.1. *Let $w(x) = |1 - |x||^\alpha$ for $x \in \mathbb{R}^n$. Then $w(x)$ is in $A_p(\mathbb{R}^n)$ whenever $p > \alpha + 1$.*

Proof. (We thank Eric Sawyer for suggesting this proof.) Since the integrals that appear in the definition of the A_p-weights are invariant under rotations for the weights under consideration, it is enough to show that

$$\sup_{\mathbb{B}} \left(\frac{1}{|\mathbb{B}|} \int_{\mathbb{B}} w(x)\, dx \right) \left(\frac{1}{|\mathbb{B}|} \int_{\mathbb{B}} [w(x)]^{\frac{1}{1-p}}\, dx \right)^{p-1} < \infty,$$

where $\mathbb{B}$ is any ball in $\mathbb{R}^n$ whose center lies on the positive real axis.

Using polar coordinates we have

$$\begin{aligned} & \left(\frac{1}{|\mathbb{B}^n(x_0, r)|} \int_{\mathbb{B}^n(x_0,r)} w(x)\, dx \right) \left(\frac{1}{|\mathbb{B}^n(x_0, r)|} \int_{\mathbb{B}^n(x_0,r)} [w(x)]^{\frac{1}{1-p}}\, dx \right)^{p-1} \\ & = \frac{c}{r^{np}} \left(\int_I \int_{\mathbb{B}^n(x_0,r)\cap S_s^{n-1}} |1 - s|^\alpha\, dS\, ds \right) \\ & \qquad \times \left(\int_I \int_{\mathbb{B}^n(x_0,r)\cap S_s^{n-1}} |1 - s|^{\frac{\alpha}{1-p}}\, dS\, ds \right)^{p-1}, \end{aligned}$$

where I is an interval of length equivalent to r and c is a constant that depends only on n. Since

$$\int_{\mathbb{B}^n(x_0,r)\cap S_s^{n-1}} dS \leq c\ r^{n-1}$$

the problem reduces to showing that, in dimension one, $|1-s|^\alpha$ is in $A_p(\mathbb{R})$ for $p > \alpha+1$. This, however, is a well known fact; see, for example, [HKM, Chapter 15]. $\square$

The following generalization to higher dimensions of a theorem of Carleson (see [C]) follows from Theorem 3 and Lemma 5.1.

THEOREM 5.2. *Let u be a monotone function in the Sobolev space*

$$W^{1,p}(\mathbb{B}^n;(1-|x|)^\alpha),$$

where $n-1 < p \leq n$ and $(p-(n-1))/(n-1) > \alpha \geq 0$. Let

$$E = \{x \in \partial\mathbb{B}^n : \text{ the nontangential limit of } u \text{ at } x \text{ does not exist }\}.$$

Then we have that $\operatorname{cap}_{p,w}(E) = 0$ and the Hausdorff dimension of the set E is less than or equal to $\alpha+n-p$.

REMARK 1. Since $\alpha+n-p < 1$, the set E is quite small on the boundary of the unit ball of $\mathbb{R}^n$.

REMARK 2. To obtain the existence of nontangential limits on a large part of the boundary it would be enough to prove that the Hausdorff dimension of E is at most $n-1$. It is therefore natural to conjecture that the theorem holds for $\alpha < p-1$. Note that, in two dimensions, this is indeed the case.

Proof of Theorem 5.2. All that remains to be done is to show that the set E has Hausdorff dimension at most $\alpha+n-p$. By a standard dimension estimate for sets of weighted capacity zero (see [HKM, Theorem 2.23]) this Hausdorff dimension can be shown to be smaller than one. However, we present here a somewhat more elaborate argument that allows us to obtain the bound $\alpha+n-p$.

We start with the weak type estimate

$$\Lambda_s^{w,\infty}\Bigg(\{y \in \mathbb{R}^n : M_{s,p}^w f(y) > t\}\Bigg) \leq \frac{c(s,w)}{t^p}\int_{\mathbb{R}^n} |f(x)|^p w(x)dx, \tag{5.3}$$

which follows from the proof of Lemma 3.1. Let u be a function in $C_0^\infty(\mathbb{B}(y,R))$. As in [HKM, Lemma 2.30], we have

$$|u(y)| \leq c\int_0^\infty \frac{1}{r^n}\int_{\mathbb{B}^n(y,r)} |\nabla u(x)|\,dx\,dr.$$

Inserting $w(x)^{1/p}\ w(x)^{-1/p}$ into the integral and applying Hölder's inequality and the A_p condition for the weight w one can easily adapt the proof of [HKM, Lemma 2.30] to obtain

$$|u(x)| \le c\, R^{1+s/p}\, M^w_{s,p}[|\nabla u|(x)] \tag{5.4}$$

for any $x \in \mathbb{R}^n$, as long as $s > -p$.

Combining (5.3) and (5.4) we have

$$\Lambda_s^{w,\infty}\Bigg(\{x \in \mathbb{B}^n(y,R) \colon |u(y)| > t\}\Bigg) \le c\, \frac{R^{s+p}}{t^p} \int_{\mathbb{B}^n(y,R)} |\nabla u(x)|^p\, w(x)\, dx. \tag{5.5}$$

The proof of [HKM, Theorem 2.26] can be carried out in the weighted case by using the estimate (5.5). Thus, $\mathrm{cap}_{p,w}(E) = 0$ implies that $\Lambda_s^{w,\infty}(E) = 0$ for all $s > -p$.

It follows that given any positive ϵ one can cover the set $E \subset \bigcup_i \mathbb{B}^n(x_i, r_i)$, such that

$$\sum_i r_i^s\, w(\mathbb{B}^n(x_i, r_i)) < \epsilon.$$

Note that the centers of the balls x_i may be taken on $\partial\mathbb{B}^n$. In view of the special nature of our weight we have $w(\mathbb{B}^n(x_i, r_i)) \approx r_i^{n+\alpha}$. Therefore,

$$\sum_i r_i^{n+\alpha+s} < c\,\epsilon.$$

Using, for example, [HKM, Lemma 2.25], we conclude that the Hausdorff dimension of the set E is at most $n + \alpha + s$ for any $s > -p$. □

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