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INFINITE PRODUCTS OF INFINITE MEASURES

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ABSTRACT. Let $(X_i, \mathcal{B}_i, m_i)$ $(i \in \mathbb{N})$ be a sequence of regular Borel measure spaces. There is a Borel measure μ on $\prod_{i\in\mathbb{N}} X_i$ such that if $K_i \subseteq X_i$ is compact for all $i \in \mathbb{N}$ and $\prod_{i\in\mathbb{N}} m_i(K_i)$ converges, then $\mu(\prod_{i\in\mathbb{N}} K_i) = \prod_{i\in\mathbb{N}} m_i(K_i)$.

1. Introduction

Let $(X_i, \mathcal{B}_i, m_i)$ $(i \in \mathbb{N})$ be a sequence of regular Borel measure spaces, where each X_i is a Hausdorff topological space. We prove the following:

THEOREM 1.1. *There is a Borel measure μ on $\prod_i X_i$ (with respect to the product topology) such that if $K_i \subseteq X_i$ is compact for all $i \in \mathbb{N}$ and $\prod_{i\in\mathbb{N}} m_i(K_i)$ converges, then $\mu(\prod_i K_i) = \prod_{i\in\mathbb{N}} m_i(K_i)$.*

We note that a special case of this result (for the case where $X_i = \mathbb{R}$ and m_i is Lebesgue measure) was proved only fairly recently ([2]). Moreover, the techniques used in [2] to prove this special case depend heavily on the metric structure of $\mathbb{R}$. The proof we use for this more general result is very different, but not more difficult, and is an adaptation of a construction from nonstandard analysis.

2. Lemmas

We begin with some useful lemmas. The first lists some properties of infinite products that are used in the proof of Theorem 1.1.

We say that a product $\prod_{i\in\mathbb{N}} a_i$ *converges to* $r \in \mathbb{R}$ (in which case we write $\prod_{i\in\mathbb{N}} a_i = r$) if $\lim_{N\to\infty} \prod_{i\le N} a_i = r$. (We remark that the definition of 'convergence' is often restricted to preclude the case $r = 0$; we make no such restriction here.)

LEMMA 2.1. *Suppose $\prod_{i\in\mathbb{N}} a_i = r$ and $\prod_{i\in\mathbb{N}} b_i = s$. Then:*

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(1) *If* $r \neq 0$ *then* $\lim_{N\to\infty} \prod_{i\geq N} a_i = 1$.
(2) $\prod_{i\in\mathbb{N}} a_i b_i = rs$.
(3) *If* $s \neq 0$ *then* $\prod_{i\in\mathbb{N}} (a_i/b_i) = r/s$.
(4) *If* $0 \leq c_i \leq a_i$ *for all* $i \in \mathbb{N}$ *then* $\prod_{i\in\mathbb{N}} c_i$ *converges (possibly to* 0*).*
(5) *If* $0 \leq a_i \leq b_i$ *for all* $i \in \mathbb{N}$ *and* $r \neq 0$ *then* $\lim_{N\to\infty} \sum_{i\geq N}(1 - a_i/b_i) = 0$.

Proof. Statements (1)-(3) are easy consequences of the definition of convergence. Statement (4) follows from (2) and the observation that $\prod_{i\in\mathbb{N}} (c_i/a_i)$ must converge. For statement (5), write $c_i = a_i/b_i$, and let $d_i = 1 - c_i$; note $d_i \geq 0$. By multiplying out the terms we get $\sum_{i\leq N} d_i \leq \prod_{i\leq N} (1 + d_i)$. Since $0 \leq 1 + x \leq (1 - x)^{-1}$ for $0 \leq x < 1$, $\prod_{i\leq N} (1 + d_i) \leq \prod_{i\leq N}(1 - d_i)^{-1}$. By (3) it follows that $\prod_i(1 - d_i)^{-1} = s/r < \infty$. Therefore, $\sum_{i\leq N} d_i$ is bounded above as $N \to \infty$, and the result follows. □

The next two lemmas are general measure-theoretic results. Recall that a *measurable space* is a pair $(X, \mathcal{B})$ where $\mathcal{B}$ is a σ-algebra on X. If μ is a measure on $(X, \mathcal{B})$ and $Z \in \mathcal{B}$, then let $\mu|_Z$ be the new measure on $(X, \mathcal{B})$ defined by $\mu|_Z(E) = \mu(E \cap Z)$.

Lemma 2.2 (Pasting Lemma). *Let* $(X, \mathcal{B})$ *be a measurable space,* $\mathcal{Z}$ *a subset of* $\mathcal{B}$ *which is closed under finite unions, and suppose* $\{\mu_Z\}_{Z\in\mathcal{Z}}$ *are finite measures on* $(X, \mathcal{B})$ *satisfying: if* $Z_1 \subseteq Z_2$ *then* $\mu_{Z_1} = (\mu_{Z_2})|_{Z_1}$. *Then* $\mu = \sup_{Z\in\mathcal{Z}} \mu_Z$ *defines a (possibly infinite) measure on* $(X, \mathcal{B})$.

Proof. The only nontrivial verification is countable additivity. Let $\{A_n\}_{n\in\mathbb{N}}$ be a sequence of disjoint elements of $\mathcal{B}$ and put $A = \bigcup_n A_n$. For any $Z \in \mathcal{Z}$, $\mu_Z(A) = \sum_{n\in\mathbb{N}} \mu_Z(A_n) \leq \sum_{n\in\mathbb{N}} \mu(A_n)$; it follows that $\mu(A) \leq \sum_{n\in\mathbb{N}} \mu(A_n)$. In particular, if $\mu(A) = \infty$, then $\sum_{n\in\mathbb{N}} \mu(A_n) = \infty$. Suppose conversely that $\mu(A) = r < \infty$, and (for a contradiction) that $\sum_{n\in\mathbb{N}} \mu(A_n) > r + \epsilon$ for some $\epsilon > 0$. Let $N \in \mathbb{N}$ be large enough that $\sum_{n\leq N} \mu(A_n) > r+\epsilon$. Since $\mathcal{Z}$ is closed under finite unions, there is a $Z \in \mathcal{Z}$ such that $\sum_{n\leq N} \mu_Z(A_n) > r + \epsilon$. Then $\mu(A) \geq \mu_Z(A) = \sum_{n\in\mathbb{N}} \mu_Z(A_n) \geq \sum_{n\leq N} \mu_Z(A_n) > r+\epsilon$, a contradiction. □

Suppose $(X, \mathcal{B})$ is a measurable space, that $\mathcal{M}$ is a family of finite measures on $(X, \mathcal{B})$, and that Y is a topological space. Call a function $f\colon X \to Y$ $\mathcal{M}$-*measurable* if for every $\mu \in \mathcal{M}$ the function f is measurable with respect to the completion of μ.

Lemma 2.3 (Forgetful Measurability). *Suppose:*

(1) $(X_1, \mathcal{B}_1)$ *and* $(X_2, \mathcal{B}_2)$ *are measurable spaces;*
(2) $(X, \mathcal{B})$ *is* $X_1 \times X_2$ *with the product sigma algebra;*
(3) $\{a\} \in \mathcal{B}_2$ *for some* $a \in X_2$*;*
(4) $\mathcal{M}_1$ *is a family of finite measures on* $(X_1, \mathcal{B}_1)$*;*

(5) $\mathcal{M} = \{\mu \times \delta_a : \mu \in \mathcal{M}_1\}$ *(where δ_a is a point mass at a);*
(6) $g : X_1 \to Y$ *for some topological space Y; and*
(7) $f(x_1, x_2) = g(x_1)$ *is $\mathcal{M}$-measurable.*

Then g is $\mathcal{M}_1$-measurable.

Proof. Let $\mu \in \mathcal{M}_1$ and $u \subseteq Y$ be open; we need to show that $g^{-1}(u)$ is completion-measurable for $(X_1, \mathcal{B}_1, \mu)$. We have $L = X_1 \times \{a\}$, and $\nu = \mu \times \delta_a$. Observe that if $E \in \mathcal{B}$ then $E \cap L = E_1 \times \{a\}$ for some $E_1 \in \mathcal{B}_1$. (This is clearly true when E is a measurable rectangle, and the property is preserved under complements and countable unions, so it holds for all of $\mathcal{B}$.) It follows that the completion of ν is the product of the completion of μ with δ_a. By hypothesis $f^{-1}(u)$, and therefore $L \cap f^{-1}(u) = g^{-1}(u) \times \{a\}$, is ν-measurable, so $g^{-1}(u)$ is μ-measurable. □

3. Proof of Theorem 1.1

Put $X = \prod_{i \in \mathbb{N}} X_i$, and let $\mathcal{B}$ be the Borel σ-algebra on X with respect to the product topology.

Call a product $\prod_{i \in \mathbb{N}} E_i$ a $\mathcal{K}$-*tube* if each E_i is a compact subset of X_i and $\prod_{i \in \mathbb{N}} m_i(E_i) = r$ for some $r \in (0, \infty)$. If instead each E_i is an *open* subset of X_i then call $\prod_{i \in \mathbb{N}} E_i$ a $\mathcal{U}$-*tube*. Note that while a $\mathcal{K}$-tube is necessarily compact, a $\mathcal{U}$-tube will not in general be open.

The problem is to find a measure on $(X, \mathcal{B})$ that assigns the 'correct' measure to $\mathcal{K}$-tubes.

Let $\mathcal{Z}$ consist of all (nonempty) finite unions of $\mathcal{K}$-tubes.

For the remainder of the paper it will be convenient to work in the framework of nonstandard analysis; we adopt the notation of [1].

We note for later reference that if $\{a_i\}_i \subseteq \mathbb{R}$ and $r \in \mathbb{R}$, then $\prod_{i \in \mathbb{N}} a_i = r$ if and only if $\prod_{i \leq H} a_i \approx r$ for any infinite $H \in^* \mathbb{N}$. (This is just the nonstandard criterion for limits, applied to the definition of $\prod_{i \in \mathbb{N}} a_i = r$.)

Fix once and for all some $H \in^* \mathbb{N} \setminus \mathbb{N}$. If $E = \prod_{i \in \mathbb{N}} E_i$ write ${}^{\#}E = {}^{*}\prod_{i \leq H} E_i$; extend this in the obvious way to finite unions of such sets. Define a partial function $\mathrm{st}_{\#} : {}^{\#}X \to X$ by $\mathrm{st}_{\#}(\langle x_i \rangle_{i \leq H}) = \langle {}^{\circ}x_i \rangle_{i \in \mathbb{N}}$.

The following is a truncated version of Tychonoff's Theorem:

Proposition 3.1. *If $Z \in \mathcal{Z}$ then ${}^{\#}Z \subseteq \mathrm{st}_{\#}^{-1}(Z)$.*

Proof. It suffices to assume that $Z = \prod_{i \in \mathbb{N}} K_i$, K_i compact. If $\langle x_i \rangle_{i \leq H} \in {}^{\#}\prod_{i \in \mathbb{N}} K_i$, then $x_i \in^* K_i$ for all standard i; since K_i is compact, ${}^{\circ}x_i$ exists in K_i, so $\mathrm{st}_{\#}(\langle x_i \rangle_i) \in \prod_{i \in \mathbb{N}} K_i$. □

Lemma 3.1. *If $Z \in \mathcal{Z}$, then $\mathrm{st}_{\#}$ is universally Loeb measurable from ${}^{\#}Z$ to Z (and hence to X).*

Proof. This follows immediately from forgetful measurability, compactness of Z, and the fact ([1, page 100]) that the standard part map on ${}^*Y$ is universally Loeb measurable for any compact Hausdorff space Y. $\square$

Let λ be the product measure $\prod_{i \le H} {}^*m_i$ on ${}^\# X$, and for $Z \in \mathcal{Z}$ let $\lambda_Z = \lambda|_{{}^\# Z}$. Note that λ_Z is finite. For each $Z \in \mathcal{Z}$ apply the Loeb measure construction (see [1, page 95]) to the internal *Borel measure λ_Z to get a standard, complete measure λ_{ZL}. By Lemma 3.1, $\mathrm{st}_\#$ is λ_{ZL}-measurable from ${}^\# Z$ to X; let μ_Z be the Borel image measure on X of λ_{ZL} under $\mathrm{st}_\#$. (Note in particular that $\mu_Z = \mu_Z|_Z$.)

The measures μ_Z $(Z \in \mathcal{Z})$ evidently satisfy the hypothesis of the Pasting Lemma, so $\mu = \sup_{Z \in \mathcal{Z}} \mu_Z$ defines a Borel measure on X. The next two lemmas show that μ gives the right measure to $\mathcal{K}$-tubes.

LEMMA 3.2. *Suppose $E = \prod_i E_i$ and $F = \prod_i F_i$ are $\mathcal{K}$-tubes, that $U = \prod_i U_i$ is a $\mathcal{U}$-tube, and $E \subseteq U$. Then $\lambda_{FL}(\mathrm{st}_\#^{-1}(E) \cap ({}^\# F \setminus {}^\# U)) = 0$.*

Proof. Note that

$$\begin{aligned}
&\mathrm{st}_\#^{-1}(E) \cap ({}^\# F \setminus {}^\# U)) \\
&\qquad = \big\{ \langle x_i \rangle_{i \le H} \in {}^\# F : \forall i \in \mathbb{N}\ {}^\circ x_i \in E_i \cap F_i \\
&\qquad\qquad \text{and } \exists \text{ infinite } i \le H\ x_i \in (F_i \setminus U_i) \big\} \\
&\qquad \subseteq \big\{ \langle x_i \rangle_{i \le H} \in {}^\# F : \forall i \in \mathbb{N}\ x_i \in U_i \cap F_i \\
&\qquad\qquad \text{and } \exists \text{ infinite } i \le H\ x_i \in (F_i \setminus U_i) \big\}.
\end{aligned}$$

We now consider two cases:

(1) $\prod_{i \in \mathbb{N}} m_i(F_i \cap U_i) = 0$. Let $\epsilon > 0$; then for sufficiently large $N_0 \in \mathbb{N}$, $\prod_{i \le N_0} m_i(F_i \cap U_i) < \epsilon$, and, since F is a $\mathcal{K}$-tube, $\prod_{i > N_0} m_i(F_i) < 1 + \epsilon$ (by Lemma 2.1). It follows from the latter inequality and the nonstandard criterion for convergence that $\prod_{N_0 < i \le H} m_i(F_i) < 1 + \epsilon$. By the properties of finite product measures, transferred to λ,

$$\lambda \left({}^*\prod_{i \le N_0} (F_i \cap U_i) \times \prod_{N_0 < i \le H} F_i \right) < \epsilon(1 + \epsilon).$$

It suffices to observe (by the note above) that

$$\mathrm{st}_\#^{-1}(E) \cap ({}^\# F \setminus {}^\# U)) \subseteq {}^*\prod_{i \le N_0} (F_i \cap U_i) \times \prod_{N_0 < i \le H} F_i.$$

(2) $\prod_{i \in \mathbb{N}} m_i(F_i \cap U_i)$ does not converge to 0. Then, by Lemma 2.1, $\prod_{i \in \mathbb{N}} m_i(F_i \cap U_i)$ converges to a positive value. Put $r = \lambda({}^\# F) \approx \prod_{i \in \mathbb{N}} m_i(F_i)$.

If $n \le H$, then

$$\begin{aligned}\lambda\left((F_n \setminus U_n) \times \prod_{n\ne i\le H} F_i\right) &= m_n(F_n \setminus U_n) \times \prod_{n\ne i\le H} m_i(F_i)\\ &= r\frac{m_n(F_n \setminus U_n)}{m_n(F_n)}\\ &= r\left(1 - \frac{m_n(F_n \cap U_n)}{m_n(F_n)}\right).\end{aligned}$$

If $N \in \mathbb{N}$, then

$$\lambda_{FL}\left(\bigcup_{N\le n\le H} (F_n \setminus U_n) \times \prod_{n\ne i\le H} F_i\right) \le \sum_{n\in\mathbb{N}, n\ge N}^{\circ} r\left(1 - \frac{m_n(F_n \cap U_n)}{m_n(F_n)}\right).$$

Since the right-hand summand tends to 0 as $N \to \infty$ (Lemma 2.1), and

$$\mathrm{st}_{\#}^{-1}(E) \cap ({}^{\#}F \setminus {}^{\#}U) \subseteq \bigcup_{N\le n\le H} (F_n \setminus U_n) \times \prod_{n\ne i\le H} F_i$$

for *any* standard $N \in \mathbb{N}$, the lemma follows. □

LEMMA 3.3. *If $E = \prod_i E_i$ is a $\mathcal{K}$-tube then $\mu(E) = \prod_i m_i(E_i)$.*

Proof. Put $r = \prod_i m_i(E_i)$. It suffices to show that $\mu_Z(E) = r$ for all $Z \in \mathcal{Z}$ containing E; so let $Z = E \cup F^1 \cup \cdots \cup F^m$, where each F^i is a $\mathcal{K}$-tube. Fix $\epsilon > 0$, and let $\langle r_i\rangle_{i\in\mathbb{N}}$ be any sequence from $(1, \infty)$ such that $\prod_i r_i \le 1 + \epsilon$, for example, $r_i = (1 + \epsilon)^{2^{-(i+1)}}$. Borel measures are outer regular with respect to open sets, so for each $i \in \mathbb{N}$ there is an open U_i with $E_i \subseteq U_i$ and $m_i(U_i) < r_i m_i(E_i)$. Put $U = \prod_i U_i$, and note (by Lemma 2.1) that $s = \prod_i m_i(U_i)$ exists and $r \le s \le r + r\epsilon$. Then:

$$\begin{aligned} r &\approx \lambda_Z({}^{\#}E)\\ &\lessapprox \lambda_{ZL}(\mathrm{st}_{\#}^{-1}(E))\\ &\le \lambda_{ZL}\left({}^{\#}U \cup \bigcup_{i=1}^{m} \mathrm{st}_{\#}^{-1}(E) \cap ({}^{\#}F^i \setminus {}^{\#}U)\right)\\ &\le s + \sum_{i=1}^{m} \lambda_{ZL}\left(\mathrm{st}_{\#}^{-1}(E) \cap ({}^{\#}F^i \setminus {}^{\#}U)\right)\\ &= s + \sum_{i=1}^{m} \lambda_{F^iL}\left(\mathrm{st}_{\#}^{-1}(E) \cap ({}^{\#}F^i \setminus {}^{\#}U)\right)\\ &\le r + r\epsilon + \sum_{i=1}^{m} 0.\end{aligned}$$

The result follows since ϵ is arbitrary. □

REMARKS 3.1. *Suppose each m_i is Haar measure on a locally compact additive group X_i. If $x = \langle x_i \rangle_i \in X$ and K is a $\mathcal{K}$-tube, then $x + K$ is a $\mathcal{K}$-tube and $\lambda(^{\#}(x + K)) = \lambda(^{\#}(K))$; it follows that the measure μ we have constructed is translation-invariant.*

4. Discussion

Suppose that each m_i has no point masses, and that infinitely many of these measures are infinite. It is easy to verify the following properties of the resulting measure on $\prod_{i \in \mathbb{N}} X_i$: (i) all open sets in the product topology will have infinite measure; and (ii) all compact sets in the box topology will be nullsets. It follows that many conventional tools for constructing Borel measures will not apply to this situation.

This is true as well for traditional nonstandard measure arguments. The usual way to create a standard measure μ from a Loeb measure ν is to obtain μ as the image, under a measurable function (usually the standard part map), of ν. It is not difficult to see that this technique will *not* work for Theorem 1.1. Finding new ways to "push down" nonstandard measures to standard ones is a major problem in nonstandard analysis; see [1]. Our technique of using a nonstandard measure to 'control' the assembly of standard measures is new with this paper.

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