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INDUCTIVE ALGEBRAS FOR $\mathrm{SL}(2,\mathbb{R})$

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Abstract. We give a characterisation of the maximal Abelian sub-algebras of the bounded operators on a Hilbert space that are normalised by the unitary representations of the group of unimodular two by two matrices.

1. Introduction

Let G be a separable locally compact group and π an irreducible unitary representation of G on a separable Hilbert space $\mathcal{H}$. Let $\mathcal{B}(\mathcal{H})$ denote the algebra of bounded operators on $\mathcal{H}$. An *inductive algebra* is a weakly closed Abelian sub-algebra of $\mathcal{B}(\mathcal{H})$ that is normalised by $\pi(G)$. If we wish to emphasise the dependence on π, we will use the term π-inductive algebra.

Unitary representations of G are typically constructed in the following way. A vector bundle $E \to X$ is given, together with a measure μ on X and a Hermitian fiber-metric. An action of G on E is given which carries fibers to fibers, is linear on fibers, and preserves the fiber metric. It follows that G acts on X (since X may be regarded as the set of fibers). This action is supposed to preserve the class of μ. Under these conditions, G acts on $L^2(X,E,\mu)$, the space of square integrable sections of E, by

$$(g \cdot s)(x) = \sqrt{\frac{d\mu(g^{-1}x)}{d\mu(x)}}\, gs(g^{-1}x), \qquad g \in G, s \in L^2(X,E,\mu).$$

A G-invariant subspace $\mathcal{H} \subseteq L^2(X,E,\mu)$ is specified, usually as the solution-space of a differential equation. When a representation is constructed in this way, we say that it is "realised" on a space of sections of E. Induced and holomorphically-induced representations are special cases of this construction.

When a representation is realised as above, we find a natural inductive algebra, namely

$$\mathcal{A}_{\mathcal{H}} = \{M_f \mid f \in L^\infty(X), M_f(\mathcal{H}) \subseteq \mathcal{H}\},$$

where $M_f : L^2(X,E,\mu) \to L^2(X,E,\mu)$ is given by $(M_f s)(x) = f(x)s(x)$.

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Conversely, given an inductive algebra $\mathcal{A}$ for an abstract representation, it is sometimes possible to realize the representation on a space $\mathcal{H}$ of sections of a vector bundle such that $\mathcal{A} = \mathcal{A}_{\mathcal{H}}$.

Mackey's results (see [4, §§2.5, 2.6, 3.7] and [8, Thm. B.10]) give a partial inverse to the previous construction: A self-adjoint inductive algebra arises from a *system of imprimitivity*, which may be thought of as a "measurable vector bundle". More precisely, there is a standard Borel G-space X, a quasi-invariant measure μ on X, a Hilbert space E and a $\mathcal{B}(E)$-valued cocycle A such that π is unitarily equivalent to the action of G on $L^2(X, E, \mu)$ given by

$$(g \cdot f)(x) = A(g, x)f(g^{-1}x),$$

and $\mathcal{A}$ corresponds to the algebra $\{M_f \mid f \in L^\infty(X)\}$ under this equivalence.

The connection between non-self-adjoint inductive algebras and realisations is somewhat tenuous, but bears out in examples, such as those considered in this paper. We hope that by systematically studying inductive algebras it will be possible to find interesting realizations of representations.

The notion of inductive algebra was introduced in [7], where the maximal inductive algebras for the Heisenberg groups were identified. In [5] (resp. [6]), Stegel found the maximal inductive algebras for the principal series representations of the full (resp. even) automorphism group of a homogeneous tree.

In this work, we identify the maximal inductive algebras for all irreducible unitary representations of $\mathrm{SL}(2, \mathbb{R})$. We use the infinitesimal approach here. If $\mathcal{A}$ is an "unknown" maximal inductive algebra, $T \in \mathcal{A}$ is smooth (i.e., $g \mapsto \pi(g)T\pi(g)^{-1}$ is smooth), and X is an element of the complexified Lie algebra, then $[T, d\pi(X)] \in \mathcal{A}$, and since $\mathcal{A}$ is Abelian it follows that

$$[T, [T, d\pi(X)]] = 0.$$

Further, if T is covariant under a maximal compact subgroup of G and X is chosen judiciously, it is possible to explicitly solve this "quadratic" equation for T, and thus identify $\mathcal{A}$.

The results are summarised in Section 2. Section 3 describes the infinitesimal structure of inductive algebras for Lie groups in general. These results are the basis for the detailed calculations in Section 4.

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2. A list of the maximal inductive algebras

Let $\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}$ be the unit disc, and let $\mathbb{T} = \{z \in \mathbb{C} \mid |z| = 1\}$ be the unit circle. If f is a function, we denote by M_f the operator of multiplication by f, i.e., $[M_f(F)](z) = f(z)F(z)$ for each function F and each z. The domain of M_f will be clear from context.

The irreducible unitary representations of $\mathrm{SL}(2,\mathbb{R})$ were classified in 1947 by Bargmann [2]. They consist of the discrete series, the principal series, and the complementary series.

Following Bargmann, we consider

$$\mathrm{SU}(1,1)=\left\{\begin{pmatrix}\alpha & \beta\\ \overline{\beta} & \overline{\alpha}\end{pmatrix}\middle| \, |\alpha|^2-|\beta|^2=1\right\},$$

which is isomorphic to $\mathrm{SL}(2,\mathbb{R})$.

We describe each representation of $\mathrm{SU}(1,1)$, and the maximal inductive algebras associated with it. That this list is exhaustive is proved in Section 4.

The discrete series: For a positive integer w let $\mathcal{H}_w$ be the Hilbert space of holomorphic functions on $\mathbb{D}$ with squared norm

$$\|F\|_w^2=\begin{cases}\int_{|z|<1}|F(z)|^2\,(1-|z|^2)^{w-2}dz & \text{for} \quad w\geq 2,\\ \sup_{0\leq r<1}\int_0^{2\pi}\left|F(re^{i\theta})\right|^2 d\theta & \text{for} \quad w=1.\end{cases}$$

The *holomorphic discrete series* representations are parametrised by a positive integer w and act on $\mathcal{H}_w$ by

$$\mathcal{D}^{+,w}\begin{pmatrix}\alpha & \beta\\ \overline{\beta} & \overline{\alpha}\end{pmatrix}F(z)=(-\beta z+\overline{\alpha})^{-w}F\left(\frac{\alpha z-\overline{\beta}}{-\beta z+\overline{\alpha}}\right).$$

Let $H^\infty(\mathbb{D})$ denote the space of bounded holomorphic functions on $\mathbb{D}$, and let $\mathcal{B}=\{M_f:\mathcal{H}_w\to\mathcal{H}_w \mid f\in H^\infty(\mathbb{D})\}$. Then $\mathcal{B}$ and $\mathcal{B}^*$ are maximal $\mathcal{D}^{+,w}$-inductive algebras for each positive integer w. They are not self-adjoint.

The *anti-holomorphic discrete series* representations are the complex-conjugates of the holomorphic discrete series representations. So the maximal inductive algebras are the complex-conjugates of the ones for the holomorphic discrete series. Clearly they are not self-adjoint.

The spherical principal series and the complementary series: The non-unitary spherical principal series representations are parametrised by a complex number w, and realized on an appropriate space $\mathcal{H}_w$ of distributions on $\mathbb{T}$ by the formula

$$\mathcal{P}^{+,w}\begin{pmatrix}\alpha & \beta\\ \overline{\beta} & \overline{\alpha}\end{pmatrix}F(z)=|-\beta z+\overline{\alpha}|^{-1-w}\,F\left(\frac{\alpha z-\overline{\beta}}{-\beta z+\overline{\alpha}}\right).$$

The representations $\mathcal{P}^{+,w}$ are unitarizable precisely when $w\in i\mathbb{R}\cup(-1,1)$, and we consider them only for such w. For a distribution F on $\mathbb{T}$ with Fourier series $F=\sum_{n=-\infty}^{\infty}\hat{F}_n e_n$ let

$$\|F\|_w^2=\sum_{n=-\infty}^{\infty}\left|\frac{\Gamma(\frac{1-w}{2}+|n|)}{\Gamma(\frac{1+w}{2}+|n|)}\right|\left|\hat{F}_n\right|^2,$$

and let

$$\mathcal{H}_w=\{F\in\mathcal{D}'(\mathbb{T})\mid \|F\|_w<\infty\}.$$

Then $\mathcal{H}_w$ is a Hilbert space and $\|.\|_w$ is preserved by $\mathcal{P}^{+,w}$. There are intertwiners $\mathcal{I}_{+,w} : \mathcal{P}^{+,w} \to \mathcal{P}^{+,-w}$ for $w \in i\mathbb{R} \cup (-1,1)$.

When $w \in i\mathbb{R}$, $\mathcal{P}^{+,w}$ is known as a *spherical principal series* representation, and when $w \in (-1,1)$, $\mathcal{P}^{+,w}$ is known as a *complementary series* representation.

By Stirling's formula, $\mathcal{H}_w = L^2(\mathbb{T})$ for $w \in i\mathbb{R}$, and the Sobolev space of order $-w/2$ for $w \in (-1,1)$.

Let $\mathcal{C} = \overline{\{M_f : \mathcal{H}_w \to \mathcal{H}_w \mid f \in C^\infty(\mathbb{T})\}}$, where the bar on top denotes weak closure. Let $\mathcal{C}_{+,w} = \mathcal{I}_{+,w}^{-1}\mathcal{C}\mathcal{I}_{+,w}$. Then $\mathcal{C}$ and $\mathcal{C}_{+,w}$ are maximal $\mathcal{P}^{+,w}$-inductive algebras for all $w \in i\mathbb{R} \cup (-1,1)$. The set $\mathcal{C}$ depends on (the real part of) w as well, because $\mathcal{H}_w$ depends on (the real part of) w, but it is notationally convenient to suppress that dependence. They are self adjoint precisely when $w \in i\mathbb{R}$. Note that if $w \in i\mathbb{R}$, then $\mathcal{C} = L^\infty(\mathbb{T})$.

The non-spherical principal series: These representations are parametrised by a non-zero imaginary number w, and realized on the Hilbert space $\mathcal{H} = L^2(\mathbb{T})$ by the formula

$$\mathcal{P}^{-,w}\begin{pmatrix}\alpha & \beta \\ \overline{\beta} & \overline{\alpha}\end{pmatrix} F(z) = \left(\frac{-\beta z + \overline{\alpha}}{|-\beta z + \overline{\alpha}|}\right) |-\beta z + \overline{\alpha}|^{-1-w} F\left(\frac{\alpha z - \overline{\beta}}{-\beta z + \overline{\alpha}}\right).$$

There are intertwiners $\mathcal{I}_{-,w} : \mathcal{P}^{-,w} \to \mathcal{P}^{-,-w}$

Let $\mathcal{C} = \{M_f : \mathcal{H} \to \mathcal{H} \mid f \in L^\infty(\mathbb{T})\}$. Let $\mathcal{C}_{-,w} = \mathcal{I}_{-,w}^{-1}\mathcal{C}\mathcal{I}_{-,w}$. Then $\mathcal{C}$ and $\mathcal{C}_{-,w}$ are maximal $\mathcal{P}^{-,w}$-inductive algebras for all $w \in i\mathbb{R}$. They are all self-adjoint.

3. Inductive algebras for Lie groups

Let G be a Lie group with Lie algebra $\mathfrak{g}$. Let π be an irreducible unitary representation of G on a Hilbert space $\mathcal{H}$. Let $\mathcal{B}(\mathcal{H})$ denote the algebra of bounded operators on $\mathcal{H}$. We give $\mathcal{B}(\mathcal{H})$ the weak (i.e., strong operator) topology. Note that for each $g \in G$ the map $\kappa(g) : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ given by $T \mapsto \pi(g)T\pi(g)^{-1}$ is a continuous algebra homomorphism. It is obvious that κ is a homomorphism and κ is strongly continuous, in the sense that for each $T \in \mathcal{B}(\mathcal{H})$ the map $g \mapsto \kappa(g)T$ is continuous.

Definition 3.1. An *inductive algebra* is a κ-invariant weakly closed Abelian sub-algebra of $\mathcal{B}(\mathcal{H})$. An inductive algebra is said to be *maximal* if it is a maximal element of the set of inductive algebras, partially ordered by inclusion.

Let $\mathcal{A}$ be a maximal inductive algebra. Let $\mathcal{H}^\infty$ and $\mathcal{A}^\infty$ denote the spaces of smooth vectors in $\mathcal{H}$ and $\mathcal{A}$, respectively. Note that $\mathcal{A}^\infty$ is an algebra. Easy modifications of arguments found in [3, Ch. III, §3] show that $\mathcal{A}^\infty$ is $d\kappa$-invariant and κ-invariant. We now prove a few preliminary results which are needed in Section 4.

LEMMA 3.2. *$\mathcal{A}^\infty$ is sequentially dense in $\mathcal{A}$.*

Proof. For $n = 1, 2, \ldots$ let φ_n be a positive smooth function on G which is supported in a ball of radius $1/n$ around 1 and such that $\int_G \varphi(g)dg = 1$. Let $T \in \mathcal{A}$ and put $T_n = \int_G \kappa(g)T\varphi_n(g)dg$. Observe that $T_n \in \mathcal{A}^\infty$. Let $v, w \in \mathcal{H}$ and $\varepsilon > 0$. Since κ is strongly continuous, the set $S = \{g \in G \mid |\langle(\kappa(g)T - T)v, w\rangle| < \varepsilon\}$ is an open neighbourhood of 1. So there exists N such that $n \geq N$ implies $\operatorname{supp}(\varphi_n) \subseteq S$, and so

$$\begin{aligned} |\langle(T_n - T)v, w\rangle| &= \left|\int_G \langle\varphi_n(g)(\kappa(g)T - T)v, w\rangle dg\right| \\ &\leq \int_G \varphi_n(g)\, |\langle(\kappa(g)T - T)v, w\rangle|\, dg \\ &\leq \varepsilon \int_G \varphi_n(g)dg = \varepsilon, \end{aligned}$$

whence $T_n \to T$. □

LEMMA 3.3. *If $T \in \mathcal{A}^\infty$ and $v \in \mathcal{H}^\infty$, then $Tv \in \mathcal{H}^\infty$.*

Proof. Let $\mathcal{H}'$ denote the linear span of the set $\{Tv \mid T \in \mathcal{A}^\infty, v \in \mathcal{H}^\infty\}$. If $X \in \mathfrak{g}$, $T \in \mathcal{A}^\infty$ and $v \in \mathcal{H}^\infty$, the product rule $d\pi(X)(Tv) = (d\kappa(X)T)v + Td\pi(v)$ implies that $\mathcal{H}'$ is stable under $d\pi(X)$. So $\mathcal{H}' \subseteq \mathcal{H}^\infty$ by a slight modification of Lemma 3.13 in [3]. □

Let K be a compact subgroup of G and let χ be in $\mathcal{C}(K)$, the set of continuous functions on K. For $T \in \mathcal{B}(\mathcal{H})$ define $\Pi_\chi(T) = \int_K (\kappa(k)T)\overline{\chi(k)}dk$.

LEMMA 3.4. *Π_χ is sequentially continuous for all $\chi \in \mathcal{C}(K)$.*

Proof. Suppose $T_n \to T$ in $\mathcal{B}(\mathcal{H})$. By the uniform boundedness principle $\|T_n\|$ is bounded, and so for all $v, w \in \mathcal{H}$ we have $\langle\Pi_\chi(T_n)v, w\rangle \to \langle\Pi_\chi(T)v, w\rangle$ by the dominated convergence theorem. □

LEMMA 3.5. *$\Pi_\chi(\mathcal{A}^\infty) \subseteq \mathcal{A}^\infty$ for all $\chi \in \mathcal{C}(K)$.*

Proof. Let $X \in \mathfrak{g}$, $T \in \mathcal{A}^\infty$ and $v, w \in \mathcal{H}$. Let $X_1, \ldots, X_n$ be a basis for $\mathfrak{g}$ and write $\mathrm{Ad}_{k^{-1}}X = \sum_{j=1}^n a_j(k)X_j$, where $a_j \in \mathcal{C}(K)$. For all $\chi \in \mathcal{C}(K)$ we have

$$\begin{aligned}
\langle (d\kappa(X)\Pi_\chi(T))v, w\rangle &= \left\langle \frac{d}{dt}\bigg|_{t=0} \kappa(\exp tX)\int_K (\kappa(k)T)v\overline{\chi(k)}dk, w\right\rangle \\
&= \frac{d}{dt}\bigg|_{t=0}\int_K \langle (\kappa((\exp tX)k)T)v, w\rangle\overline{\chi(k)}dk \\
&= \int_K \frac{\partial}{\partial t}\bigg|_{t=0}\langle (\kappa((\exp tX)k)T)v, w\rangle\overline{\chi(k)}dk \\
&\qquad \text{(by the dominated convergence theorem)} \\
&= \int_K \langle (d\kappa(X)\kappa(k)T)v, w\rangle\overline{\chi(k)}dk \\
&= \int_K \langle (\kappa(k)d\kappa(\mathrm{Ad}_{k^{-1}}X)T)v, w\rangle\overline{\chi(k)}dk \\
&= \sum_{j=1}^n \int_K \langle (\kappa(k)d\kappa(X_j)T)v, w\rangle a_j(k)\overline{\chi(k)}dk \\
&= \left\langle \left(\sum_{j=1}^n \Pi_{\overline{a_j}\chi}(d\kappa(X_j)T)\right)v, w\right\rangle.
\end{aligned}$$

Let $\mathcal{A}'$ denote the linear span of $\{\Pi_\chi(T) \mid T \in \mathcal{A}^\infty, \chi \in \mathcal{C}(K)\}$. The above calculation shows that $\mathcal{A}'$ is stable under $d\kappa(X)$ for all $X \in \mathfrak{g}$. So $\mathcal{A}' \subseteq \mathcal{A}^\infty$ by a slight modification of Lemma 3.13 in [3]. □

From Lemmas 3.2, 3.4 and 3.5 we obtain the following important corollary, which is repeatedly used in Section 4.

Corollary 3.6. *$\Pi_\chi(\mathcal{A}) \cap \mathcal{A}^\infty$ is dense in $\Pi_\chi(\mathcal{A})$ for all $\chi \in \mathcal{C}(K)$.*

If χ is the character of an irreducible unitary representation ρ of K, define $\Pi_\rho = \dim(\rho)\Pi_\chi$. By the Schur orthogonality relations we have

$$\Pi_\rho\Pi_{\rho'} = \begin{cases} \Pi_\rho & \text{if } \rho \approx \rho', \\ 0 & \text{if } \rho \not\approx \rho'. \end{cases}$$

For all $T \in \mathcal{B}(\mathcal{H})$ we have $\sum_{\rho\in\hat{K}} \Pi_\rho(T) = T$ by the Peter–Weyl Theorem.

4. Calculations for $\mathrm{SL}(2,\mathbb{R})$

In this section $G = \mathrm{SU}(1,1)$. Note that $K = \{\left(\begin{smallmatrix} \alpha & 0 \\ 0 & \overline{\alpha}\end{smallmatrix}\right) \mid |\alpha| = 1\}$ is a maximal compact subgroup of G and that $h = \left(\begin{smallmatrix} i & 0 \\ 0 & -i\end{smallmatrix}\right)$, $e = \left(\begin{smallmatrix} 0 & 2 \\ 0 & 0\end{smallmatrix}\right)$ and $f = \left(\begin{smallmatrix} 0 & 0 \\ 2 & 0\end{smallmatrix}\right)$ form a $\mathbb{C}$-basis for $\mathbb{C} \otimes \mathfrak{g}$, the complexified Lie algebra of G. Let $e_n(z) = z^n$. In the following calculations we consider certain operators denoted by T_e and T_f. We warn the reader that they depend on the representation being considered, as well as on the parameter w.

4.1. The discrete series. The K-types of $\mathcal{D}^{+,w}$ are $\mathbb{C}e_n$, $n = 0, 1, 2, \ldots$, since $\mathcal{D}^{+,w}\left(\begin{smallmatrix}\alpha & 0\\ 0 & \bar{\alpha}\end{smallmatrix}\right) e_n = \alpha^{w+2n} e_n$. The $\mathbb{C} \otimes \mathfrak{g}$ action is given by the formulae

$$\begin{aligned}
\left(d\mathcal{D}^{+,w}(h)\right) e_n &= (w + 2n) i e_n,\\
\left(d\mathcal{D}^{+,w}(e)\right) e_n &= (2w + 2n) e_{n+1},\\
\left(d\mathcal{D}^{+,w}(f)\right) e_n &= -2n e_{n-1}.
\end{aligned}$$

Let $\mathcal{A}$ be a maximal inductive algebra. Note that if l is odd then $\Pi_l(\mathcal{A}) = 0$. For $m \in \mathbb{Z}$ let $\mathcal{A}_m = \Pi_{2m}(\mathcal{A})$. Note that $T \in \mathcal{A}_m$ implies $Te_n = a_n e_{n+m}$ for some $a_n \in \mathbb{C}$ and $n = 0, 1, 2, \ldots$, and $a_n = 0$ if $n < -m$. For $T \in \mathcal{A}^\infty$ let $T_e = [T, d\mathcal{D}^{+,w}(e)]$ and $T_f = [T, d\mathcal{D}^{+,w}(f)]$. For $T \in \mathcal{A}_m \cap \mathcal{A}^\infty$ we have

$$\begin{aligned}
T_e e_n &= ((2w + 2n)a_{n+1} - (2w + 2n + 2m)a_n) e_{n+m+1},\\
T_f e_n &= (-2n a_{n-1} + (2n + 2m)a_n) e_{n+m-1}.
\end{aligned}$$

Lemma 4.1. *$\mathcal{A}_0 = \mathbb{C}I$, where I is the identity operator.*

Proof. It is clear that $\mathbb{C}I \subseteq \mathcal{A}_0$. Let $T \in \mathcal{A}_0 \cap \mathcal{A}^\infty$. Then

$$\begin{aligned}
0 &= [T, T_e] e_n\\
&= (2w + 2n)(a_{n+1} - a_n)^2 e_{n+1}.
\end{aligned}$$

Since $w \geq 1$, it follows that $a_{n+1} = a_n$ for $n = 0, 1, 2, \ldots$. So $\mathcal{A}_0 \cap \mathcal{A}^\infty = \mathbb{C}I$, which is dense, and being finite dimensional, closed in $\mathcal{A}_0$. ☐

Lemma 4.2. *Let $m \in \mathbb{Z}$.*

(1) *For $m > 0$ we have $\mathcal{A}_{m-1} = 0 \implies \mathcal{A}_m = 0$.*
(2) *For $m < 0$ we have $\mathcal{A}_{m+1} = 0 \implies \mathcal{A}_m = 0$.*

Proof. (1) Assume $m > 0$ and $\mathcal{A}_{m-1} = 0$. Let $T \in \mathcal{A}_m \cap \mathcal{A}^\infty$. Then there exist $a_n \in \mathbb{C}$ such that $Te_n = a_n e_{n+m}$ for $n = 0, 1, 2, \ldots$. Since $T_f \in \mathcal{A}_{m-1}$, we have

$$n a_{n-1} - (n + m) a_n = 0$$

for $n = 0, 1, 2, \ldots$. Taking $n = 0$ gives $a_0 = 0$. Also, $a_{n-1} = 0$ implies $a_n = 0$. By induction it follows that $a_n = 0$ for $n = 0, 1, 2, \ldots$. So $T = 0$. As before, it follows that $\mathcal{A}_m = 0$.

(2) Assume $m < 0$ and $\mathcal{A}_{m+1} = 0$. Let $T \in \mathcal{A}_m \cap \mathcal{A}^\infty$. Then there exist $a_n \in \mathbb{C}$ such that $Te_n = a_n e_{n+m}$ for $n = 0, 1, 2, \ldots$. Observe that $a_0 = 0$. Since $T_e \in \mathcal{A}_{m+1}$, we have

$$(w + n) a_{n+1} - (w + n + m) a_n = 0.$$

Since $(w + n) > 0$ for all n, it now follows that $a_n = 0$ for all n. So $T = 0$. As before, it follows that $\mathcal{A}_m = 0$. ☐

Since $\mathcal{A} \neq \mathcal{A}_0$, the Peter-Weyl theorem implies that there exists $m \neq 0$ such that $\mathcal{A}_m \neq 0$. It now follows from Lemma 4.2 that $\mathcal{A}_1 \neq 0$ or $A_{-1} \neq 0$. Define $T_{\mathcal{B}} : \mathcal{H} \to \mathcal{H}$ by the formula

$$T_{\mathcal{B}} e_n = e_{n+1}$$

for $n = 0, 1, 2, \dots$.

Lemma 4.3. $\mathcal{A}_1 \neq 0 \implies T_{\mathcal{B}} \in \mathcal{A}_1$.

Proof. Let $T \in \mathcal{A}_1 \setminus 0$. Then there exist $a_n \in \mathbb{C}$ such that $Te_n = a_n e_{n+1}$ for $n = 0, 1, 2, \dots$. Now $T_f = kI$ for some $k \in \mathbb{C}$. Replacing T by a constant multiple, we may assume that $k = 2$. So for each $n = 0, 1, 2, \dots$ we have

$$(n+1)a_n - na_{n-1} = 1.$$

So $a_n = 1$ for $n = 0, 1, 2, \dots$. So $T = T_{\mathcal{B}}$. In particular, $T_{\mathcal{B}} \in \mathcal{A}_1$ □

It follows that if $\mathcal{A}_1 \neq 0$ then $\mathcal{B} \subseteq \mathcal{A}$ and, by maximality, $\mathcal{A} = \mathcal{B}$.

Lemma 4.4. $\mathcal{A}_{-1} \neq 0 \implies T_{\mathcal{B}}^* \in \mathcal{A}_{-1}$.

Proof. Let $T \in \mathcal{A}_{-1} \setminus 0$. Then there exist $a_n \in \mathbb{C}$ such that $Te_n = a_n e_{n-1}$ for $n = 0, 1, 2, \dots$. Observe that $a_0 = 0$. Now $T_e = kI$ for some $k \in \mathbb{C}$. Replacing T by a constant multiple, we may assume that $k = 2$. So for each $n = 0, 1, 2, \dots$ we have

$$(w+n)a_{n+1} - (w+n-1)a_n = 1.$$

It is easy to solve this recursion, and we find that

$$a_n = \frac{n}{w+n-1}.$$

So $T = T_{\mathcal{B}}^*$. In particular, $T_{\mathcal{B}}^* \in \mathcal{A}_{-1}$. □

It follows that if $\mathcal{A}_{-1} \neq 0$ then $\mathcal{B}^* \subseteq \mathcal{A}$ and, by maximality, $\mathcal{A} = \mathcal{B}^*$.

4.2. The spherical principal series and the complementary series. The K-types are $\mathbb{C}e_n$, $n \in \mathbb{Z}$, because $\mathcal{P}^{+,w} \left(\begin{smallmatrix} \alpha & 0 \\ 0 & \bar{\alpha} \end{smallmatrix} \right) e_n = \alpha^{2n} e_n$. The $\mathbb{C} \otimes \mathfrak{g}$ action is given by the formulae

$$\begin{aligned} \left(d\mathcal{P}^{+,w}(h)\right) e_n &= 2ine_n, \\ \left(d\mathcal{P}^{+,w}(e)\right) e_n &= (w+2n+1)e_{n+1}, \\ \left(d\mathcal{P}^{+,w}(f)\right) e_n &= (w-2n+1)e_{n-1}. \end{aligned}$$

The action of the intertwiners on the K-types is given by

$$\mathcal{I}_{+,w}(e_n) = \frac{\Gamma(\frac{1-w}{2} + |n|)}{\Gamma(\frac{1+w}{2} + |n|)} e_n.$$

Let $\mathcal{A}$ be a maximal inductive algebra. Note that if l is odd then $\Pi_l(\mathcal{A}) = 0$. For $m \in \mathbb{Z}$ let $\mathcal{A}_m = \Pi_{2m}(\mathcal{A})$. Note that $T \in \mathcal{A}_m$ implies $Te_n = a_n e_{n+m}$

for some $a_n \in \mathbb{C}$ and all $n \in \mathbb{Z}$. For $T \in \mathcal{A}^\infty$ let $T_e = [T, d\mathcal{P}^{+,w}(e)]$ and $T_f = [T, d\mathcal{P}^{+,w}(f)]$. For $T \in \mathcal{A}_m \cap \mathcal{A}^\infty$ we have

$$T_e e_n = ((w+2n+1)a_{n+1} - (w+2n+2m+1)a_n)e_{n+m+1},$$
$$T_f e_n = ((w-2n+1)a_{n-1} - (w-2n-2m+1)a_n)e_{n+m-1}.$$

Lemma 4.5. *$\mathcal{A}_0 = \mathbb{C}I$, where I is the identity operator.*

Proof. It is clear that $\mathbb{C}I \subseteq \mathcal{A}_0$. Let $T \in \mathcal{A}_0 \cap \mathcal{A}^\infty$. Then

$$\begin{aligned} 0 &= [T, T_e]e_n \\ &= (w+2n+1)(a_{n+1} - a_n)^2 e_{n+1}. \end{aligned}$$

Since w is not an odd integer, it follows that $a_{n+1} = a_n$ for all $n \in \mathbb{Z}$. As before, it follows that $\mathcal{A}_0 = \mathbb{C}I$. □

Lemma 4.6. *Let $m \in \mathbb{Z}$.*

(1) *For $m \neq -1$ we have $\mathcal{A}_m = 0 \implies \mathcal{A}_{m+1} = 0$.*
(2) *For $m \neq 1$ we have $\mathcal{A}_m = 0 \implies \mathcal{A}_{m-1} = 0$.*

Proof. We will prove the first part. The argument for the second part is identical, except that the roles of T_e and T_f are reversed.

Assume $\mathcal{A}_m = 0$. Let $T \in \mathcal{A}_{m+1} \cap \mathcal{A}^\infty$. Then there exist $a_n \in \mathbb{C}$ such that $Te_n = a_n e_{n+m+1}$ for all $n \in \mathbb{Z}$. Since $T_f \in \mathcal{A}_m$, we have

$$(w-2n-1)a_n - (w-2n-2m-3)a_{n+1} = 0 \tag{1}$$

for all $n \in \mathbb{Z}$. Since T commutes with T_e, we have $[T, T_e]e_n = 0$ for all $n \in \mathbb{Z}$. Therefore

$$\begin{aligned} &(w+2n+1)a_{n+1}a_{n+m+2} \\ &-2(w+2n+2m+3)a_n a_{n+m+2} \\ &+(w+2n+4m+5)a_n a_{n+m+1} \\ &= 0. \end{aligned}$$

Multiplying by $w-2n-2m-3$ and using identity (1), we get

$$\begin{aligned} \big\{[w^2 - (2n+1)^2] - 2[w^2 - (2n+2m+3)^2] \\ + [w^2 - (2n+4m+5)^2]\big\} a_n a_{n+m+2} = 0. \end{aligned}$$

Upon simplification we get

$$(m+1)^2 a_n a_{n+m+2} = 0.$$

Therefore, for all $n \in \mathbb{Z}$ we have $a_{n+1} = 0$ or $a_{n+m+2} = 0$. Since w is not an odd integer, repeated use of identity (1) shows that $a_n = 0$ for all $n \in \mathbb{Z}$, and hence $T = 0$. As in the previous lemma, this implies $\mathcal{A}_{m+1} = 0$. □

Since $\mathcal{A} \neq \mathcal{A}_0$, the Peter-Weyl theorem implies that there exists $m \neq 0$ such that $\mathcal{A}_m \neq 0$. It now follows from Lemma 4.6 that $\mathcal{A}_1 \neq 0$ or $\mathcal{A}_{-1} \neq 0$. First assume that $\mathcal{A}_1 \neq 0$. Define $T_{\mathcal{C}}, T_{\mathcal{C}_{+,w}} : \mathcal{H} \to \mathcal{H}$ by the formulae

$$T_{\mathcal{C}} e_n = e_{n+1},$$
$$T_{\mathcal{C}_{+,w}} e_n = \frac{-w-2n-1}{w-2n-1} e_{n+1},$$

for each $n \in \mathbb{Z}$. Note that $T_{\mathcal{C}_{+,w}} = \mathcal{I}_{+,w}^{-1} T_{\mathcal{C}} \mathcal{I}_{+,w}$.

Lemma 4.7. *$T_{\mathcal{C}} \in \mathcal{A}_1$ or $T_{\mathcal{C}_{+,w}} \in \mathcal{A}_1$.*

Proof. Let $T \in \mathcal{A}_1 \setminus 0$. Then there exist $a_n \in \mathbb{C}$ such that $Te_n = a_n e_{n+1}$ for all $n \in \mathbb{Z}$. Now $T_f = kI$ for some $k \in \mathbb{C}$. Replacing T by a constant multiple, we may assume that $k = 2$. So for each $n \in \mathbb{Z}$ we have

$$(w-2n-1)a_n - (w-2n-3)a_{n+1} = 2, \tag{2}$$

for all $n \in \mathbb{Z}$. Since T commutes with T_e, we have $[T, T_e]e_n = 0$ for all $n \in \mathbb{Z}$. Therefore

$$(w+2n+1)a_{n+1}a_{n+2} - 2(w+2n+3)a_n a_{n+2} + (w+2n+5)a_n a_{n+1} = 0.$$

Using the identity (2) we get

$$\begin{aligned}(w+2n+1)&[(w-2n-1)a_n - 2][(w-2n-1)a_n - 4]\\ -2(w+2n+3)&a_n[(w-2n-1)a_n - 4](w-2n-3)\\ +(w+2n+5)&a_n[(w-2n-1)a_n - 2](w-2n-5)\\ &= 0.\end{aligned}$$

Upon simplification we get

$$(w-2n-1)a_n^2 + 2(2n+1)a_n - (w+2n+1) = 0.$$

The solutions of this quadratic equation are

$$a_n = 1, \qquad \text{and} \qquad a_n = \frac{-w-2n-1}{w-2n-1}.$$

The identity (2) forces us to choose either the first or the second solution consistently for each n. So $T = T_{\mathcal{C}}$ or $T = T_{\mathcal{C}_{+,w}}$. In particular, $T_{\mathcal{C}} \in \mathcal{A}_1$ or $T_{\mathcal{C}_{+,w}} \in \mathcal{A}_1$ □

It follows from the maximality of $\mathcal{A}$ that if T is invertible in $\mathcal{B}(\mathcal{H})$ and $T \in \mathcal{A}$ then $T^{-1} \in \mathcal{A}$. Since $T_{\mathcal{C}}$ and $T_{\mathcal{C}_{+,w}}$ are invertible, it now follows that $\mathcal{A}_{-1} \neq 0$. Similarly, the assumption that $\mathcal{A}_{-1} \neq 0$ leads to the conclusion that $\mathcal{A}_1 \neq 0$, and Lemma 4.7 is valid without any assumption about $\mathcal{A}_1$. If $T_{\mathcal{C}} \in \mathcal{A}_1$, then $\mathcal{C} \subseteq \mathcal{A}$ and, by maximality, $\mathcal{A} = \mathcal{C}$. If $T_{\mathcal{C}_{+,w}} \in \mathcal{A}_1$, then $\mathcal{C}_{+,w} \subseteq \mathcal{A}$ and, by maximality, $\mathcal{A} = \mathcal{C}_{+,w}$.

4.3. The non-spherical principal series. The K-types are $\mathbb{C}e_n$ for $n \in \mathbb{Z}$ because $\mathcal{P}^{-,w}\left(\begin{smallmatrix}\alpha & 0\\ 0 & \alpha\end{smallmatrix}\right) e_n = \alpha^{2n-1}e_n$. The $\mathbb{C}\otimes\mathfrak{g}$ action is given by the formulae

$$\begin{aligned}\left(d\mathcal{P}^{-,w}(h)\right) e_n &= (2n-1)ie_n,\\ \left(d\mathcal{P}^{-,w}(e)\right) e_n &= (w+2n)e_{n+1},\\ \left(d\mathcal{P}^{-,w}(f)\right) e_n &= (w-2n)e_{n-1}.\end{aligned}$$

The action of the intertwiners on the K-types is given by

$$\mathcal{I}_{-,w}(e_n) = r_n e_n,$$

where $r_0 = 1$ and

$$r_{n+1} = \frac{-w+2n}{w+2n} r_n.$$

Let $\mathcal{A}$ be a maximal inductive algebra. Note that if l is odd then $\Pi_l(\mathcal{A}) = 0$. For $m \in \mathbb{Z}$ let $\mathcal{A}_m = \Pi_{2m}(\mathcal{A})$. Note that $T \in \mathcal{A}_m$ implies $Te_n = a_n e_{n+m}$ for some $a_n \in \mathbb{C}$ and all $n \in \mathbb{Z}$. For $T \in \mathcal{A}^\infty$ let $T_e = [T, d\mathcal{P}^{-,w}(e)]$ and $T_f = [T, d\mathcal{P}^{-,w}(f)]$. For $T \in \mathcal{A}_m \cap \mathcal{A}^\infty$ we have

$$\begin{aligned}T_e e_n &= ((w+2n)a_{n+1} - (w+2n+2m)a_n)e_{n+m+1},\\ T_f e_n &= ((w-2n+2)a_{n-1} - (w-2n-2m+2)a_n)e_{n+m-1}.\end{aligned}$$

Lemma 4.8. *$\mathcal{A}_0 = \mathbb{C}I$, where I is the identity operator.*

Proof. It is clear that $\mathbb{C}I \subseteq \mathcal{A}_0$. Let $T \in \mathcal{A}_0 \cap \mathcal{A}^\infty$. Then

$$\begin{aligned}0 &= [T, T_e]e_n\\ &= (w+2n)(a_{n+1}-a_n)^2 e_{n+1}.\end{aligned}$$

Since w is non-zero and imaginary, it follows that $a_{n+1} = a_n$ for all n. As before, it follows that $\mathcal{A}_0 = \mathbb{C}I$. □

Lemma 4.9. *Let $m \in \mathbb{Z}$.*

(1) *For $m \neq -1$ we have $\mathcal{A}_m = 0 \implies \mathcal{A}_{m+1} = 0$.*
(2) *For $m \neq 1$ we have $\mathcal{A}_m = 0 \implies \mathcal{A}_{m-1} = 0$.*

Proof. We will prove the first part. The argument for the second part is identical, except that the roles of T_e and T_f are reversed.

Assume $\mathcal{A}_m = 0$. Let $T \in \mathcal{A}_{m+1} \cap \mathcal{A}^\infty$. Then there exist $a_n \in \mathbb{C}$ such that $Te_n = a_n e_{n+m+1}$ for all $n \in \mathbb{Z}$. Since $T_f \in \mathcal{A}_m$, we have

$$(w-2n)a_n - (w-2n-2m-2)a_{n+1} = 0 \tag{3}$$

for all $n \in \mathbb{Z}$. Since T commutes with T_e, we have $[T, T_e]e_n = 0$ for all $n \in \mathbb{Z}$. Therefore

$$\begin{gathered}(w+2n)a_{n+1}a_{n+m+2}\\ -2(w+2n+2m+2)a_n a_{n+m+2}\\ +(w+2n+4m+4)a_n a_{n+m+1}\\ =0.\end{gathered}$$

Multiplying by $w-2n-2m-2$ and using identity (3), we get

$$\{[w^2-(2n)^2]-2[w^2-(2n+2m+2)^2]+[w^2-(2n+4m+4)^2]\}a_n a_{n+m+2}=0.$$

Upon simplification we get

$$(m+1)^2 a_n a_{n+m+2} = 0.$$

Therefore, for all $n \in \mathbb{Z}$ we have $a_{n+1} = 0$ or $a_{n+m+2} = 0$. Since w is imaginary and non-zero, repeated use of identity (3) shows that $a_n = 0$ for all $n \in \mathbb{Z}$, and hence $T = 0$. As before, it follows that $\mathcal{A}_{m+1} = 0$. □

Since $\mathcal{A} \neq \mathcal{A}_0$, the Peter-Weyl theorem implies that there exists $m \neq 0$ such that $\mathcal{A}_m \neq 0$. It now follows from Lemma 4.9 that $\mathcal{A}_1 \neq 0$ or $\mathcal{A}_{-1} \neq 0$. First assume that $\mathcal{A}_1 \neq 0$. Define $T_{\mathcal{C}}, T_{\mathcal{C}_{-,w}} : \mathcal{H} \to \mathcal{H}$ by the formulae

$$T_{\mathcal{C}} e_n = e_{n+1},$$
$$T_{\mathcal{C}_{-,w}} e_n = \frac{-w-2n}{w-2n} e_{n+1},$$

for each $n \in \mathbb{Z}$. Note that $T_{\mathcal{C}_{-,w}} = \mathcal{I}_{-,w}^{-1} T_{\mathcal{C}} \mathcal{I}_{-,w}$.

LEMMA 4.10. *$T_{\mathcal{C}} \in \mathcal{A}_1$ or $T_{\mathcal{C}_{-,w}} \in \mathcal{A}_1$.*

Proof. Let $T \in \mathcal{A}_1 \setminus 0$. Then there exist $a_n \in \mathbb{C}$ such that $Te_n = a_n e_{n+1}$ for all $n \in \mathbb{Z}$. Now $T_f = kI$ for some $k \in \mathbb{C}$. Replacing T by a constant multiple, we may assume that $k = 2$. So for each $n \in \mathbb{Z}$ we have

$$(w-2n)a_n - (w-2n-2)a_{n+1} = 2 \tag{4}$$

for all $n \in \mathbb{Z}$. Since T commutes with T_e, we have $[T, T_e]e_n = 0$ for all $n \in \mathbb{Z}$. Therefore

$$(w+2n)a_{n+1}a_{n+2} - 2(w+2n+2)a_n a_{n+2} + (w+2n+4)a_n a_{n+1} = 0.$$

Using the identity (4) we get

$$\begin{gathered}(w+2n)[(w-2n)a_n-2][(w-2n)a_n-4]\\ -2(w+2n+2)a_n[(w-2n)a_n-4](w-2n-2)\\ +(w+2n+4)a_n[(w-2n)a_n-2](w-2n-4)\\ =0.\end{gathered}$$

Upon simplification we get

$$(w - 2n)a_n^2 + 2(2n)a_n - (w + 2n) = 0.$$

The solutions of this quadratic equation are

$$a_n = 1, \qquad \text{and} \qquad a_n = \frac{-w - 2n}{w - 2n}.$$

The identity (4) forces us to choose either the first or the second solution consistently for each n. So $T = T_{\mathcal{C}}$ or $T = T_{\mathcal{C}_{-,w}}$. In particular, $T_{\mathcal{C}} \in \mathcal{A}_1$ or $T_{\mathcal{C}_{-,w}} \in \mathcal{A}_1$ □

It follows from the maximality of $\mathcal{A}$ that if T is invertible in $\mathcal{B}(\mathcal{H})$ and $T \in \mathcal{A}$ then $T^{-1} \in \mathcal{A}$. Since $T_{\mathcal{C}}$ and $T_{\mathcal{C}_{-,w}}$ are invertible, it now follows that $\mathcal{A}_{-1} \neq 0$. Similarly, the assumption that $\mathcal{A}_{-1} \neq 0$ leads to the conclusion that $\mathcal{A}_1 \neq 0$, and Lemma 4.10 is valid without any assumption about $\mathcal{A}_1$. If $T_{\mathcal{C}} \in \mathcal{A}_1$, then $\mathcal{C} \subseteq \mathcal{A}$ and, by maximality, $\mathcal{A} = \mathcal{C}$. If $T_{\mathcal{C}_{-,w}} \in \mathcal{A}_1$, then $\mathcal{C}_{-,w} \subseteq \mathcal{A}$ and, by maximality, $\mathcal{A} = \mathcal{C}_{-,w}$.

5. Concluding remarks

Recently we came across the paper [1], some of whose results can be deduced from our results in this paper. Indeed, the principal result of [1], which characterises the homogeneous shifts can be recovered by generalising the results of this paper to the universal cover of SL(2, ℝ). The details will appear elsewhere.

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