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A SPECIAL THIN TYPE

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Abstract. We answer a question of Pillay and Ziegler and construct a type of U-rank 1 (of the theory of separably closed fields) which is thin but not very thin.

1. Introduction

Fix a prime number p. Let K be a field of characteristic p. We say that $B \subset K$ is p-independent if the set of all p-monomials, i.e., the monomials of the form $b_1^{i_1} \dots b_n^{i_n}$ with $b_j \in B$ and $0 \leq i_j \leq p-1$, is linearly independent in the K^p-vector space K. A maximal p-independent subset B of K is called a p-basis. For results on p-bases and p-independence, we refer to [4].

Let $\mathcal{L} = \{+, -, \cdot, 0, 1\}$ be the language of fields. For $e \in \omega \cup \{\infty\}$ we denote by SCF_e the $\mathcal{L}$-theory of separably closed fields of characteristic p and degree of imperfection e (e is the cardinality of a p-basis). For any e, SCF_e is complete [6]. Throughout this paper we assume that $e > 0$.

We define the so-called λ-functions $\lambda_{i,n}$, $n \in \omega, i \in p^n$, as follows. For an n-tuple $\overline{y}$ and an element x we set $\lambda_{i,n}(\overline{y}, x) = 0$ if $\overline{y}$ is p-dependent or the $(n+1)$-tuple $\overline{y}, x$ is p-independent; otherwise, we define $\lambda_{i,n}(\overline{y}, x)$ by

$$x = \sum_{i \in p^n} \lambda_{i,n}(\overline{y}, x)^p y_0^{i(0)} \dots y_{n-1}^{i(n-1)}.$$

Consider $\mathcal{L}_\lambda = \mathcal{L} \cup \{\lambda_{i,n} : n \in \omega, i \in p^n\}$, and let $SCF_{e,\lambda}$ be the expansion of SCF_e obtained by adding the axioms defining the λ-functions. Then $SCF_{e,\lambda}$ is complete and has quantifier elimination [3].

Fix a separably closed field K of degree of imperfection e. Let L be a monster model of $SCF_{e,\lambda}$ containing K (i.e., a very large separably closed field containing K, separable over K and of the same degree of imperfection). For any tuple $\overline{c} \in L$ we denote by $K\langle \overline{c} \rangle$ the prime model over $K \cup \{\overline{c}\}$, that is,

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the separable closure of the smallest subfield of L containing $K(\overline{c})$ and which is closed under the λ-functions.

Fix some p-independent elements $b_0, \dots, b_{n-1}$ of K. For $i \in p^n$ we define $\lambda_i(x) = \lambda_{i,n}(b_0, \dots, b_{n-1}, x)$. When $\eta = (i_0, \dots, i_{t-1}) \in (p^n)^t$ we define λ_η as the composition $\lambda_{i_{t-1}} \circ \cdots \circ \lambda_{i_0}$. By quantifier elimination, if $a \in L^{p^m}[b_0, \dots, b_{n-1}]$ for all $m \in \omega$, then $q = \mathrm{tp}(a/K)$ is determined by the $\mathcal{L}$-isomorphism type over K of the infinite tuple $(\lambda_\eta(a) : \eta \in (p^n)^{<\omega})$. In the definitions below we assume that $a \in L^{p^m}[b_0, \dots, b_{n-1}]$ for all $m \in \omega$.

Definition 1.1.

(i) A type $\mathrm{tp}(a/K)$ is thin if the transcendence degree of $K(\lambda_\eta(a) : \eta \in (p^n)^{<\omega})$ over K is finite.

(ii) A type $\mathrm{tp}(a/K)$ is very thin if $F = K(\lambda_\eta(a)^{p^{|\eta|}} : \eta \in (p^n)^{<\omega})$ is finitely separably generated over K, that is, there is a finite tuple $\overline{c}$ from F such that F is separably algebraic over $K(\overline{c})$.

The above definition is formulated in terms of λ-functions and is different from the definition which is given in terms of Hasse derivations in [9]. To see that both definitions are equivalent it is enough to use a suitable Wronskian, as in Lemma 4.3 of [10].

If $q = \mathrm{tp}(a/K)$ we denote by $\mathrm{trdeg}(q)$ (the transcendence degree of q) the transcendence degree of $K(\lambda_\eta(a) : \eta \in (p^n)^{<\omega})$ over K. Recall that the U-rank of a thin type is less than or equal to its transcendence degree. We do not define U-rank in general, but, for readers who are not experts in model theory, we give the following equivalent definition of U-rank 1 types (i.e., minimal types) for types realised in L:

Definition 1.2. A type $\mathrm{tp}(a/K)$ is of U-rank 1 if $a \notin K$ and for any $\overline{c} \in L$ the following holds: either $K(\lambda_\eta(a) : \eta \in (p^n)^{<\omega})$ is algebraically independent from $K\langle\overline{c}\rangle$ over K, or $a \in K\langle\overline{c}\rangle$.

In [9] the authors give a new proof of the Mordell-Lang conjecture in the characteristic 0 case. This proof eliminates the most complicated part of Hrushovski's proof [7], namely Zariski geometries [8]. In positive characteristic, Hrushovski's proof is the only known proof. Under the additional assumption that generic types of some particular type-definable groups (in a model of SCF_e) are very thin, Pillay and Ziegler are able to extend their result to the positive characteristic case (see Question 3.3). This motivates the question whether there exists a type which is thin but not very thin. It is easy to give an example of such a type of U-rank 2 [9]. Thus in [9] the following question is formulated.

($*$) Does there exist a type of U-rank 1 which is thin but not very thin?

In this paper we answer this question in the affirmative. More precisely, in Section 1, we give a construction of a thin but not very thin type of U-rank

1 and of arbitrary transcendence degree > 1, using the construction from [5]. Moreover, it turns out that our type can be chosen as the generic type of a connected type-definable subgroup of $(L, +)$. In Section 2, we describe a simpler example of a thin but not very thin type of U-rank 1 and of transcendence degree 2.

2. The main example

To get our example we will modify the construction of a type of U-rank 1 and of arbitrary transcendence degree given by Chatzidakis and Wood [5]. More precisely, after each step of their construction we will add a new step to make the arising type not very thin.

We recall once again that L is a large model of $SCF_{e,\lambda}$ for $e > 0$ and K is a submodel of L. We fix an element $u \in K \setminus K^p$. We take the notation for λ-functions from the introduction for $\{b_0, \dots, b_{n-1}\} = \{u\}$. Moreover, for $\eta \in p^{<\omega}$ and $a \in L$ we define $a_\eta = \lambda_\eta(a)$ and $a_{\leq m} = (a_\sigma)_{\sigma \in p^{\leq m}}$. If we have a sequence $a_0, \dots, a_d$ of elements (or variables) of L, then $a_{i,\eta}$ denotes $\lambda_\eta(a_i)$. We will construct a type $q = \mathrm{tp}(a/K) \in S_1(K)$ satisfying our demands and such that for all $n \in \omega$ we have $a \in L^{p^n}[u]$. So q is determined by the quantifier-free $\mathcal{L}$-type over K of the infinite tuple $(a_\eta : \eta \in p^{<\omega})$. The type q is called additive if $a + b$ realise q for all realisations a and b of q such that $K\langle a\rangle$ and $K\langle b\rangle$ are algebraically independent over K. In [2] one can also find the result that q is additive iff q is a generic type of a connected type-definable subgroup of $(L, +)$.

For a partial type $p = p(x_1, \dots, x_k)$ over K such that for any realisation $a_1, \dots, a_k$ in L, $a_i \in L^{p^m}[u]$ for $1 \leq i \leq k$, we denote by $p \restriction_m$ the quantifier-free $\mathcal{L}$-type over K in the variables $(x_{1,\leq m}, \dots, x_{k,\leq m})$ implied by p. We say that such a type p is complete up to level m if whenever $a_1, \dots, a_k$ and $b_1, \dots, b_k$ realise p in L, then the tuples $(a_{i,\leq m} : 1 \leq i \leq k)$ and $(b_{i,\leq m} : 1 \leq i \leq k)$ have the same quantifier-free $\mathcal{L}$-type over K. If $p = p \restriction_m$ is complete up to level m, we denote by $\mathrm{trdeg}(p)$ the transcendence degree of $K(a_{\leq m})$ over K for any realisation a of p and we say that p is additive if $a + b$ realise q for every realisations a and b of q such that $K(a_{\leq m})$ and $K(b_{\leq m})$ are algebraically independent over K.

First, we recall the following lemma [4, Lemma 2.5].

Lemma 2.1. *If $\overline{z}$ is a tuple from L such that $\overline{z} \subseteq L^{p^n}[u]$ and y is separably algebraic over $K(\overline{z})$, then $y \in L^{p^n}[u]$ and $y_{\leq n} \subseteq K(\overline{z}_{\leq n}, y)$.*

We will also need the following lemma.

Lemma 2.2. *Let $p = p \restriction_m$ be a 1-type over K which is complete up to level m, and assume that $\sigma(1), \dots, \sigma(k) \in p^m$ are such that if $a \models p$, then $\{a_{\sigma(1)}, \dots, a_{\sigma(k)}\}$ is a separating transcendence basis of $K(a_{\leq m})$ over K. Let*

$q = q \restriction_n$ be a k-type over K which is complete up to level n and which is realisable by a k-tuple of elements that are algebraically independent over K. Then the partial type

$$r(x) = p(x) \cup q(x_{\sigma(1)}, \ldots, x_{\sigma(k)})$$

is consistent, and

- $a \in L^{p^{m+n}}[u]$,
- $\mathrm{trdeg}(\mathrm{tp}(a/K) \restriction_{m+n}) = \mathrm{trdeg}(q)$,

for any realisation a of r.

Moreover, if p and q are additive, there is a realisation a of r such that $\mathrm{tp}(a/K) \restriction_{m+n}$ *is additive.*

Proof. The first part of the proof is exactly the proof of [5, Lemma 3]. Let $b_1, \ldots, b_k \in L$ realise q. Choose $a_{\leq m}$ (in some algebraic closure of L) such that $a_{\sigma(j)} = b_j$ for $j = 1, \ldots, k$, and such that for any c realising p the tuples $a_{\leq m}$ and $c_{\leq m}$ have the same quantifier-free $\mathcal{L}$-type over K.

Then $a_{\leq m}$ is separably algebraic over $K(b_1, \ldots, b_k)$, and therefore is in L. By Lemma 2.1, it follows that $a \in L^{p^{m+n}}[u]$ and $\mathrm{trdeg}(\mathrm{tp}(a/K) \restriction_{m+n}) = \mathrm{trdeg}(q)$.

Now, if p and q are additive, consider an algebraic closure L^a of L and the algebraic subgroup G of the cartesian power $(L^a, +)^{\times(p^{m+n+1}-1)/(p-1)}$ defined by all the equations over K from r on $x_{\leq m+n}$. Choose $a_{\leq m+n}$ realising the principal generic of G such that $a_{\sigma(j)} = b_j$ for $j = 1, \ldots, k$. Then $a \in L$ and $\mathrm{tp}(a/K) \restriction_{m+n}$ is additive. □

Remark 2.3. The type r is not necessarily complete up to level $m+n$. Let $p(x)$ be defined by $x = x^p_{\langle 0\rangle} + x^p_{\langle 1\rangle} u$, $x_{\langle 0\rangle} = x_{\langle 1\rangle} + x^p_{\langle 1\rangle} u$ and x transcendental over K; let $q(y)$ be defined by $y = y^p_{\langle 0\rangle} + y^{p^2}_{\langle 0\rangle} u$ and y transcendental over K. Then $p = p \restriction_1$ and $q = q \restriction_1$ are both complete up to level 1; if $a \models p$, then $\{a_{\langle 0\rangle}\}$ is a separating transcendence basis of $K(a_{\leq 1})$ over K; but $r(x) = p(x) \cup q(x_{\langle 0\rangle})$ is not complete up to level 2. In fact, for each $\alpha \in K$ such that $\alpha + \alpha^p u = 0$, the type

$$r_\alpha(x) = r(x) \cup \{x_{\langle 1\rangle} = x^p_{\langle 0,0\rangle} + \alpha\}$$

is consistent and complete up to level 2.

Now we can prove the main theorem.

Theorem 2.4. *For any $d \in \omega \setminus \{0, 1\}$ there is a thin but not very thin type $q \in S_1(K)$ of U-rank 1 and of transcendence degree d.*

Proof. Let $\overline{Y}$ be a countable infinite tuple of variables and let $X_\infty = \{X_\sigma : \sigma \in p^{<\omega}\}$ and $X_{\leq n} = \{X_\sigma : \sigma \in p^{\leq n}\}$. Fix an enumeration f_n, $n \in \omega$, of the elements of $\mathbb{F}_p[X_\infty, \overline{Y}]$ such that $f_n \in \mathbb{F}_p[X_{\leq n}, \overline{Y}]$ for every n.

We will build our type q by induction on n. More precisely, we construct types $q_n = q_n \restriction_{N(n)}$ complete up to level $N(n)$ and satisfying:

(1) $\operatorname{trdeg}(q_n) = d$ and q_n extends q_{n-1};
(2) for any tuple $\overline{c}$ from L and $a \models q_n$, if $f_{n-1}(a_{\leq n-1}, \overline{c}) = 0$, then either the polynomial $f_{n-1}(X_{\leq n-1}, \overline{c})$ vanishes on all realisations of q_n or $a \in K\langle\overline{c}\rangle$;
(3) there is a $\sigma \in p^{N(n)}$ such that for any $a \models q_n$, the element $a_\sigma^{p^{N(n)}}$ is inseparable over $K(a_\eta^{p^{|\eta|}} : \eta \in p^{<N(n)})$.

In [5] only (1) and (2) are required. Having a type q_n satisfying (1) and (2), the authors perform the induction step (consisting of two non-trivial steps) to get q_{n+1}. We are not going to repeat their proof here and the reader is referred to [5, Theorem 1].

Assume we have constructed types $q_0, \dots, q_{n-1}$ satisfying (1), (2) and (3). The construction given in [5] gives us a type $q'_n \supset q_{n-1}$, which is complete up to some level $N'(n)$ and satisfies (1) and (2). We will extend this type q'_n to a type q_n which is complete up to level $N(n) = N'(n) + 1$ and satisfies (1) and (3). Since $q_n \subset q'_n$, q_n will also satisfy (2).

Choose $\tau(1), \dots, \tau(d) \in p^{N'(n)}$ such that the set $\{a_{\tau(1)}, \dots, a_{\tau(d)}\}$ is a separating transcendence basis of $K(a_{\leq N'(n)})$ over K for $a \models q'_n$.

Consider the type $s = s \restriction_1$ in the variables $x_1, \dots, x_d$ consisting of the following formulas (as everywhere, we evaluate the λ-functions with respect to the p-independent set $\{u\}$):

(i) $x_1 = x_{1,\langle 0\rangle}^p$,
(ii) $x_2 = x_{2,\langle 0\rangle}^p + x_{2,\langle 1\rangle}^p u$,
(iii) $x_{1,\langle 0\rangle} = x_{2,\langle 0\rangle}^p$,
(iv) $x_i = x_{i,\langle 0\rangle}^p$, for $3 \leq i \leq d$,
(v) the elements $x_{2,\langle 0\rangle}, x_{2,\langle 1\rangle}, x_{3,\langle 0\rangle}, \dots, x_{d,\langle 0\rangle}$ are algebraically independent over K.

Of course s is consistent, complete up to level 1 and it implies that $x_1, \dots, x_d$ are algebraically independent over K. Hence by Lemma 2.2 we get that the type

$$r(x) := q'_n(x) \cup s(x_{\tau(1)}, \dots, x_{\tau(d)})$$

is consistent, and we can take $q_n = q_n \restriction_{N(n)}$ as an arbitrary completion of r to a type complete up to level $N(n)$. (In fact, in this particular case, we do not have to take here a completion of r, since r is already complete up to level $N(n)$.) Since $\operatorname{trdeg}(s) = d$, the type q_n satisfies (1). So to finish the construction it is enough to show the following claim.

Claim. *The type q_n satisfies (3).*

Proof. Note that for any m and any $a \in L^{p^m}[u]$ one has

$$K(a_\eta^{p^{|\eta|}} : \eta \in p^{\leq m}) = K(a_\eta^{p^m} : \eta \in p^m) = K(a_\eta^{p^m} : \eta \in p^{\leq m}).$$

If a satisfies q_n, then $\{a_{\tau(1)}, a_{\tau(2)}, a_{\tau(3)}, \dots, a_{\tau(d)}\}$ is a separating transcendence basis of $K(a_\eta : \eta \in p^{\leq N'(n)})$ over K, so that

$$\{a_{\tau(1)}^{p^{N'(n)}}, a_{\tau(2)}^{p^{N'(n)}}, a_{\tau(3)}^{p^{N'(n)}}, \dots, a_{\tau(d)}^{p^{N'(n)}}\}$$

is a separating transcendence basis of $K(a_\eta^{p^{|\eta|}} : \eta \in p^{\leq N'(n)})$ over K.

Now, $a_{\tau(2),\langle 0\rangle}^{p^2} = a_{\tau(1)}$, and this implies that $a_{\tau(2),\langle 0\rangle}^{p^{N'(n)+1}}$ is purely inseparable of degree p over $K(a_{\tau(1)}^{p^{N'(n)}}, a_{\tau(2)}^{p^{N'(n)}}, a_{\tau(3)}^{p^{N'(n)}}, \dots, a_{\tau(d)}^{p^{N'(n)}})$. □

Now we define

$$q = \bigcup_{n\in\omega} q_n.$$

Then one can easily check that q is of transcendence degree d by (1), has U-rank 1 by (2), and is not very thin by (3). □

Remark 2.5. If the types one starts from in the construction of [5] are additive, then by Lemma 2.2 we can assume that the types q'_n constructed in [5] are additive as well, and therefore also the types q_n constructed in Theorem 2.4. Then q is a generic type of a type definable connected subgroup of $(L, +)$. Recall that by [2, Lemma 2.7] it is indeed possible to start from additive types in this construction.

3. A simpler example and final comments

In the construction of [5], the authors use a compactness argument in order to obtain U-rank 1 types. Now, without using their construction, we give an explicit U-rank 1 type of transcendence degree 2. Moreover, by using the same idea as in the proof of Theorem 2.4, we obtain a not very thin type.

Example 3.1. We define a type $q = \mathrm{tp}(a/K)$ by describing the tree of the a_η's. This tree is described by the condition that there are sequences $(x_i)_{i\in\omega}$, $(y_i)_{i\in\omega}$, $(x'_i)_{i\in\omega}$, $(y'_i)_{i\in\omega}$, and $(z_i)_{i\in\omega}$ such that

(i) $a = x_0^p + y_0^p u$,

and for all $i \in \omega$:

(ii) We have

$$\begin{cases} x_i = ({x'_i}^{p^{2\cdot 8^{i+1}}})^p, \\ y_i = ({y'_i}^{p^{2\cdot 8^{i+1}}})^p + ({y'_i}^{p^{6\cdot 8^{i+1}+1}})^p u. \end{cases}$$

(iii) We have

$$\begin{cases} x_i' = (x_{i+1}^{p^2})^p + z_i^p u, \\ y_i' = z_i^p, \\ z_i = x_{i+1}^p + y_{i+1}^p u. \end{cases}$$

(iv) The elements x_i, y_i are algebraically independent over K.

Proof. The fact that q is not very thin follows from (iii) and (iv), in the same way as in Theorem 2.4. We prove that q has U-rank 1 by using (ii) and (iii).

Let a be a realisation of q in L. Note that $x_i = Q_i(x_{i+1}, y_{i+1})$ and $y_i = R_i(x_{i+1}, y_{i+1})$, where $Q_i, R_i \in K[X,Y]$, $\deg Q_i = p^{2\cdot 8^{i+1}+4}$ and $\deg R_i = p^{6\cdot 8^{i+1}+4}$, for all i. By an easy induction, $x_i = Q_{i,j}(x_j, y_j)$ and $y_i = R_{i,j}(x_j, y_j)$, where $Q_{i,j}, R_{i,j} \in K[X,Y]$, $\deg Q_{i,j} \leq p^{8^{j+1}}$ and $\deg R_{i,j} \leq p^{8^{j+1}}$, for all $j > i$.

Let $\bar{c} \in L$ be such that $K(a_\eta : \eta \in p^{<\omega})$ is not algebraically independent from $K' := K\langle \bar{c} \rangle$ over K. Then there are $i < \omega$ and $P \in K'[X,Y] \setminus \{0\}$ such that $P(x_i, y_i) = 0$. Let $j > i$ be such that $\deg P \leq p^{8^{j+1}}$ and consider $S(X,Y) := P(Q_{i,j}(X,Y), R_{i,j}(X,Y))$. Then $S(x_j, y_j) = 0$, $S \neq 0$ and $\deg S \leq p^{2\cdot 8^{j+1}}$.

Suppose that x_j and y_j are not in K'. By using the λ-functions, y_j is separably algebraic over $K'(x_{j,\leq 2\cdot 8^{j+1}})$ since $\deg S \leq p^{2\cdot 8^{j+1}}$. Now we repeat the arguments of the proof of [2, Lemma 2.7]. We have $y_{j,\leq 1} \subset K'(x_{j,\leq 2\cdot 8^{j+1}+1}, y_j)$ from Lemma 2.1. By an easy computation, it follows that

$$(1) \quad [K'(y_{j,\leq 1}) : K'(y_j)] \leq [K'(x_{j,\leq 2\cdot 8^{j+1}+1}) : K'(x_j)][K'(x_j, y_j) : K'(y_j)].$$

Moreover, since x_j and y_j are transcendental over K', $\deg S \leq p^{2\cdot 8^{j+1}}$,

$$x_j = {x_j'}^{p^{2\cdot 8^{j+1}+1}} \text{ and } y_j = \left({y_j'}^{p^{2\cdot 8^{j+1}}}\right)^p + \left({y_j'}^{p^{2\cdot 8^{j+1}}}\right)^{p^{4\cdot 8^{j+1}+2}} u,$$

one can deduce that

$$[K'(x_j, y_j) : K'(y_j)] \leq p^{2\cdot 8^{j+1}},$$
$$[K'(x_{j,\leq 2\cdot 8^{j+1}+1}) : K'(x_j)] = p^{2\cdot 8^{j+1}+1},$$
$$[K'(y_{j,\leq 1}) : K'(y_j)] = p^{4\cdot 8^{j+1}+2}.$$

This gives a contradiction with (1).

Thus x_j or y_j lies in K'. Then x_j' or y_j' lies also in K'. By (iii), if x_j' lies in K', then so do z_j and y_j'; if y_j' lies in K', then so do z_j, x_{j+1} and x_j'. We conclude that x_j and y_j are in K', and then $a \in K'$. □

We do not know if this example can be generalized to yield effective examples of thin types of arbitrary transcendence degree and of U-rank 1.

In the first version of this paper, we asked if there exists a type of transcendence degree 1 which is not very thin. This question has a negative answer. We include the following proof suggested by the referee. It should be noted that an independent proof of the same fact had also been communicated to us by F. Benoist [1].

FACT 3.2. *The types of transcendence degree 1 are very thin.*

Proof. Fix a p-independent tuple $\bar{b} = (b_0, \dots, b_{n-1})$ in K. We will first prove the following claim.

CLAIM. *If $a \in L^p[\bar{b}]$ and $K(a, \lambda_i(a) : i \in p^n)$ is of transcendence degree 1 over K, then $K(\lambda_i(a)^p : i \in p^n) \subset K(a)^s$ (= separable closure of $K(a)$).*

Proof. Let $m \geq 0$ be minimal such that all $\lambda_i(a)^{p^m}$ are separably algebraic over $K(a)$. If $m = 0$, then there is nothing to prove, so we will assume that $m > 0$. Fix $i \in p^n$ such that $\lambda_i(a)^{p^{m-1}}$ is not in $K(a)^s$, and let $f(X, Y)$ be an irreducible polynomial over K such that $f(\lambda_i(a)^{p^m}, a) = 0$. Since $K(a, \lambda_i(a)^{p^{m-1}})$ is a separable extension of K of transcendence degree 1, and by minimality of m, we have that $a \in K(\lambda_i(a)^{p^m})^s$, i.e., $\partial f/\partial Y(\lambda_i(a)^{p^m}, Y) \neq 0$.

Let D be any derivation on K. We can extend D to L since L is a separable extension of K. Then $K(\lambda_i(a)^{p^m})$ is closed under D, so that $K(\lambda_i(a)^{p^m})^s = K(a)^s$ is a differential subfield of L.

The proof of the claim will be an induction on n, and we will first consider the case $n = 1$. Let D be a derivation on K satisfying $D(b_0) = 1$, and extend D to L. We write $a = \sum_{j=0}^{p-1} \lambda_j(a)^p b_0^j$. Then $D(a) = \sum_{j=1}^{p-1} j\lambda_j(a)^p b_0^{j-1}$, $\dots$, $D^{p-1}(a) = (p-1)!\lambda_{p-1}(a)^p$.

Then $D(a), D^2(a), \dots, D^{p-1}(a) \in K(a)^s$, from which one obtains successively $\lambda_{p-1}(a)^p$, $\lambda_{p-2}(a)^p$,$\dots$,$\lambda_0(a)^p \in K(a)^s$. This gives $m = 1$ and finishes the proof when $n = 1$.

In the general case, write $a = \sum_{j=0}^{p-1} a_j b_0^j$, where each $a_j \in L^p(b_1, \dots, b_{n-1})$, and let D be a derivation on K satisfying $D(b_0) = 1$, $D(b_j) = 0$ if $j > 0$. Extend D to L. Then $D(a) = \sum_{j=1}^{p-1} j a_j b_0^{j-1}$, and as above we get that all a_j's are in $K(a)^s$. By the induction hypothesis applied to $a_j \in L^p(b_1, \dots, b_{n-1})$, we obtain that $K(\lambda_i(a_j)^p : i \in p^{n-1}) \subset K(a_j)^s \subset K(a)^s$, which shows that $m = 1$ and finishes the proof of the claim. □

Now, assume that a realises a type of transcendence degree 1 over K, and that $a \in L^{p^m}[\bar{b}]$ for every m. Then we check, by induction on m, that $\lambda_\eta(a)^{p^m} \in K(a)^s$ for $\eta \in (p^n)^m$, i.e., $\mathrm{tp}(a/K)$ is very thin.

Indeed, if $\eta \in (p^n)^m$, then $\lambda_\eta(a) \in L^p[\bar{b}]$, and either $K(\lambda_\eta(a), \lambda_i(\lambda_\eta(a)) : i \in p^n)$ is of transcendence degree 1 over K, or $\lambda_\eta(a) \in K$.

Hence, $K(\lambda_i(\lambda_\eta(a))^p : i \in p^n) \subset K(\lambda_\eta(a))^s$, so that $K(\lambda_i(\lambda_\eta(a))^{p^{m+1}} : i \in p^n) \subset K(\lambda_\eta(a)^{p^m})^s \subset K(a)^s$. □

In [2] there is a description of minimal groups in separably closed fields of finite degree of imperfection. From this description we have that each minimal group G orthogonal to L^{p^∞} is definably isomorphic either to a type-definable subgroup of $(L, +)$ or to $\bigcap_n p^n A(L)$ for some abelian variety A defined over L.

Example 3.1 is a thin but not very thin type of U-rank 1 which is a generic type of a type-definable subgroup of $(L, +)$. It is reasonable to consider the second part of the above dichotomy and to ask the following question.

QUESTION 3.3. *Does there exist a semiabelian variety A defined over K such that the generic type of $\bigcap_n p^n A(L)$ is not very thin?*

There is no reason to require here the type to be thin, since from [7] we know that the generic type of a group of the form $\bigcap_n p^n A(L)$ is always thin.

The importance of Question 3.3 is due to the fact that if the answer to this question is “no”, then the proof by Pillay and Ziegler of the Mordell-Lang conjecture [9] would work in the positive characteristic case.

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