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MODULUS OF CONTINUITY OF THE MAZUR MAP BETWEEN UNIT BALLS OF ORLICZ SPACES AND APPROXIMATION BY HÖLDER MAPPINGS

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ABSTRACT. Under some regularity assumptions, we compute the modulus of continuity of the generalized Mazur map between unit balls of Orlicz spaces. Our estimate coincides with the known estimates in the setting of $L_p(\mu)$-spaces. We apply this estimate to approximate uniformly continuous mappings between balls of reflexive Orlicz spaces by α-Hölder maps, with α as large as possible. We also relate this optimal value of α to the Boyd indices of the spaces and to the problem of isomorphic extension of Hölder maps.

1. Introduction

Concerning the uniform classification of the unit spheres of infinite-dimensional Banach spaces (see [2, p. 197]), the most general result known is due to F. Chaatit [3]. Quantitative versions of this result assert that if X and Y are separable infinite-dimensional Banach lattices which admit q- and q'-concavity constants $C_q(X) < \infty$ and $C_{q'}(Y) < \infty$ for some $q, q' < \infty$, then there exists a uniform homeomorphism between the unit spheres, $F : S(X) \to S(Y)$, such that F and F^{-1} have moduli of continuity which depend only on $q, q', C_q(X)$ and $C_{q'}(Y)$. Recall that the modulus of continuity ω_f of a map $f : (X, d_X) \to (Y, d_Y)$ between two metric spaces is the function $\omega_f(t) = \sup\{d_Y(f(x), f(y)) : x, y \in X \text{ and } d_X(x, y) \le t\}$. If $\omega_f(t) \le Ct^\alpha$ for some constant C and $\alpha \in (0, 1]$, we say that f is α-Hölder; if this holds with $\alpha = 1$, we say that f is Lipschitz.

Historically, as mentioned in [2], the earliest result in this setting has been obtained using the Mazur map (see [12]), which is an explicit uniform homeomorphism, with explicit modulus of continuity, between the unit spheres of different $L_p(\mu)$-spaces, for $p \ge 1$. More precisely, consider a measure μ and fix $1 \le p, q < \infty$. Suppose that $L_p(\mu)$ is infinite-dimensional. Define the Mazur map $\phi_{pq} : L_p(\mu) \to L_q(\mu)$ by $\phi_{pq}(x) = |x|^{p/q} \operatorname{sign}(x)$. Then ϕ_{pq} provides a

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uniform homeomorphism between the unit spheres such that $\phi_{pq}^{-1} = \phi_{qp}$ and ϕ_{pq} is Lipschitz on the unit sphere if $p \geq q$, and p/q-Hölder if $p \leq q$. If ν is another measure such that $L_p(\mu)$ and $L_q(\nu)$ have the same density character, then, as $L_2(\mu)$ and $L_2(\nu)$ are isometric, the Mazur map provides a uniform homeomorphism between the unit spheres of $L_p(\mu)$ and $L_q(\nu)$. Weston [17] remarked that the properties of the modulus of continuity of ϕ_{pq} remain valid when $0 < p, q \leq 1$.

The explicit knowledge of the modulus of continuity of ϕ_{pq} plays a key role in the proof of a result due to I. G. Tsaŕkov [16] (see also [2, p. 36]) on the uniform approximation of uniformly continuous mappings from the unit ball of $L_p(\mu)$ to $L_q(\nu)$ by α-Hölder mappings, with $\alpha \in (0, 1]$ as large as possible. We give a precise statement of this result in Section 3.

Our aim here is to obtain similar results in the setting of Orlicz spaces. Let M and N be two Orlicz functions and let μ be a measure. Following an idea of M. S. Kaczmarz [8] we consider the map $\phi_{MN} : L_M(\mu) \to L_N(\mu)$ defined by $\phi_{MN}(x) = N^{-1} \circ M(|x|)\,\mathrm{sign}(x)$. Under some regularity assumptions on $N^{-1} \circ M$, we obtain that ϕ_{MN} is again an explicit uniform homeomorphism between the closed unit balls, with explicit modulus of continuity. Moreover, as ϕ_{MN} maps the unit sphere of $L_M(\mu)$ onto the unit sphere of $L_N(\mu)$, it is in fact a uniform homeomorphism between the unit spheres.

E. Odell and T. Schlumprecht [15] gave a very general version of the Mazur map. Using our explicit form for maps between Orlicz spaces, we are able to compute directly the modulus of continuity of ϕ_{MN} without factorization through any other space. Thus we obtain a similar modulus of continuity as the one for ϕ_{pq} in the setting of L_p-spaces. In agreement with F. Chaatit [3], the moduli of continuity of ϕ_{MN} and $\phi_{MN}^{-1} = \phi_{NM}$ depend only on the p-convexity and the q-concavity of the spaces. The explicit knowledge of these moduli of continuity then leads to an approximation result in the setting of Orlicz spaces which can be related to the above-mentioned result of Tsaŕkov.

This paper is organized as follows. In Section 2, we recall some results about Orlicz spaces and we state our main result. Section 3 deals with the uniform approximation of uniformly continuous mappings between balls of Orlicz spaces. Section 4 is devoted to the problem of isomorphic extension of Hölder maps.

2. Orlicz spaces and statement of the main result

Orlicz spaces. We recall here some definitions and basic facts about Orlicz spaces. For further details, see [4] and [9].

Let $M : \mathbb{R} \to \mathbb{R}$ be such that

(i) M is even, convex and continuous,
(ii) $M(1) = 1$, and $M(u) = 0$ if and only if $u = 0$,
(iii) $M(u)/u \to 0$ as $u \to 0$ and $M(u)/u \to \infty$ as $u \to \infty$.

Then M is called an *Orlicz function.* Such a function admits a non-decreasing right derivative, which we denote by M'_r, and satisfies $M(u) = \int_0^u M'_r(t)dt$ for all $u \geq 0$.

Let (G, Σ, μ) be a measure space. For a μ-measurable function x on G we define its modulus by

$$\rho_M(x) = \int_G M(x(t))d\mu(t).$$

We define the *Orlicz space* by

$$L_M(G) = \{x : \rho_M(\lambda x) < \infty \text{ for some } \lambda > 0\}$$

and the *Luxemburg norm* by

$$\|x\|_M = \inf\{\lambda > 0 : \rho_M(x/\lambda) \leq 1\}.$$

The space $(L_M(G), \|\,.\,\|_M)$ is a Banach space.

We say that M satisfies condition Δ_2^0 (resp. Δ_2^∞), and we write $M \in \Delta_2^0$ (resp. $M \in \Delta_2^\infty$), if there exist $u_0 > 0$ and $K > 1$ such that $M(2u) \leq KM(u)$ for all $0 \leq u \leq u_0$ (resp. $u \geq u_0$). If there exists $K \geq 1$ such that $M(2u) \leq KM(u)$ for all $u \geq 0$, then $M \in \Delta_2^0$ and $M \in \Delta_2^\infty$ and we say that M satisfies condition Δ_2 *everywhere*.

Recall that, by the definition of the norm and by Fatou's theorem, we have $\rho_M(x/\|x\|_M) \leq 1$ if $\|x\|_M > 0$. The equality sign occurs when M satisfies the appropriate Δ_2 condition, but this no longer holds if $M \notin \Delta_2$ (see [9, p. 78]).

Two Orlicz functions M_1 and M_2 are equivalent at 0 (resp. at ∞) if there exist constants $C \geq 1$ and $u_0 > 0$ such that $C^{-1}M_2(u) \leq M_1(u) \leq CM_2(u)$ for all $0 \leq u \leq u_0$ (resp. $u \geq u_0$). If M_1 and M_2 are equivalent at 0 and at ∞, we say that M_1 and M_2 are equivalent *everywhere*.

We are mainly interested in the following three cases: $G = [0, 1]$ or $G = (0, \infty)$ with μ the Lebesgue measure on G, or $G = \mathbb{N}$ with μ the counting measure. The study of $L_M[0, 1]$ is associated with Δ_2^∞ and with the equivalence at ∞; that of $L_M(\mathbb{N})$ is associated with Δ_2^0 and the equivalence at 0; similarly, the study of $L_M(0, \infty)$ is related to Δ_2 *everywhere* and to the equivalence *everywhere*. In the following, G will always be one of the above three sets. We denote by Δ_2^G the corresponding Δ_2 condition, and by $\sim_G$ the corresponding equivalence of Orlicz functions. A set $I \subset \mathbb{R}$ is called *corresponding to the Orlicz spaces built on* G if, when $G = [0, 1]$ (resp. $G = \mathbb{N}$), there exists $u_0 > 0$ such that $I = (u_0, \infty)$ (resp. $I = (0, u_0)$), and, when $G = (0, \infty)$, there exist $u_0, u_1 > 0$ such that $I = (0, u_0) \cup (u_1, \infty)$.

When $M \in \Delta_2^G$, we have

$$L_M(G) = \{x : \rho_M(\lambda x) < \infty \text{ for all } \lambda > 0\}.$$

If $M_1 \sim_G M_2$, then $L_{M_1}(G)$ and $L_{M_2}(G)$ are isomorphic and the identity map is an isomorphism.

We denote by $M^*(v) = \sup\{uv - M(u) : u \in \mathbb{R}\}$ the complementary function of M. The function M^* is also an Orlicz function. When the spaces are endowed with the Luxemburg norm, we have the following version of the Hölder inequality: For all $x \in L_M(G)$ and $y \in L_{M^*}(G)$ we have

$$\int_G x(t)y(t)d\mu(t) \leq 2\|x\|_M \|y\|_{M^*}.$$

The constant 2 appears here because the usual Hölder inequality uses another, equivalent, norm on $L_M(G)$ (see [9, p. 80]).

It is known that $L_M(G)$ is reflexive if and only if $M \in \Delta_2^G$ and $M^* \in \Delta_2^G$. We denote by $M \in \Delta_2^G \cap \nabla_2^G$ this last condition.

We recall next the notions of p-convexity and p-concavity; we refer to [6], [7] and [10] for more details.

Let $(X, \|\,.\,\|)$ be a Banach lattice and let $1 \leq p < \infty$. We say that X is p-convex if there exists a constant $C^p(X) < \infty$ so that for every choice of vectors $\{x_i\}_{i=1}^n$ we have

$$\left\| \left(\sum_{i=1}^n |x_i|^p \right)^{1/p} \right\| \leq C^p(X) \left(\sum_{i=1}^n \|x_i\|^p \right)^{1/p}.$$

We say that X is p-concave if there exists a constant $C_p(X)$ so that the reverse inequality holds.

The following notions can be found in [6, p. 88]. Note that the setting of [6] is more general than our definition of *Orlicz functions*. We call *quasi-Orlicz function* every non-decreasing continuous function φ defined for $u \geq 0$ such that $\varphi(0) = 0$ and $\varphi(1) = 1$. In [6] such functions are called *Orlicz functions*. Here we work with Banach spaces $L_M(G)$ with convex functions M.

Given $0 < \alpha \leq \beta < \infty$, a quasi-Orlicz function φ is said to be α-convex (resp. β-concave) if $\varphi(u^{1/\alpha})$ is a convex function (resp. if $\varphi(u^{1/\beta})$ is a concave function) of $u \geq 0$.

As stated in [7, p. 168], for an Orlicz function M, the Banach lattice $L_M(G)$ is p-convex (resp. q-concave) if and only if there exists a p-convex (resp. q-concave) Orlicz function $\tilde{M}$ such that $M \sim_G \tilde{M}$. This occurs if and only if $M \in \Delta_2^G$ (resp. $M^* \in \Delta_2^G$).

Definition 2.1. Let $0 < \alpha < \beta < \infty$ and $G \in \{[0,1], (0,\infty), \mathbb{N}\}$. A quasi-Orlicz function φ is said to be in the class $\mathcal{K}(\alpha, \beta)$ if $\varphi(u)/u^\alpha$ is a non-decreasing function of $u > 0$ and if $\varphi(u)/u^\beta$ is a non-increasing function of $u > 0$. For $G \in \{[0,1], (0,\infty), \mathbb{N}\}$, φ is said to be in the class $\mathcal{K}_G(\alpha, \beta)$ if the above statements hold for u in a neighbourhood corresponding to the Orlicz spaces built on G.

Remark 2.2. (i) If M is an Orlicz function such that $M \in \Delta_2^G \cap \nabla_2^G$, then there exist α and β such that $M \in \mathcal{K}_G(\alpha, \beta)$.

(ii) A quasi-Orlicz function which is α-convex and β-concave belongs to the class $\mathcal{K}(\alpha,\beta)$ and thus to the classes $\mathcal{K}_G(\alpha,\beta)$ for all $G \in \{[0,1],(0,\infty),\mathbb{N}\}$.

(iii) A quasi-Orlicz function φ with continuous derivative belongs to the class $\mathcal{K}_G(\alpha,\beta)$ if and only if there exists a neighbourhood V corresponding to the Orlicz spaces built on G such that

$$\alpha \le u\varphi'(u)/\varphi(u) \le \beta, \text{ for all } u \in V. \tag{2.1}$$

The equivalence of quasi-Orlicz functions is defined in the same way as that of Orlicz functions. We have the following result.

Proposition 2.3 ([6, p. 89]). *Let $0<\alpha<\beta<\infty$ and let φ be a quasi-Orlicz function which belongs to the class $\mathcal{K}_G(\alpha,\beta)$. Then there exists an α-convex and β-concave quasi-Orlicz function $\tilde\varphi$ (so that $\tilde\varphi \in \mathcal{K}(\alpha,\beta)$), with continuous second derivative, such that $\varphi \sim_G \tilde\varphi$. Moreover, $\tilde\varphi$ satisfies (2.1) for all $u>0$.*

The main result. As above, let G denote one of the sets $[0,1]$, $\mathbb{N}$, and $(0,\infty)$, with the associated measure μ.

Let M and N be two Orlicz functions. We consider the mapping

$$\begin{aligned}\phi_{MN} : L_M(G) &\longrightarrow L_N(G),\\ x &\longmapsto \phi_{MN}(x) = N^{-1}\circ M(|x|)\operatorname{sign}(x).\end{aligned}$$

We denote by φ the quasi-Orlicz function $N^{-1}\circ M$ associated to ϕ_{MN}. The inverse $\phi_{MN}^{-1} = \phi_{NM} : L_N(G) \to L_M(G)$ is associated to the quasi-Orlicz function $\varphi^{-1} = M^{-1}\circ N$. Denote by $B_M(G), B_N(G)$ the closed unit balls, and by $S_M(G), S_N(G)$ the unit spheres.

Theorem 2.4. *Suppose that the quasi-Orlicz function $\varphi = N^{-1}\circ M$ is in the class $\mathcal{K}(\alpha,\beta)$. Then ϕ_{MN} is $\alpha\wedge 1$-Hölder on $B_M(G)$ and $\phi_{MN}^{-1} = \phi_{NM}$ is $(1/\beta)\wedge 1$-Hölder on $B_N(G)$.*

Before proving Theorem 2.4 we make some comments and we state a corollary.

As mentioned above, ϕ_{MN} maps $S_M(G)$ onto $S_N(G)$ and provides a uniform homeomorphism between *unit spheres*. Indeed, if $\|x\|_M = 1$, then Fatou's theorem ensures that $\|\phi_{MN}(x)\|_N \le 1$, and a direct computation shows that in fact $\|\phi_{MN}(x)\|_N = 1$.

According to [3] and [15], the homeomorphy of the unit spheres of $L_M(G)$ and $L_N(G)$ is well known when $M,N \in \Delta_2^G$. The following result gives an explicit modulus of continuity for the homeomorphism under some regularity assumptions.

Corollary 2.5. *Fix $G \in \{[0,1],(0,\infty),\mathbb{N}\}$. Let M and N be two Orlicz functions and suppose that there exist $1 \le p_M \le q_M < \infty$ and $1 \le p_N \le$*

$q_N < \infty$ such that $M \in \mathcal{K}_G(p_M, q_M)$ and $N \in \mathcal{K}_G(p_N, q_N)$. Then, up to an equivalent renorming, $\phi_{MN} : S_M(G) \to S_N(G)$ is a uniform homeomorphism with explicit modulus of continuity, namely:

- *ϕ_{MN} is $(p_M/q_N) \wedge 1$-Hölder on $S_M(G)$,*
- *$\phi_{MN}^{-1} = \phi_{NM}$ is $(p_N/q_M) \wedge 1$-Hölder on $S_N(G)$.*

Proof of Corollary 2.5. We use Proposition 2.3 to obtain two Orlicz functions $\tilde{M} \in \mathcal{K}(p_M, q_M)$ and $\tilde{N} \in \mathcal{K}(p_N, q_N)$ such that $M \sim_G \tilde{M}$ and $\tilde{N} \sim_G N$ and such that the quasi-Orlicz function $\tilde{N}^{-1} \circ \tilde{M}$ satisfies the assumptions of Theorem 2.4. Indeed, we have $\tilde{N}^{-1} \in \mathcal{K}(1/q_N, 1/p_N)$ and $\tilde{N}^{-1} \circ \tilde{M} \in \mathcal{K}(p_M/q_N, q_M/p_N)$.

We know, by equivalence, that the identity mapping between $L_M(G)$ (resp. $L_N(G)$) and $L_{\tilde{M}}(G)$ (resp. $L_{\tilde{N}}(G)$) is an isomorphism. With this renorming, Corollary 2.5 follows. □

REMARK 2.6. (i) Fix $p, q \geq 1$. Take $M(u) = u^p$ and $N(u) = u^q$ for all $u \geq 0$. Then $\phi_{MN} = \phi_{pq}$ is the usual Mazur map between $L_p(G)$ and $L_q(G)$. We have $p_M = q_M = p$ and $p_N = q_N = q$, and Theorem 2.4 gives the usual estimate for the modulus of continuity of ϕ_{pq} on the unit ball of $L_p(G)$.

(ii) We give now an example for which the moduli of continuity obtained in Theorem 2.4 are both sharp.

Take $G = (0, \infty)$ and consider the Orlicz functions $M(u) = u^2 \vee u^4$ and $N(u) = u^2$ defined for $u \in \mathbb{R}^+$. The quasi-Orlicz function $\varphi(u) = N^{-1} \circ M(u) = u \vee u^2$ is in the class $\mathcal{K}(1, 2)$. Theorem 2.4 gives that $\phi_{MN} : B_M(0, \infty) \to B_N(0, \infty)$ is Lipschitz, and this is the best-possible estimate for the modulus of continuity. Moreover, the inverse mapping $\phi_{MN}^{-1} = \phi_{NM} : B_N(0, \infty) \to B_M(0, \infty)$ is 1/2-Hölder on the unit ball $B_N(0, \infty)$. Using the indicator functions $\chi_{[0,1/n]}$ with $n \in \mathbb{N}$, one can show that the exponent 1/2 is also best-possible.

Let us now prove Theorem 2.4. The proof relies on the following lemma.

LEMMA 2.7. *Let φ be a quasi-Orlicz function in the class $\mathcal{K}(\alpha, \beta)$. Then, for all $a, b \in \mathbb{R}$ such that $(a, b) \neq (0, 0)$, we have:*

- *If $\beta \leq 1$ or $\operatorname{sign}(a) \neq \operatorname{sign}(b)$, then*
$$\big|\varphi(|a|) \operatorname{sign}(a) - \varphi(|b|) \operatorname{sign}(b)\big| \leq 2\varphi\left(|a - b|\right).$$
- *If $1 \leq \beta$, then*
$$|\varphi(|a|) - \varphi(|b|)| \leq 2^{(1-\alpha)\vee 0} \beta \frac{\varphi\left(|a| + |b|\right)}{|a| + |b|} |a - b|.$$

Proof. If $\operatorname{sign}(a) \neq \operatorname{sign}(b)$, then $\varphi(|a|) + \varphi(|b|) \leq 2\varphi(|a| + |b|)$ because φ is non-decreasing. But in this case $|a| + |b| = |a - b|$, so the result follows.

Next, assume $\beta \le 1$ and $\operatorname{sign}(a) = \operatorname{sign}(b)$. We can suppose that $|b| \ge |a|$. Since $\varphi(u)/u$ is a non-increasing function of u, we have

$$\frac{\varphi(|b|)}{|b|} \le \frac{\varphi(|a|)}{|a|} \text{ and } \frac{\varphi(|b|)}{|b|} \le \frac{\varphi(|b|-|a|)}{|b|-|a|}.$$

We write $\varphi(|b|) = \lambda\varphi(|b|) + (1-\lambda)\varphi(|b|)$ and apply the above bounds for $\varphi(|b|)/|b|$ to the two terms on the right, with the choice $\lambda = |a|/|b|$. Using the inequality $|b| - |a| \le |b-a|$, we obtain the result.

Finally, assume $1 \le \beta$ and $\operatorname{sign}(a) = \operatorname{sign}(b)$. Again we can suppose that $|b| \ge |a| > 0$. As $\varphi(|a|)/|a|^\beta \ge \varphi(|b|)/|b|^\beta$ and $\varphi(|b|)/|b|^\alpha \le \varphi(|a|+|b|)/(|a|+|b|)^\alpha$, we obtain

$$\varphi(|b|) - \varphi(|a|) \le \left(1 - \left(\frac{|a|}{|b|}\right)^\beta\right)\left(\frac{|b|}{|a|+|b|}\right)^\alpha \varphi(|a|+|b|).$$

Using the inequality $1 - u^\beta \le \beta(1-u)$ for $u \in [0,1]$, the result follows. □

Proof of Theorem 2.4. If $\varphi \in \mathcal{K}(\alpha,\beta)$, then $\varphi^{-1} \in \mathcal{K}(1/\beta, 1/\alpha)$. Thus, to prove Theorem 2.4, it suffices to consider the three cases $\alpha \le \beta \le 1$, $1 \le \alpha \le \beta$, and $\alpha \le 1 \le \beta$.

Denote by $B_M(G)$ the closed unit ball of $L_M(G)$ and by φ the quasi-Orlicz function $N^{-1} \circ M$. Fix $x, y \in B_M(G)$ such that $\|x-y\|_M > 0$.

FACT 1. *It suffices to prove the result for x and y satisfying $\|x-y\|_M < K$, where $K > 0$ is any given constant.*

Indeed, as $x, y \in B_M(G)$ and as ϕ_{MN} takes values in $B_N(G)$, we have, for $\|x-y\|_M \ge K$,

$$\|\phi_{MN}(x) - \phi_{MN}(y)\|_N \le 2 \le \frac{2}{K}\|x-y\|_M \le \frac{2}{K}2^{1-\alpha}\|x-y\|_M^\alpha,$$

so the desired inequality holds trivially for such x and y.

Now, define, for $t \in G$,

$$\begin{aligned}\Delta_{MN}(t) &= \left|\phi_{MN}(x)(t) - \phi_{MN}(y)(t)\right| \\ &= \left|\varphi(|x(t)|)\operatorname{sign}(x(t)) - \varphi(|y(t)|)\operatorname{sign}(y(t))\right|.\end{aligned}$$

Under the assumption that φ is in the class $\mathcal{K}(\alpha,\beta)$, our aim is to estimate

$$\|\Delta_{MN}\|_N = \inf\left\{\lambda > 0 : \int_G N\left(\frac{\Delta_{MN}(t)}{\lambda}\right) d\mu(t) \le 1\right\}.$$

Case 1: $\alpha \le \beta \le 1$.

In view of Fact 1, we can suppose $0 < \|x-y\|_M < 1$. Set $\lambda = 2\|x-y\|_M^\alpha$. Lemma 2.7 gives, for all $t \in G$,

$$\frac{\Delta_{MN}(t)}{\lambda} \le \frac{2}{\lambda}\varphi\left(|x(t)-y(t)|\right).$$

But $\varphi(u)/u^\alpha$ is a non-decreasing function of u and $2/\lambda \geq 1$, so we have for all $t \in G$

$$\frac{\Delta_{MN}(t)}{\lambda} \leq \varphi\left(\left(\frac{2}{\lambda}\right)^{1/\alpha} |x(t)-y(t)|\right).$$

This gives the estimate

$$\int_G N\left(\frac{\Delta_{MN}(t)}{\lambda}\right) d\mu(t) \leq \int_G M\left(\frac{|x(t)-y(t)|}{\|x-y\|_M}\right) d\mu(t).$$

By the definition of the norm this implies

$$\|\phi_{MN}(x) - \phi_{MN}(y)\|_N \leq 2\|x-y\|_M^\alpha.$$

Hence ϕ_{MN} is α-Hölder on $B_M(G)$.

Case 2: $1 \leq \alpha \leq \beta$.
Define the sets

$$G^+ = \{t \in G : \operatorname{sign}(x(t)) = \operatorname{sign}(y(t))\},$$
$$G^- = \{t \in G : \operatorname{sign}(x(t)) \neq \operatorname{sign}(y(t))\}.$$

Set $\lambda = 3 \times 2 \times \beta \times 4^\beta \|x-y\|_M$. In view of Fact 1, we can suppose that $0 < \lambda \leq 2$ (and thus $0 < \|x-y\|_M < 1$).

Consider first the case $t \in G^-$. According to Lemma 2.7,

$$\frac{\Delta_{MN}(t)}{\lambda} \leq \frac{2}{\lambda}\varphi(|x(t)-y(t)|)$$

Now $\varphi(u)/u$ is a non-decreasing function of u and $2/\lambda \geq 1$, so

$$\int_{G^-} N\left(\frac{\Delta_{MN}(t)}{\lambda}\right) d\mu(t) \leq \int_G M\left(\frac{2}{\lambda}|x(t)-y(t)|\right) d\mu(t) \leq \frac{1}{3}, \tag{2.2}$$

by the convexity of M (note that $2/\lambda \leq 1/(3\|x-y\|_M)$) and because $0 < \|x-y\|_M < 1$.

Next, consider the case $t \in G^+$. Using Lemma 2.7 we obtain

$$\Delta_{MN}(t) \leq \beta\frac{\varphi(s(t))}{s(t)}|x(t)-y(t)|,$$

with $s(t) = |x(t)| + |y(t)|$. Note that we may assume that $s(t) \neq 0$ because if $s(t) = 0$ then $\Delta_{MN}(t) = 0$. As $\varphi(u)/u^\beta$ is non-increasing, we have for all $t \in G^+$ such that $s(t) \neq 0$,

$$N\left(\frac{\Delta_{MN}(t)}{\lambda}\right) \leq N\left[\varphi(s(t))\frac{\beta}{\lambda}\frac{|x(t)-y(t)|}{s(t)}\right] \leq N\left[\varphi\left(s(t)/4\right) f(t)\right],$$

where

$$f(t) = 4^\beta \frac{\beta}{\lambda}\frac{|x(t)-y(t)|}{s(t)}.$$

Define the sets

$$G_1^+ = \{t \in G^+ : s(t) \neq 0 \text{ and } f(t) \geq 1\},$$
$$G_2^+ = \{t \in G^+ : s(t) \neq 0 \text{ and } f(t) < 1\}.$$

For all $t \in G_1^+$, as $\varphi(u)/u$ is non-decreasing,

$$N\left[\varphi\left(s(t)/4\right) f(t)\right] \leq N\left[\varphi\left(\frac{s(t)}{4} f(t)\right)\right].$$

So we have

$$\int_{G_1^+} N\left(\frac{\Delta_{MN}(t)}{\lambda}\right) d\mu(t) \leq \int_G M\left(4^\beta \frac{\beta}{\lambda}|x(t) - y(t)|\right) d\mu(t) \leq \frac{1}{3}, \tag{2.3}$$

using again the convexity of M and the choice of λ.

For all $t \in G_2^+$, by the convexity of N and the fact that $\varphi(u)/u^\beta$ is non-increasing, we have

$$N\left(\varphi\left(s(t)/4\right) f(t)\right) \leq f(t) N\left[\varphi(s(t)/4)\right] = 4^{\beta-1}\frac{\beta}{\lambda}\frac{M(s(t)/4)}{s(t)/4}|x(t) - y(t)|.$$

We are going to apply the generalized Hölder inequality to the function on the right. It is easy to check that for all $u > 0$, $M^*(M(u)/u) \leq M(u)$ (see [9, p. 13]). Since $\|s(.)/4\|_M < 1$, we have as above

$$\int_G M^*\left(\frac{M(s(t)/4)}{s(t)/4}\right) d\mu(t) \leq 1,$$

and thus

$$\left\|\frac{M(s(.)/4)}{s(.)/4}\right\|_{M^*} \leq 1.$$

Applying the Hölder inequality, we obtain

$$\begin{aligned} \int_{G_2^+} N\left(\frac{\Delta_{MN}(t)}{\lambda}\right) d\mu(t) &\leq 2 \times 4^{\beta-1}\left\|\frac{M(s(.)/4)}{s(.)/4}\right\|_{M^*}\left\|\frac{\beta}{\lambda}(x-y)\right\|_M \\ &\leq 2 \times 4^{\beta-1}\frac{\beta}{\lambda}\|x-y\|_M \leq \frac{1}{3}. \end{aligned} \tag{2.4}$$

From inequalities (2.2), (2.3) and (2.4) we obtain

$$\int_G N\left(\frac{\Delta_{MN}(t)}{\lambda}\right) d\mu(t) = \int_{G^- \cup G_1^+ \cup G_2^+} N\left(\frac{\Delta_{MN}(t)}{\lambda}\right) d\mu(t) \leq 1.$$

This means that

$$\|\phi_{MN}(x) - \phi_{MN}(y)\|_N \leq \lambda = 6\beta 4^\beta \|x - y\|_M,$$

so ϕ_{MN} is Lipschitz on $B_M(G)$.

Case 3: $\alpha \leq 1 \leq \beta$.

If $t \in G^-$, we argue as above using the fact that $\varphi(u)/u^\alpha$ is a non-decreasing function of u.

If $t \in G_2^+$, the argument is again the same as above, using the estimate $\|x-y\|_M \le 2^{1-\alpha}\|x-y\|_M^\alpha$.

If $t \in G_1^+$, a different argument is required. As $\varphi(u)/u^\alpha$ is non-decreasing, we have

$$\begin{aligned} N\left[\varphi\left(s(t)/4\right) f(t)\right] &\le N\left[\varphi\left(\frac{s(t)}{4} f(t)^{1/\alpha}\right)\right], \\ &= N\left[\varphi\left(\left(\frac{\beta}{\lambda}\right)^{1/\alpha} 4^{(\beta/\alpha)-1} s(t)^{1-(1/\alpha)}|x(t)-y(t)|^{1/\alpha}\right)\right]. \end{aligned}$$

But

$$\begin{aligned} |x(t)-y(t)|^{1/\alpha} &= |x(t)-y(t)|^{(1/\alpha)-1}|x(t)-y(t)| \\ &\le s(t)^{(1/\alpha)-1}|x(t)-y(t)|, \end{aligned}$$

because $1/\alpha - 1 \ge 0$. Thus, for $t \in G_1^+$ we have

$$N\left(\frac{\Delta_{MN}(t)}{\lambda}\right) \le M\left[\left(\frac{\beta}{\lambda}\right)^{1/\alpha} 4^{(\beta/\alpha)-1}|x(t)-y(t)|\right].$$

Combining the estimates on G^-, G_2^+ and G_1^+, we obtain that in this case, ϕ_{MN} is α-Hölder on $B_M(G)$.

Summarizing, we have proved:

- If $\alpha \le \beta \le 1$, ϕ_{MN} is α-Hölder on $B_M(G)$.
- If $1 \le \alpha \le \beta$, ϕ_{MN} is Lipschitz on $B_M(G)$.
- If $\alpha \le 1 \le \beta$, ϕ_{MN} is α-Hölder on $B_M(G)$.

This completes the proof of Theorem 2.4. □

3. Application to the approximation of uniformly continuous mappings

The main result. Let $(X, \|\,.\,\|_X)$ and $(Y, \|\,.\,\|_Y)$ be two Banach spaces. Denote by $B_X = \{x \in X : \|x\|_X \le 1\}$ the closed unit ball of $(X, \|\,.\,\|_X)$. Let $f : B_X \to Y$ be uniformly continuous. We consider the problem of approximating f uniformly by α-Hölder mappings, with $\alpha \in (0,1]$ as large as possible. This can be formulated in the following way: We define

$$UC(B, Y) = \{f : B_X \to Y : f \text{ is uniformly continuous}\},$$

and

$$\mathcal{H}^\alpha(B_X, Y) = \{f : B_X \to Y : f \text{ is } \alpha\text{-Hölder}\}$$

for $\alpha \in (0,1]$, and set $\mathcal{H}^0(B_X, Y) = UC(B, Y)$. We have the inclusions $\mathcal{H}^\beta(B_X, Y) \subset \mathcal{H}^\alpha(B_X, Y) \subset UC(B, Y)$ for $1 \ge \beta \ge \alpha \ge 0$. Moreover, $UC(B_X, Y)$ endowed with the norm $\|f\|_\infty = \sup\{\|f(x)\|_Y : x \in B_X\}$ is a Banach space. Thus we are interested in the closure of $\mathcal{H}^\alpha(B_X, Y)$ with respect to the norm $\|\,.\,\|_\infty$ in $UC(B_X, Y)$.

We set

$$\alpha(B_X,Y)=\sup\left\{\alpha\in[0,1],\quad \overline{\mathcal{H}^\alpha(B_X,Y)}^{\|\,.\,\|_\infty}=UC(B_X,Y)\right\}.$$

The value $\alpha=\alpha(B_X,Y)$ is best-possible in the sense that if $\beta>\alpha$ then there exist $\varepsilon>0$ and a uniformly continuous map $f:B_X\to Y$ such that for every β-Hölder function $\psi:B_X\to Y$ we have $\|f-\psi\|_\infty>\varepsilon$.

For general resuls on such approximations problems we refer to the book [2].

We now consider the case when $(X,\|\,.\,\|_X)=(L_p(G),\|\,.\,\|_p)$ and $(Y,\|\,.\,\|_Y)=(L_q(G'),\|\,.\,\|_q)$ with $p,q\geq 1$ and G,G' equal to $[0,1]$ or $\mathbb{N}$ or $(0,\infty)$. We denote by $B_p(G)$ the closed unit ball of $(L_p(G),\|\,.\,\|_p)$. The following theorem of I. G. Tsaŕkov [16] gives the exact value of $\alpha(B_p(G),L_q(G'))$.

Theorem 3.1 ([16] and [2, p. 36]). *Let $G,G'\in\{[0,1],(0,\infty),\mathbb{N}\}$ with adapted measures. Then we have:*

$$\alpha(B_p(G),L_q(G'))=\begin{cases}1 & \text{if } p\geq 2\geq q \text{ or } q=\infty,\\ \min(1/2,1/q) & \text{if } p=\infty \text{ and } q<\infty,\end{cases}$$

$$\alpha(B_p(G),L_q(G'))\geq\begin{cases}2/q & \text{if } p,q\geq 2,\\ p/2 & \text{if } p,q\leq 2,\\ p/q & \text{if } p\leq 2\leq q.\end{cases}$$

Moreover, equality holds in the following cases:

- *If $G\neq\mathbb{N}$, then $\alpha(B_p(G),L_q(G'))=2/q$ if $p,q\geq 2$.*
- *If $G'\neq\mathbb{N}$, then $\alpha(B_p(G),L_q(G'))=p/2$ if $p,q\leq 2$.*
- *If $G=G'=\mathbb{N}$ or $G\neq\mathbb{N}$ and $G'\neq\mathbb{N}$, then $\alpha(B_p(G),L_q(G'))=p/q$ if $p\leq 2\leq q$.*

Remark 3.2. The different cases for G and G' involving $\mathbb{N}$, to obtain upper bounds for the exponent of the approximation in the second part of Theorem 3.1, implicitly appear in the proof given in [2] when isomorphic copies of L_2 are needed in $L_p(G)$ and in $L_q(G')$.

Our main result of this section is the following theorem, which extends Tsaŕkov's result to the setting of Orlicz spaces.

Theorem 3.3. *Let $G,G'\in\{[0,1],(0,\infty),\mathbb{N}\}$ with adapted measures. Let M and N be two Orlicz functions such that $M\in\mathcal{K}_G(p_M,q_M)$ and $N\in$*

$\mathcal{K}_G(p_N, q_N)$, with $p_M, p_N > 1$ and $q_M, q_N < \infty$. Then we have:

$$\alpha(B_M(G), L_N(G')) = 1 \text{ if } p_M \geq 2 \geq q_N,$$

$$\alpha(B_M(G), L_N(G')) \geq \begin{cases} 2/q_N & \text{if } p_M, q_N \geq 2, \\ p_M/2 & \text{if } p_M, q_N \leq 2, \\ p_M/q_N & \text{if } p_M \leq 2 \leq q_N. \end{cases}$$

Moreover, we have:

- *If $G \neq \mathbb{N}$, then $\alpha(B_M(G), L_N(G')) \leq 2/p_N$ if $q_M, p_N \geq 2$.*
- *If $G' \neq \mathbb{N}$, then $\alpha(B_M(G), L_N(G')) \leq q_M/2$ if $q_M, p_N \leq 2$.*
- *If $G = G' = \mathbb{N}$ or $G \neq \mathbb{N}$ and $G' \neq \mathbb{N}$, then $\alpha(B_M(G), L_N(G')) \leq q_M/p_N$ if $q_M \leq 2 \leq p_N$.*

REMARK 3.4. (i) When G and G' are in the appropriate cases, Theorem 3.3 gives the estimates

$$\frac{p_M \wedge 2}{q_N \vee 2} \leq \alpha(B_M(G), L_N(G')) \leq \frac{q_M \wedge 2}{p_N \vee 2}.$$

(ii) Our method of proof is the same as that of Tsaŕkov [16]. However, the more general setting of Orlicz spaces clearly shows the central role played by the notions of p-convexity and of q-concavity, which was hidden in the setting of L_p-spaces.

For the proof of Theorem 3.3 we will need the following lemma, which essentially is proved in [2]. For the sake of completeness we give a proof here.

LEMMA 3.5. *Let M and N be two Orlicz functions. Suppose that $L_M[0,1]$ and $L_M(0,\infty)$ are reflexive. Let X be a normed linear space and denote by B_X its closed unit ball. Then*

$$\alpha(B_X, L_N(0,\infty)) \leq \alpha(B_X, L_N[0,1]),$$
$$\alpha(B_M(0,\infty), X) \leq \alpha(B_M[0,1], X).$$

Proof of Lemma 3.5. Define the maps

$$\varphi : L_N[0,1] \longrightarrow L_N(0,\infty) \qquad\qquad P : L_N(0,\infty) \longrightarrow L_N[0,1],$$

$$x \longmapsto \begin{cases} x(t) & \text{if } t \in [0,1], \\ 0 & \text{else,} \end{cases} \qquad\qquad x \longmapsto x\big|_{[0,1]}.$$

Then φ and P are linear and satisfy $\|\varphi(x)\| = \|x\|$ and $\|P(x)\| \leq \|x\|$ with x in the appropriate Orlicz space and $\| \, . \, \|$ the associated norm. Moreover, $P \circ \varphi$ is the identity on $L_N[0,1]$.

Let $f : B_X \to L_N[0,1]$ be uniformly continuous. Fix $\alpha < \alpha(B_X, L_N(0,\infty))$. We seek to approximate f by an α-Hölder map. Fix $\varepsilon > 0$. As $\varphi \circ f : B_X \to L_N(0,\infty)$ is uniformly continuous, there exists $g_\varepsilon \in \mathcal{H}^\alpha(B_X, L_N(0,\infty))$

such that for all $x \in B_X$, $\|\varphi \circ f(x) - g_\varepsilon(x)\| \le \varepsilon$. Thus, for all $x \in B_X$, $\|P \circ \varphi \circ f(x) - P \circ g_\varepsilon(x)\| \le \varepsilon$. But $P \circ \varphi \circ f(x) = f(x)$, and so $P \circ g_\varepsilon$ approximates f and is α-Hölder. By definition, we have $\alpha \le \alpha(B_X, L_N[0,1])$. Since this holds for all $\alpha < \alpha(B_X, L_N(0,\infty))$, we obtain $\alpha(B_X, L_N(0,\infty)) \le \alpha(B_X, L_N[0,1])$. Thus the first inequality is proved.

Now consider the map φ defined as above with M instead of N. The spaces $B_M[0,1]$ and $\varphi(B_M[0,1])$ are Lipschitz equivalent (with φ and φ^{-1} defined on $\varphi(L_M[0,1])$), so it is easy to check that $\alpha(B_M[0,1], X) = \alpha(\varphi(B_M[0,1]), X)$. Moreover, by [5], as the space $L_M(0,\infty)$ is reflexive, endowed with the Luxemburg norm, it has a uniformly normal structure. Now, by a result in [2, p. 28], every closed, convex and bounded subset $A \subset L_M(0,\infty)$ is an absolute uniform retract (i.e., for every metric space (Y, d_Y) containing A, there exists a uniformly continuous map $r : Y \to A$ which is the identity on A). We apply this with the subset $\varphi(B_M[0,1]) \subset B_M(0,\infty)$ to obtain a uniformly continuous map $r : B_M(0,\infty) \to \varphi(B_M[0,1])$ such that, for all $x \in \varphi(B_M[0,1])$, $r(x) = x$.

To prove the second inequality, let $f : \varphi(B_M[0,1]) \to X$ be uniformly continuous, and let $\alpha < \alpha(B_M(0,\infty), X)$ and $\varepsilon > 0$. Then $f \circ r : B_M(0,\infty) \to X$ is also uniformly continuous and there exists a map $g_\varepsilon : B_M(0,\infty) \to X$, which is α-Hölder, such that, for all $x \in B_M(0,\infty)$, we have $\|f \circ r(x) - g_\varepsilon(x)\| \le \varepsilon$. This gives, for all $x \in \varphi(B_M[0,1])$, $\|f(x) - g_\varepsilon(x)\| \le \varepsilon$. Also, g_ε restricted to $\varphi(B_M[0,1])$ is still α-Hölder. It follows that $\alpha \le \alpha(\varphi(B_M[0,1]), X)$, and since this holds for any $\alpha < \alpha(B_M(0,\infty), X)$, we obtain $\alpha(B_M(0,\infty), X) \le \alpha(\varphi(B_M[0,1]), X) = \alpha(B_M[0,1], X)$, which is the desired inequality. □

Proof of Theorem 3.3. The proof uses the same method as in [16] and [2, p. 36], with the map ϕ_{MN} in place of ϕ_{pq}. We apply Proposition 2.3 to obtain maps $\tilde{M}$ and $\tilde{N}$ such that $\tilde{M} \sim_G M$ and $\tilde{N} \sim_{G'} N$ and such that $\tilde{M}$ is p_M-convex and q_M-concave and $\tilde{N}$ is p_N-convex and q_N-concave. The map $\tilde{\varphi} = \tilde{N}^{-1} \circ \tilde{M}$ then satisfies the assumptions of Theorem 2.4. Also, using the isomorphisms between the Orlicz spaces, it is easy to see that $\alpha(B_M(G), L_N(G')) = \alpha(B_{\tilde{M}}(G), L_{\tilde{N}}(G'))$. Thus, it suffices to prove the result with $\tilde{M}$ and $\tilde{N}$ in place of M and N. To simplify notations, we drop the "tilde" symbol in $\tilde{M}$, $\tilde{N}$, and $\tilde{\varphi}$, and assume that M and N are such that the map $N^{-1} \circ M$ satisfies the assumptions of Theorem 2.4.

Let $f : B_M(G) \to L_N(G')$ be uniformly continuous and fix $\varepsilon > 0$. Consider the composition $\Phi = \phi_{N2} \circ f \circ \phi_{2M}$, where $\phi_{2M} : L_2(G) \to L_M(G)$ and $\phi_{N2} : L_N(G') \to L_2(G')$ are uniformly continuous on balls. Then $\Phi : B_2(G) \to L_2(G')$ is a uniformly continuous map between Hilbert spaces. Hence, according to [13], there exists a Lipschitz mapping $g : B_2(G) \to L_2(G')$ such that $\|\Phi - g\|_\infty \le \varepsilon$. It is clear that $(\phi_{N2})^{-1} \circ g \circ (\phi_{2M})^{-1} = \phi_{2N} \circ g \circ \phi_{M2}$ is a uniform approximation of f on $B_M(G)$. Moreover, according to Theorem

2.4, this approximation is in the appropriate space $\mathcal{H}^\alpha(B_M(G), L_N(G'))$ corresponding to the position of p_M and q_N relative to the number 2. By the definition of $\alpha(B_M(G), L_N(G'))$, this proves the lower estimates of Theorem 3.3.

As $\alpha(B_M(G), L_N(G')) \leq 1$, the upper estimate holds in the case when $q_M \geq 2 \geq p_N > 1$, so it suffices to deal with the remaining cases.

First, suppose that $G = G'$. We have to consider three cases: (1) $q_M \leq 2$ and $p_N \geq 2$; (2) $q_M \geq 2$ and $p_N \geq 2$, and (3) $q_M \leq 2$ and $p_N \leq 2$.

Case 1: $q_M \leq 2$ and $p_N \geq 2$.

We have to prove that $\alpha(B_M(G), L_N(G)) \leq q_M/p_N$. Fix $\delta > 0$. The proof consists in finding a uniformly continuous map $\phi : B_M(G) \to L_N(G)$ satisfying

$$\inf\{\|\phi - g\|_\infty : g \in \mathcal{H}^{q_M/p_N+\delta}(B_M(G), L_N(G))\} > 0.$$

In fact, we will show that the map ϕ_{MN} itself has this property. This follows from the following lemma, which is similar to the result in [2, p. 36].

LEMMA 3.6. *Let M, N be two Orlicz functions satisfying the assumptions of Corollary 2.5, with the same notations. Suppose that $q_M \leq p_N$. Fix $\alpha \in (0,1]$. Then for all $g \in \mathcal{H}^\alpha(B_M(G), L_N(G))$ there exists $C > 0$ such that*

$$2\|\phi_{MN} - g\|_\infty \geq \sup_{n\in\mathbb{N}, n \text{ even}} \{1 - Cn^{1/p_N} M^{-1}(1/n)^\alpha\}, \text{ if } G \in \{(0,\infty), \mathbb{N}\},$$

$$2\|\phi_{MN} - g\|_\infty \geq \sup_{n\in\mathbb{N}, n \text{ even}} \{1 - Cn^{1/p_N} 1/M^{-1}(n)^\alpha\}, \text{ if } G = [0,1].$$

In particular, if $f_n(\alpha) = n^{1/p_N} M^{-1}(1/n)^\alpha \to 0$ as $n \to \infty$ (resp. $f_n(\alpha) = n^{1/p_N} 1/M^{-1}(n)^\alpha \to 0$), then the set $\mathcal{H}^\alpha(B_M(G), L_N(G))$ is not dense in $UC(B_M(G), L_N(G))$ when $G \in \{(0,\infty), \mathbb{N}\}$ (resp. when $G = [0,1]$).

Proof of Lemma 3.6. Case 1: $G \in \{(0,\infty), \mathbb{N}\}$.

Let $g \in \mathcal{H}^\alpha(B_M(G), L_N(G))$. Fix $n \in \mathbb{N}$ and set $l_M^{2n} = (\mathbb{R}^{2n}, \|\,.\,\|_M)$, where

$$\|x\|_M = \inf\left\{\lambda > 0 : \sum_{i=1}^{2n} M(|x_i|/\lambda) \leq 1\right\}$$

and $l_N^{2n} = (\mathbb{R}^{2n}, \|\,.\,\|_N)$. Denote by $B(l_M^{2n})$ and $B(l_N^{2n})$ the closed unit balls in these spaces. The space l_M^{2n} (resp. l_N^{2n}) can be represented as the subspace of $L_M(G)$ (resp. $L_N(G)$) consisting of functions that are constant on some fixed $2n$ disjoint sets, with measure 1 each.

There exists a norm-one projection $P : L_N(G) \to l_N^{2n}$ such that $P(x) = x$ for all $x \in l_N^{2n}$. So, for all $x \in B(l_M^{2n})$,

$$\begin{aligned}\|\phi_{MN}(x) - P \circ g(x)\|_N &= \|P \circ \phi_{MN}(x) - P \circ g(x)\|_N \\ &\leq \|\phi_{MN}(x) - g(x)\|_N \leq \|\phi_{MN} - g\|_\infty.\end{aligned}$$

Given a permutation σ of $\{1,\ldots,2n\}$ and a choice of signs $\theta=(\theta_1,\ldots,\theta_{2n})$, we consider the operator

$$\begin{aligned} U_{\sigma,\theta} : \mathbb{R}^{2n} &\to \mathbb{R}^{2n} \\ x &\mapsto (\theta_1 x_{\sigma^{-1}(1)},\ldots,\theta_i x_{\sigma^{-1}(i)},\ldots,\theta_{2n} x_{\sigma^{-1}(2n)}). \end{aligned}$$

This operator defines an isometry on both l_M^{2n} and l_N^{2n} and induces the norm-one operator

$$\begin{aligned} V_{\sigma,\theta} : \big(UC(B(l_M^{2n}), l_N^{2n}), \|\,.\,\|_\infty\big) &\to \big(UC(B(l_M^{2n}), l_N^{2n}), \|\,.\,\|_\infty\big) \\ f &\mapsto U_{\sigma,\theta}\circ f\circ U_{\sigma,\theta}^{-1}, \end{aligned}$$

and the average operator

$$V=\frac{1}{(2n)!2^{2n}}\sum_{\sigma,\theta} V_{\sigma,\theta}.$$

It is not difficult to see that $V_{\sigma,\theta}(\phi_{MN})=\phi_{MN}$. So we have

$$\begin{aligned} \|\phi_{MN}-V(P\circ g)\|_\infty &= \|V(\phi_{MN})-V(P\circ g)\|_\infty \\ &\le \|\phi_{MN}-P\circ g\|_\infty \text{ (because } V \text{ is a norm-one operator)} \\ &\le \|\phi_{MN}-g\|_\infty \text{ (as before).} \end{aligned}$$

To simplify notations, we set $h=V(P\circ g)$. Then $h\in\mathcal{H}^\alpha(B(l_M^{2n}),l_N^{2n})$, with a constant C_h that is independent of n.

We have $V_{\sigma,\theta}(h)=h$ for all σ,θ, which means that $U_{\sigma,\theta}\circ h=h\circ U_{\sigma,\theta}$. Thus h preserves the support and, if $c>0$ and if χ_A is the indicator function of a subset A of $\{1,\ldots,2n\}$, then $h(c\chi_A)=c'\chi_A$, where the constant c' depends only on c and on the cardinality of A.

Now, for all $x,y\in B(l_M^{2n})$, we have

$$2\|\phi_{MN}-g\|_\infty \ge \|\phi_{MN}(x)-\phi_{MN}(y)\|_N-\|h(x)-h(y)\|_N.$$

We will apply this with a judicious choice of x and y.

Set $x_k = M^{-1}(1/2n)\chi(k,\ldots,k+n-1)$, for $1\le k\le n+1$. Then $x_k\in B(l_M^{2n})$ and $\|x_k-x_{k+1}\|_M = M^{-1}(1/2n)/M^{-1}(1/2)$. A direct calculation gives $\|\phi_{MN}(x_1)-\phi_{MN}(x_{n+1})\|_N=1$. Moreover, the vectors $(h(x_k)-$

$h(x_{k+1}))_{1\le k\le n+1}$ are disjointly supported and satisfy

$$\begin{aligned}
\|h(x_1)-h(x_{n+1})\|_N &= \left\|\sum_{k=1}^{n}(h(x_k)-h(x_{k+1}))\right\|_N \\
&= \left\|\left(\sum_{k=1}^{n}|h(x_k)-h(x_{k+1})|^{p_N}\right)^{1/p_N}\right\|_N \quad \text{(by the disjointness of supports)} \\
&\le C\left(\sum_{k=1}^{n}\|h(x_k)-h(x_{k+1})\|_N^{p_N}\right)^{1/p_N} \quad \text{(by the } p_N\text{-convexity of } L_N(G)) \\
&\le C_1\left(\sum_{k=1}^{n}(\|x_k-x_{k+1}\|_M^{\alpha})^{p_N}\right)^{1/p_N} \\
&= C_2 n^{1/p_N} M^{-1}(1/2n)^{\alpha},
\end{aligned}$$

where C, C_1, and C_2 are constants independent of n.

Case 2: $G=[0,1]$.

Since the spaces l_M^{2n} are in general not subspaces of the Orlicz space $L_M[0,1]$, the above argument does not work here and needs to be modified, in contrast to the case $M(u)=u^p$. Again we fix $n\in\mathbb{N}$, a permutation σ of $\{1,\dots,2n\}$ and a choice of signs $\theta=(\theta_1,\dots,\theta_{2n})$. We divide the interval $[0,1]$ into $2n$ subintervals $I_k=[\frac{k-1}{2n},\frac{k}{2n}]$, $1\le k\le 2n$. Let

$$\begin{aligned}
&T_{\sigma,k}: I_k \longrightarrow I_{\sigma^{-1}(k)} \\
t=\lambda\frac{k-1}{2n}+(1-\lambda)\frac{k}{2n} &\longmapsto T_{\sigma,k}(t)=\lambda\frac{\sigma^{-1}(k)-1}{2n}+(1-\lambda)\frac{\sigma^{-1}(k)}{2n}.
\end{aligned}$$

As above, we consider, for $\varphi=M$ or $\varphi=N$, the operator

$$U_{\sigma,\theta}: L_\varphi[0,1]\to L_\varphi[0,1]\,,\ x\mapsto U_{\sigma,\theta}(x),$$

defined by $U_{\sigma,\theta}(x)(t)=\theta_k x(T_{\sigma,k}(t)$ for all $t\in I_k$. This operator is an isometry because

$$\int_{I_k} x(T_{\sigma,k}(t))dt=\int_{I_{\sigma^{-1}(k)}} x(t)dt,$$

and it satisfies $U_{\sigma,\theta}^{-1}=U_{\sigma^{-1},\theta_\sigma}$, where $\theta_\sigma=(\theta_{\sigma(1)},\dots,\theta_{\sigma(2n)})$. The associated norm-one operator is given by

$$\begin{aligned}
V_{\sigma,\theta}: UC(B_M[0,1],L_N[0,1]) &\to UC(B_M[0,1],B_N[0,1]) \\
f &\mapsto U_{\sigma,\theta}\circ f\circ U_{\sigma,\theta}^{-1},
\end{aligned}$$

and the average operator V is defined in the same way as before.

Using these operators $V_{\sigma,\theta}$, the proof proceeds as before, with x_k defined by $x_k=\chi_{[\frac{k-1}{2n},\frac{k+n-1}{2n}]}$, $1\le k\le n+1$, so that $\|x_k-x_{k+1}\|_M=1/M^{-1}(n)^\alpha$ and $f_n(\alpha)=n^{1/p_N}1/M^{-1}(n)^\alpha$. Hence the lemma is proved. □

We now continue with the proof of Theorem 3.3. We fix $\delta > 0$ and let $f_n(\alpha)$ be defined as in Lemma 3.6. According to the lemma, it suffices to show that $f_n(q_M/p_N + \delta) \to 0$ as $n \to \infty$. By Proposition 2.3, $M(u^{1/q_M})$ is a concave function of u, so its inverse $M^{-1}(u)^{q_M}$ is a convex function of u. Thus, recalling that $M^{-1}(1) = 1$, we obtain $M^{-1}(1/n)^{q_M/p_N} \leq (1/n)^{1/p_N}$ and $M^{-1}(n)^{q_M/p_N} \geq n^{1/p_N}$. Hence, for all $n \in \mathbb{N}$,

$$f_n(q_M/p_N + \delta) \leq M^{-1}(1/n)^{\delta} \text{ , if } G \in \{(0,\infty), \mathbb{N}\},$$
$$f_n(q_M/p_N + \delta) \leq 1/M^{-1}(n)^{\delta} \text{ , if } G = [0,1].$$

Since $M^{-1}(1/n)$ and $1/M^{-1}(n)$ go to 0 as n increases to infinity, we obtain that $\|\phi_{MN} - g\|_\infty \geq 1/2$, for all $g \in \mathcal{H}^{q_M/p_N+\delta}(B_M(G), L_N(G))$. By the definition of the index $\alpha(B_M(G), L_N(G))$, this implies $\alpha(B_M(G), L_N(G)) \leq q_M/p_N + \delta$, for all $\delta > 0$, and the estimate follows. Using Lemma 3.5 we obtain the desired estimates for the case when $G = [0,1]$ and $G' = (0,\infty)$ and when $G' = [0,1]$ and $G = (0,\infty)$.

For the remaining cases of q_M and p_N we follow the arguments in [2, p. 38].

Case 2: $q_M \geq 2$ and $p_N \geq 2$.
According to [10, p. 134], $L_2[0,1]$, and thus $L_2(G')$, is isomorphic to a subspace of $L_M[0,1]$. Moreover, it is known (see [2]) that $L_2(G')$ has uniformly normal structure. The arguments used in the proof of Lemma 3.5 show that for every normed linear space Y we have $\alpha(B_M[0,1], Y) \leq \alpha(B_2(G'), Y)$. Taking $Y = L_N(G')$, we obtain $\alpha(B_2(G'), L_N(G')) \leq 2/p_N$. by the upper estimate proved above. Therefore $\alpha(B_M[0,1], L_N(G') \leq 2/p_N$. By Lemma 3.5, the same holds with $[0,1]$ replaced by $(0,\infty)$,

Case 3: $q_M \leq 2$ and $p_N \leq 2$.
Since, by [10], $L_2(G)$ is isomorphic to a complemented subspace of $L_N[0,1]$. it is easy to see, using the same arguments as in the proof of Lemma 3.5, that for every normed linear space X with closed unit ball B_X we have $\alpha(B_X, L_N[0,1]) \leq \alpha(B_X, L_2(G))$. Taking $X = L_M(G)$, the upper estimate already proved gives $\alpha(B_M(G), L_2(G)) \leq q_M/2$. Thus $\alpha(B_M(G), L_N[0,1]) \leq q_M/2$. Again, by Lemma 3.5, the same estimate holds with $[0,1]$ replaced by $(0,\infty)$ □

Connection with the Boyd indices. We now relate our approximation result to the usual Boyd indices. In the setting of Orlicz spaces, the Boyd indices coincide with the Matuszewska-Orlicz indices of the corresponding Orlicz function, which are defined as follows:

Definition 3.7 (see [11, p. 21]). Let M be an Orlicz function and $G \in \{[0,1],(0,\infty),\mathbb{N}\}$. We define

$$\alpha_M(G) = \sup\left\{p : \inf_{\lambda\geq 1 : u\in I(G)} M(\lambda u)/M(\lambda)u^p > 0\right\},$$

$$\beta_M(G) = \inf\left\{q : \sup_{\lambda\geq 1 : u\in I(G)} M(\lambda u)/M(\lambda)u^q < \infty\right\},$$

with $I[0,1] = [1,\infty)$, $I(\mathbb{N}) = (0,1]$ and $I(0,\infty) = (0,\infty)$. These indices are called the *Matuszewska-Orlicz indices* of the Orlicz function M.

As stated above, we can identify Matuszewska-Orlicz indices with Boyd indices, and we shall make this identification in the sequel.

Theorem 3.8. *Let $L_M(G)$ and $L_N(G')$ be two Orlicz spaces such that $1 < \alpha_M(G), \alpha_N(G')$ and $\beta_M(G), \beta_N(G') < \infty$. Then Theorem 3.3 holds with $p_M = \alpha_M(G)$, $q_M = \beta_M(G)$, $p_N = \alpha_N(G')$ and $q_N = \beta_N(G')$, provided the indices q_M and p_N are strictly greater than or strictly less than the number* 2.

Remark 3.9. The conditions on q_M and p_N break down into the following four cases: (1) $q_M > 2 > p_N > 1$; (2) $q_M, p_N > 2$; (3) $1 < q_M, p_N < 2$; (4) $1 < q_M < 2 < p_N$.

Proof. According to [10, p. 139,141] we have

$$\alpha_M(G) = \sup\{p \geq 1 : L_M(G) \text{ satisfies an upper } p\text{-estimate}\},$$
$$\beta_M(G) = \inf\{q \geq 1 : L_M(G) \text{ satisfies a lower } q\text{-estimate}\}.$$

Recall (see [10, p. 82]) that a Banach lattice $(X, \|\,.\,\|)$ is said to satisfy an upper p-estimate (resp. lower q-estimate) if the property of p-convexity (resp. q-concavity) holds for every choice of pairwise disjoint vectors $\{x_i\}_{i=1}^n$.

We set $p_M(G) = \alpha_M(G)$ and $q_M(G) = \beta_M(G)$, and fix $\varepsilon > 0$ small enough such that $1 < p_M(G)-\varepsilon < q_M(G)+\varepsilon$, and we do the same with N. According to [10, p. 100,101], $L_M(G)$ is $(p_M(G)-\varepsilon)$-convex and $(q_M(G)+\varepsilon)$-concave. Thus, by Proposition 2.3 there exists a $(p_M(G)-\varepsilon)$-convex and $(q_M(G)+\varepsilon)$-concave Orlicz function M_ε such that $M \sim_G M_\varepsilon$. Set $p_{M_\varepsilon} = p_M(G)-\varepsilon$ and $q_{M_\varepsilon} = q_M(G)+\varepsilon$. We define N_ϵ and q_{N_ϵ}, p_{N_ϵ} analogously, so that $N_\varepsilon \sim_{G'} N$. It is clear that $\alpha(B_M(G), L_N(G')) = \alpha(B_{M_\varepsilon}(G), L_{N_\varepsilon}(G'))$ because the spaces are isomorphic.

Suppose, for example, that $1 < q_M(G) < 2 < p_N(G')$. Then for ε small enough we have $1 < q_{M_\varepsilon} < 2 < p_{N_\varepsilon}$ and Theorem 3.3 gives $p_{M_\varepsilon}/q_{N_\varepsilon} \leq \alpha(B_{M_\varepsilon}(G), L_{N_\varepsilon}(G')) \leq q_{M_\varepsilon}/p_{N_\varepsilon}$. Thus, for ε small enough,

$$\frac{p_M(G)-\varepsilon}{q_N(G)+\varepsilon} \leq \alpha(B_M(G), L_N(G')) \leq \frac{q_M(G)+\varepsilon}{p_N(G)-\varepsilon}.$$

By letting ε go to 0, we obtain the desired estimate for the case $1 < q_M(G) < 2 < p_N(G')$. The other cases can be dealt with by similar arguments. □

REMARK 3.10. In particular, if the Orlicz spaces are p-convex and q-concave with p and q as their Boyd indices, then Theorem 3.3 applies directly.

Examples and comments. We now use Remark 2.2 to compute the classes $\mathcal{K}_G$ for some examples.

EXAMPLE 3.11. Consider the function $M(u) = u^2(1 + |\ln(u)|)$ for $u > 0$. This function is an Orlicz function for u in a neighbourhood of 0 and for large values of u. We define $N(u) = u^2$ if $u \geq 0$.

First, suppose that $G = G' = [0,1]$. As for large values of u we have $uM_r'(u)/M(u) = 2+1/(1+\ln(u)) \geq 2$, Theorem 3.3 gives $\alpha(B_M[0,1], L_2[0,1]) = 1$ and the proof shows that Lipschitz maps are dense (i.e., the bound $\alpha(B_M[0,1], L_2[0,1])$ is attained).

Next, suppose that $G = G' = \mathbb{N}$. We write l_2 for $L_2(\mathbb{N})$. For u in a neighbourhood of 0 we have $uM_r'(u)/M(u) = 2 - 1/(1 - \ln(u))$. Fix $\varepsilon > 0$. There exists $u_0 = u_0(\varepsilon)$ such that, for all $u \leq u_0$, $uM_r'(u)/M(u) \geq 2 - \varepsilon$. Then Theorem 3.3 gives $\alpha(B_M(\mathbb{N}), l_2) \geq (2-\varepsilon)/2$. This is true for all $\varepsilon > 0$, so $\alpha(B_M(\mathbb{N}), l_2) = 1$. But in this case Lipschitz maps are not dense (i.e., the bound $\alpha(B_M(\mathbb{N}), l_2)$ is not attained). Indeed, by Proposition 2.3 there exists a $(2-\varepsilon)$-convex and 2-concave Orlicz function M_ε, with continuous second derivative, such that $M \sim_G M_\varepsilon$. Then $\alpha(B_{M_\varepsilon}(\mathbb{N}), l_2) = \alpha(B_M(\mathbb{N}), l_2) = 1$ because the spaces are isomorphic. We apply Lemma 3.6 to the function M_ε. We note that

$$f_n(1) = n^{1/2} M_\varepsilon^{-1}(1/n) = n^{1/2} c_n,$$

where $c_n = M_\varepsilon^{-1}(1/n)$ satisfies $M_\varepsilon(c_n) = 1/n$. But $M_\varepsilon \sim_G M$, so there exists $C > 0$ independent of n such that $M_\varepsilon(c_n) \geq CM(c_n) = Cc_n^2(1 - \ln(c_n))$ for n large enough. Hence,

$$f_n(1) \leq \frac{1}{\sqrt{C}} \frac{1}{(1-\ln(c_n))^{1/2}} \to 0 \text{ as } n \to \infty.$$

Lemma 3.6 asserts that Lipschitz maps are not dense in $UC(B_{M_\varepsilon}(\mathbb{N}), l_2)$. Hence they are not dense in $UC(B_M(\mathbb{N}), l_2)$ even if $\alpha(B_M(\mathbb{N}), l_2) = 1$, a bound which is not attained here.

EXAMPLE 3.12. Suppose that $G = G' = \mathbb{N}$ and fix $p > 1 + \sqrt{2}$. Consider the Orlicz function $M(u) = u^{p+\sin(\ln|\ln(u)|)}$ in some neighbourhood of 0. Then

$$uM_r'(u)/M(u) = p + \sin(\ln|\ln(u)|) + \cos(\ln|\ln(u)|)$$

and we have $p_M = p - 1 \leq q_M = p + 1$. In this case Theorem 3.3 does not provide the precise value of α, but only upper and lower estimates for it.

4. Connection between approximation and extension

We use the following well known general principle: An extension theorem for a class of functions implies an approximation theorem by functions in this class. Consider two metric spaces (X, d_X) and (Y, d_Y). Following the terminology of A. Naor [14], we denote by $\mathcal{B}(X, Y)$ the set of all $\alpha > 0$ such that there is a constant C such that, for all $D \subset X$ and for any α-Hölder function $g : D \to Y$ with constant K, there is an α-Hölder function $\tilde{g} : X \to Y$ with constant less than CK which extends g. Such a function $\tilde{g}$ is called an isomorphic extension of g.

Suppose that $(X, \|\,.\,\|_X)$ and $(Y, \|\,.\,\|_Y)$ are two Banach spaces. Without showing that $\mathcal{B}(X, Y)$ is non-empty (which may occur), we now prove some inclusion results, which provide the supremum of the values α for which an isomorphic extension can exist.

A result of the form $\mathcal{B}(X, Y) \subset (0, \alpha_0)$ means that for $\alpha > \alpha_0$ there exists an α-Hölder map between a subset D of X and Y which admits no isomorphic extension. The following proposition gives such a result.

Proposition 4.1. *If $\alpha \in \mathcal{B}(X, Y)$, then $\mathcal{H}^\alpha(B_X, Y)$ is dense in $UC(B_X, Y)$ endowed with $\|\,.\,\|_\infty$. Hence*

$$\mathcal{B}(X, Y) \subset (0, \alpha(B_X, Y)].$$

Proof (see [2, p. 35]*).* Let $\alpha \in \mathcal{B}(X, Y)$ and let C be the corresponding constant. Let $\varepsilon > 0$ and $f : B_X \to Y$ be uniformly continuous with modulus of continuity ω. Let $\mathcal{A}$ be a maximal ε-separated set in B_X. For all $x, y \in \mathcal{A}$,

$$\|x - y\|_X \leq \left(\left[\frac{\|x - y\|_X}{\varepsilon^\alpha} \right] + 1 \right) \varepsilon^\alpha,$$

where $[\,.\,]$ denotes the integral part. As ω is nondecreasing and subadditive, we obtain

$$\begin{aligned} \|f(x) - f(y)\|_Y \leq \omega(\|x - y\|_X) &\leq \omega \left(\left(\left[\frac{\|x - y\|_X}{\varepsilon^\alpha} \right] + 1 \right) \varepsilon^\alpha \right) \\ &\leq \frac{\omega(\varepsilon^\alpha)}{\varepsilon^\alpha} (2^{1-\alpha} + 1) \|x - y\|_X^\alpha, \end{aligned}$$

because $x, y \in \mathcal{A} \subset B_X$. Thus the restriction of f to $\mathcal{A}$ is α-Hölder with constant $K = \omega(\varepsilon^\alpha)\varepsilon^{-\alpha}(2^{1-\alpha} + 1)$. Denote by $\tilde{f}$ its isomorphic extension with constant CK. Let $x \in B_X$. There exists $y \in \mathcal{A}$ such that $\|x - y\|_X \leq \varepsilon$ and thus

$$\begin{aligned} \|f(x) - \tilde{f}(x)\|_Y &\leq \|f(x) - f(y)\|_Y + \|\tilde{f}(y) - \tilde{f}(x)\|_Y \\ &\leq \omega(\varepsilon) + C(2^{1-\alpha} + 1)\omega(\varepsilon^\alpha). \end{aligned}$$

Thus $f \in \overline{\mathcal{H}^\alpha(B_X, Y)}^{\|\,.\,\|_\infty}$, and by definition $\alpha \leq \alpha(B_X, Y)$. □

Claim 4.2. *There exist Banach spaces X and Y such that $\alpha(B_X, Y) = 0$ and thus $\mathcal{B}(X, Y) = \emptyset$.*

Indeed, fix $p \geq 1$ and take $X = L_p[0,1]$ and $Y = (\sum L_{q_k}[0,1])_2$ with $q_k \geq 2$ for all k and $q_k \to \infty$. Since there exists a norm-one projection from Y onto L_{q_k} for all k, by arguing as in the proof of Lemma 3.5 one easily sees, using Theorem 3.1, that $\alpha(B_p, Y) \leq \alpha(B_p, L_{q_k}) = \min\{p, 2\}/q_k$ for all $k \in \mathbb{N}$. The right-hand side here tends to 0 as k goes to ∞, so we have $\alpha(B_X, Y) = 0$, and the above proposition gives $\mathcal{B}(X, Y) = \emptyset$.

Claim 4.3. *Theorem 3.3 and Proposition 4.1 provide examples of normed spaces Y such that $1 \notin \mathcal{B}(H, Y)$, where H stands for the Hilbert spaces L_2 or l_2.*

This fact, stated in [14] in the setting of L_p-spaces, provides another answer to a question posed by K. Ball in [1] and solved in [14].

Example 4.4. Consider our previous Orlicz function $M(u) = u^2(1 + |\ln(u)|)$ in some neighbourhood of 0. Set $l_M = L_M(\mathbb{N})$. Then we have

$$\mathcal{B}(l_M, l_2) \subset (0, 1[.$$

It may be possible to extend isomorphically every α-Hölder maps with $\alpha < 1$, but not every Lipschitz map. Indeed, we have seen that Lipschitz maps are not dense in $UC(B_M(\mathbb{N}), l_2)$ endowed with $\|\,.\,\|_\infty$.

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