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THE SPECTRUM OF DIFFERENTIAL OPERATORS IN H^p SPACES

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Abstract. This paper is concerned with linear partial differential operators with constant coefficients in $H^p(\mathbf{R}^n)$. In the case $0 < p \leq 1$, we establish some basic properties and the spectral mapping property, and determine completely the essential spectrum, point spectrum, approximate point spectrum, continuous spectrum, and residual spectrum of such differential operators. In the case $p > 2$, we show that the point spectrum of such differential operators in $L^p(\mathbf{R}^n)$ is the empty set for $p \in (2, \frac{2n}{n-1})$, but not for $p > \frac{2n}{n-1}$ in general. Moreover, we make some remarks on the case $p > 1$ and give several examples.

1. Introduction

The spectrum of linear partial differential operators (PDOs) with constant coefficients in $L^p(\mathbf{R}^n)$ has been extensively studied (cf. [19], [15]). In particular, the spectral mapping property holds for all PDOs in $L^2(\mathbf{R}^n)$, and for many classes of PDOs (e.g., elliptic PDOs) in $L^p(\mathbf{R}^n)$ $(p > 1)$. On the other hand, the application of the theory of H^p spaces has become an interesting subject in harmonic analysis since the 1980s. However, as yet we do not know any results related to the spectrum of PDOs in $H^p(\mathbf{R}^n)$ $(0 < p \leq 1)$.

Since $H^p(\mathbf{R}^n)$ $(0 < p < 1)$ is not a Banach space, one might expect that the spectral theory of PDOs with constant coefficients in $L^p(\mathbf{R}^n)$ is more complete than in $H^p(\mathbf{R}^n)$. However, our work shows that the opposite is true. One of the main reasons is that the structure of the eigenvalues of such PDOs is well understood in $H^p(\mathbf{R}^n)$, but not in $L^p(\mathbf{R}^n)$ $(p > 2)$. There are also many technical differences in the study of such PDOs on those two spaces, especially for the problem of the essential spectrum. The theory of $H^p(\mathbf{R}^n)$ spaces (see [4]) and Fourier multipliers in such spaces (see [14]) provide the fundamental knowledge that we need in this subject.

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This paper is organized as follows.

In Section 2, we start with some preliminaries on $H^p(\mathbf{R}^n)$ $(0 < p \le 1)$ and Fourier multipliers in such spaces. The main results of this section give some general properties of PDOs with constant coefficients in $H^p(\mathbf{R}^n)$ $(0 < p \le 1)$. In particular, we show that the space $\mathcal{S}_c$ (see Lemma 2.1(b) below) is a core of PDOs. This space plays a key role in the treatment of PDOs in $H^p(\mathbf{R}^n)$, analogous to the role of the Schwartz space in the case of $L^p(\mathbf{R}^n)$.

Section 3 is concerned with the spectrum of PDOs with constant coefficients in $H^p(\mathbf{R}^n)$ $(0 < p \le 1)$. We first prove the spectral mapping theorem for some classes of coercive PDOs, in which a practical sufficient condition is given by using coercive and hypoelliptic indices of their symbols. Next, we show that the essential spectrum is always consistent with their spectra; the proof is based on the structure of an important sequence in $H^p(\mathbf{R}^n)$ (see Lemma 3.4 below). Finally, we prove that such PDOs have no eigenvalues, and that the approximate point spectrum is also consistent with their spectra. In particular, a characterization of the residual spectrum is obtained by using their symbols.

Section 4 is devoted to the spectrum of PDOs with constant coefficients in $L^p(\mathbf{R}^n)$ $(p > 1)$. We first consider general spectral results, and then deal with the eigenvalues of such PDOs in $L^p(\mathbf{R}^n)$ $(p > 2)$. The main result of this section shows that such PDOs have no eigenvalues in $L^p(\mathbf{R}^n)$ $(2 < p < \frac{2n}{n-1})$. We give an example showing that there exist PDOs having eigenvalues in $L^p(\mathbf{R}^n)$ $(p > \frac{2n}{n-1})$. We find that the existence of eigenvalues of PDOs depends on the geometrical property of level surfaces associated with their symbols. As a result in this direction we prove that if the level surfaces are all contained in planes, then the corresponding PDO has no eigenvalues in $L^p(\mathbf{R}^n)$ $(p > \frac{2n}{n-1})$. We have made some progress in this direction, and further results will be given in a forthcoming paper.

Finally, in Section 5, we provide four examples. The first one corrects a result given in [1]. The second one shows that the spectral mapping property may not hold for certain non-coercive PDOs in $L^p(\mathbf{R}^n)$ for all $p > 0$ $(p \neq 2)$, and also corrects the form of a polynomial that appeared in several papers ([19], [6], [1], [15]). The third one shows that there are PDOs which satisfy the spectral mapping property in $L^p(\mathbf{R}^n)$ for larger p-values, but not for smaller p-values. The last example deals with semi-elliptic PDOs. We conclude by posing two questions.

Throughout this paper, $\mathcal{S}$ (resp. C_c^∞, $\mathcal{S}'$) denotes the space of rapidly decreasing functions (resp. C^∞ functions with compact support, tempered distributions) on $\mathbf{R}^n$. We denote by $\mathcal{F}\phi$ (or $\hat{\phi}$) the Fourier transform of $\phi \in \mathcal{S}$; that is,

$$(\mathcal{F}\phi)(y) = \hat{\phi}(y) = \int_{\mathbf{R}^n} e^{-iy\cdot x}\phi(x)dx \quad \text{for } y \in \mathbf{R}^n,$$

and $\mathcal{F}^{-1}\phi$ (i.e., $\hat{\phi}(-\cdot)$) is the inverse transform. We let $D_j = -i\partial/\partial x_j$ ($1 \le j \le n$), $D^\alpha = D_1^{\alpha_1} \cdots D_n^{\alpha_n}$, and $|\alpha| = \alpha_1 + \cdots + \alpha_n$ for $\alpha \in \mathbf{N}_0^n$, where $\mathbf{N}_0 = \mathbf{N} \cup \{0\}$. Moreover, we set $n_p = n|\frac{1}{2} - \frac{1}{p}|$ for $p > 0$.

2. General properties of PDOs in H^p $(0 < p \le 1)$

We first recall the definition of H^p ($p > 0$). For fixed $\varphi \in \mathcal{S}$ with $\hat{\varphi}(0) \neq 0$, let $\varphi_t = t^{-n}\varphi(\frac{\cdot}{t})$ ($t > 0$) and $f^+(x) = \sup_{t>0} |(f * \varphi_t)(x)|$ ($x \in \mathbf{R}^n,\ f \in \mathcal{S}'$), where $*$ denotes the convolution. Define $H^p = \{f \in \mathcal{S}';\ f^+ \in L^p\}$ with norm $\|f\|_{H^p} = \|f^+\|_{L^p}$.

It is known that, if we replace the function φ by another function in $\mathcal{S}$, then the space H^p remains the same and the norm changes to an equivalent one. If $p > 1$, then $H^p = L^p$ with equivalent norms. If $p = 1$, then H^1 is a Banach space and $H^1 \subset L^1$; If $0 < p < 1$, we consider H^p merely as a Fréchet space, and $\|\cdot\|_{H^p}^p$ is subadditive and so gives a metric on H^p. In the remainder of this section we always assume $0 < p \le 1$.

The following lemma collects several properties of H^p spaces (see [21], [2]), which are used later. Define

$$\mathcal{S}_c = \left\{ f \in \mathcal{S};\ \ \hat{f} \text{ is in } C_c^\infty \text{ and vanishes in a neighborhood of the origin} \right\}.$$

LEMMA 2.1.

(a) *$H^p \hookrightarrow \mathcal{S}'$, i.e., H^p is continuously embedded in $\mathcal{S}'$.*
(b) *$C_c^\infty \cap H^p$ and $\mathcal{S}_c$ both are dense subspaces of H^p.*
(c) *Let $f \in H^p$, and let $\varphi \in \mathcal{S}$ with $\hat{\varphi}(0) = 1$. Then $\|\varphi_t * f\|_{H^p} \le M\|f\|_{H^p}$ and $\varphi_t * f \to f$ $(t \to 0)$ in H^p.*

We refer to [14] for Fourier multipliers in H^p. Define

$$\mathcal{M}_p = \{u \in L^\infty;\ \|u\|_{\mathcal{M}_p} < \infty\},$$

where

$$\|u\|_{\mathcal{M}_p} = \sup\{\|\mathcal{F}^{-1}(u\hat{f})\|_{H^p};\ f \in \mathcal{S} \cap H^p,\ \|f\|_{H^p} \le 1\}.$$

LEMMA 2.2.

(a) *If $u \in \mathcal{M}_p$, then $u \in C(\mathbf{R}^n \setminus \{0\})$.*
(b) *Let $u \in C^k(\mathbf{R}^n)$ with $k = [n_p]+1$. If there exist constants $a \ge -1$ and $b \ge (a+1)n_p$ such that $|D^\alpha u(\xi)| = O(|\xi|^{a|\alpha|-b})$ as $|\xi| \to \infty$, where $|\alpha| \le k$, then $u \in \mathcal{M}_p$.*
(c) *Let T be a bounded linear operator on H^p which commutes with translations. Then there exists a unique $u \in \mathcal{M}_p$ such that $Tf = \mathcal{F}^{-1}(u\hat{f})$ for $f \in \mathcal{S} \cap H^p$.*

Proof. (a) This is well known (see [22]).

(b) By the assumption there exist constants $M > 0$, $L > 1$ such that

$$|D^\alpha u(\xi)| \le M|\xi|^{a|\alpha|-b} \le M|\xi|^{a|\alpha|-(a+1)n_p} \quad \text{for } |\xi| \ge L \text{ and } |\alpha| \le k.$$

Let $\phi \in C_c^\infty$ be such that

$$\phi(\xi) = \begin{cases} 1, & |\xi| \le L, \\ 0, & |\xi| \ge L+1. \end{cases}$$

Then by Leibniz's formula there exists a constant $M_1 > 0$ such that

$$|D^\alpha(u(\xi)(1-\phi(\xi)))| \le M_1|\xi|^{a|\alpha|-(a+1)n_p}$$

for $\xi \in \mathbf{R}^n$ and $|\alpha| \le k$, and so Theorem G(i) in [14] implies that $u(1-\phi) \in \mathcal{M}_p$. On the other hand, there exists a constant $M_2 > 0$ such that

$$|D^\alpha(u(\xi)\phi(\xi))| \le M_2\chi_{\{\xi;\ |\xi|\le L+1\}} \le M_2((L+1)/|\xi|)^{|\alpha|}$$

for $\xi \in \mathbf{R}^n$ and $|\alpha| \le k$. It follows thus from a generalization of Mihlin's multiplier theorem (see [2]) that $u\phi \in \mathcal{M}_p$. The proof is complete.

(c) By Remark 2.4 in [14] there exists $K \in \mathcal{S}'$ such that $Tf = K * f$ for $f \in \mathcal{S} \cap H^p$. Since T is bounded on H^p, it follows from Theorem 3.5 in [14] that there exists a polynomial P of degree $\le [n/p-n]$ such that $f \mapsto (K-P)*f$ is a bounded operator on L^2, and so $u := \hat{K} - \hat{P} \in L^\infty$. Noting that supp $\hat{P} \subset \{0\}$, we have $Tf = \mathcal{F}^{-1}(\hat{K}\hat{f}) = \mathcal{F}^{-1}(u\hat{f})$ for $f \in \mathcal{S} \cap H^p$. Since $\mathcal{M}_p \subset L^\infty$, the uniqueness is obvious. □

In the sequel, we always assume that $P : \mathbf{R}^n \to \mathbf{C}$ is a polynomial of degree $m > 0$. The corresponding PDO in H^p is defined by $P_p = P(D)$ with maximal domain in H^p in the distributional sense. Equivalently, $P_p f = \mathcal{F}^{-1}(P\hat{f})$ with $D(P_p) = \{f \in H^p;\ \mathcal{F}^{-1}(P\hat{f}) \in H^p\}$. For $s > 0$ define

$$H_s^p = \{f \in H^p;\ \mathcal{F}^{-1}((1+|\cdot|^2)^{s/2}\hat{f}) \in H^p\}$$

with norm $\|f\|_{H_s^p} := \|\mathcal{F}^{-1}((1+|\cdot|^2)^{s/2}\hat{f})\|_{H^p}$. Then H_s^p is a Fréchet space.

P is called coercive if $|P(\xi)| \to \infty$ as $|\xi| \to \infty$. For $r \in (0, m]$, P is called r-coercive if $|P(\xi)|^{-1} = O(|\xi|^{-r})$ as $|\xi| \to \infty$. It is known that P is coercive if and only if it is r-coercive for some $r \in (0, m]$ (cf. [8]). Furthermore, P is called elliptic if the principal part of P never vanishes outside of the origin. This is equivalent to P being m-coercive. Moreover, we say that $\mathcal{D}(\subset D(P_p))$ is a core of P_p if P_p is the closure of $P(D)|_{\mathcal{D}}$ in H^p, where $P(D)|_{\mathcal{D}}$ is the operator $P(D)$ defined on $\mathcal{D}$.

Theorem 2.3.

(a) *P_p is a closed and densely defined operator, and $\mathcal{S} \cap H^p \subset H_m^p \subset D(P_p)$.*
(b) *If P is r-coercive, then $D(P_p) \subset H_s^p$, where $s = \max\{0, r-(m-r)n_p\}$. In particular, if P is elliptic, then $D(P_p) = H_m^p$.*

(c) *$\mathcal{S}_c$ and $C_c^\infty \cap H^p$ both are cores of P_p.*

Proof. (a) The closedness of P_p is a direct consequence of Lemma 2.1(a). Since $\mathcal{S} \cap H^p = \{f \in \mathcal{S};\ (D^\alpha \hat{f})(0) = 0 \text{ for } |\alpha| \le [n/p - n]\}$, it is not hard to check that $\mathcal{S} \cap H^p \subset H_m^p$. On the other hand, $\xi \mapsto \xi^\alpha/(1+|\xi|^2)^{m/2} \in \mathcal{M}_p$ for $|\alpha| \le m$ by Lemma 2.2(b), and thus $H_m^p \subset D(P_p)$. Noting that $\mathcal{S}_c \subset \mathcal{S} \cap H^p$, we have by Lemma 2.1(b) that P_p is densely defined.

(b) By assumption there exist constants $M, L > 0$ such that $|P(\xi)| \ge M|\xi|^r$ for $|\xi| \ge L$. Let $f \in D(P_p)$, and let $\phi \in C_c^\infty$ be such that

$$\phi(\xi) = \begin{cases} 1, & |\xi| \le L, \\ 0 & |\xi| \ge 2L. \end{cases}$$

Then $(1+|\cdot|^2)^{s/2}\phi \in \mathcal{M}_p$, and thus $\mathcal{F}^{-1}((1+|\cdot|^2)^{s/2}\phi\hat{f}) \in H^p$. On the other hand, since we may check that

$$|D^\alpha((1+|\xi|^2)^{s/2}(1-\phi(\xi))P^{-1}(\xi))| = O(|\xi|^{(m-r-1)|\alpha|+s-r})\ (|\xi| \to \infty)$$

for $\alpha \in \mathbf{N}_0^n$, Lemma 2.2(b) leads to $(1+|\cdot|^2)^{s/2}(1-\phi)P^{-1} \in \mathcal{M}_p$. Consequently,

$$\mathcal{F}^{-1}((1+|\cdot|^2)^{s/2}(1-\phi)\hat{f}) = \mathcal{F}^{-1}((1+|\cdot|^2)^{s/2}(1-\phi)P^{-1}\mathcal{F}(P_p f)) \in H^p,$$

and thus $f \in H_s^p$.

(c) Let $\varphi \in \mathcal{S}$ with $\hat{\varphi}(0) = 1$. Then for $g \in D(P_p)$,

$$\mathcal{F}^{-1}(P\mathcal{F}(\varphi_t * g)) = \mathcal{F}^{-1}(\hat{\varphi}_t P \hat{g}) = \varphi_t * (P_p g),$$

which implies by Lemma 2.1(c) that $D(P_p) \ni \varphi_t * g \to g$ and $P_p(\varphi_t * g) \to P_p g$ $(t \to 0)$ in H^p. Since (a) and Lemma 2.1(b) implies that $\mathcal{S}_c$ is a dense subset in $D(P_p)$, it follows that for $h \in \mathcal{S}_c$,

$$\|\varphi_t * h - \varphi_t * g\|_{H^p} \le \|\hat{\varphi}_t\|_{\mathcal{M}_p}\|h - g\|_{H^p},$$
$$\|P_p(\varphi_t * h) - P_p(\varphi_t * g)\|_{H^p} \le \|\hat{\varphi}_t P\|_{\mathcal{M}_p}\|h - g\|_{H^p}.$$

Combining these inequalities and noting that $\varphi_t * h \in \mathcal{S}_c$, we see that $\mathcal{S}_c$ is a core of P_p. Similarly, we can show that $C_c^\infty \cap H^p$ is also a core of P_p. □

3. The spectrum of PDOs in H^p $(0 < p \le 1)$

Denote by $\rho(P_p)$ the resolvent set of P_p, i.e.,

$$\rho(P_p) = \{\lambda \in \mathbf{C};\ \text{the range } R(\lambda - P_p) \text{ is dense in } H^p \text{ and there exists } M > 0 \text{ such that } \|f\|_{H^p} \le M\|(\lambda - P_p)f\|_{H^p} \text{ for } f \in D(P_p)\}.$$

The spectrum of P_p is $\sigma(P_p) := \mathbf{C} \setminus \rho(P_p)$. Since H^p is a Fréchet space, the closed graph theorem yields that for every $\lambda \in \rho(P_p)$, $(\lambda - P_p)^{-1}$ is bounded, i.e., $\sup\{\|(\lambda - P_p)^{-1}f\|_{H^p};\ \|f\|_{H^p} = 1\} < \infty$. Furthermore, $\rho(P_p)$ is an open set.

We start with the spectral inclusion theorem and a characterization of $\lambda \in \rho(P_p)$.

THEOREM 3.1.
(a) $\overline{P(\mathbf{R}^n)} \subset \sigma(P_p)$, *where* $P(\mathbf{R}^n) = \{P(\xi);\ \xi \in \mathbf{R}^n\}$.
(b) $\lambda \in \rho(P_p)$ *if and only if* $(\lambda - P)^{-1} \in \mathcal{M}_p$.

Proof. (a) Let $\lambda \in \rho(P_p)$. Since $(\lambda - P_p)^{-1}$ is translation invariant, by Lemma 2.2(c) there exists $u \in \mathcal{M}_p$ such that $(\lambda - P_p)^{-1}\phi_k = \mathcal{F}^{-1}(u\hat{\phi}_k)$, where $\phi_k \in \mathcal{S}_c$ with $\hat{\phi}_k(\xi) = 1$ for $1/k \le |\xi| \le k$ $(k \in \mathbf{N})$. This implies that $u(\xi)(\lambda - P(\xi)) = 1$ for $\xi \ne 0$, and thus $\lambda \notin P(\mathbf{R}^n \setminus \{0\})$ by Lemma 2.2(a). The claim follows now from the closedness of $\sigma(P_p)$.

(b) We may assume $\lambda = 0$. If $0 \in \rho(P_p)$, then from the proof of (a) one sees easily that $P^{-1} = u \in \mathcal{M}_p$. Conversely, since $f = P_p(\mathcal{F}^{-1}(P^{-1}\hat{f}))$ for $f \in H^p$ and since $g = \mathcal{F}^{-1}(P^{-1}\mathcal{F}(P_p g))$ for $g \in D(P_p)$, we deduce from $P^{-1} \in \mathcal{M}_p$ that $0 \in \rho(P_p)$. □

As for the spectral mapping property of P_p (i.e., $\overline{P(\mathbf{R}^n)} = \sigma(P_p)$), we assume that P is coercive. Then $\overline{P(\mathbf{R}^n)} = P(\mathbf{R}^n)$. When $\rho(P_p) = \emptyset$, Theorem 3.1(a) implies that the spectral mapping property for P_p holds if and only if $P(\mathbf{R}^n) = \mathbf{C}$. When $\rho(P_p) \ne \emptyset$, we have the following theorem.

THEOREM 3.2. *Let P be an r-coercive polynomial of degree m. Suppose one of the following conditions is satisfied:*
(a) $\rho(P_p) \ne \emptyset$.
(b) $n_p(m - r) \le r$. *In particular, P is elliptic.*
(c) *There exists $s \in [1 - r/n_p, 1]$ such that*

$$\left|\frac{D^\alpha P(\xi)}{P(\xi)}\right| = O(|\xi|^{-s|\alpha|})\ (|\xi| \to \infty) \quad \text{for } 0 < |\alpha| \le [n_p] + 1. \tag{3.1}$$

Then $\sigma(P_p) = P(\mathbf{R}^n)$.

Proof. We first note that if P is r-coercive, then (3.1) is satisfied with $s = r + 1 - m$ (see [19, p. 67]), and so (b) implies (c). If (c) is satisfied, we may assume $P(\mathbf{R}^n) \ne \mathbf{C}$. Otherwise, the spectral mapping property follows from Theorem 3.1(a) immediately. Let $\lambda \notin P(\mathbf{R}^n)$. Since

$$|D^\alpha(\lambda - P(\xi))^{-1}| = O(|\xi|^{-s|\alpha| - r})\ (|\xi| \to \infty) \quad \text{for } |\alpha| \le [n_p] + 1,$$

we have $(\lambda - P)^{-1} \in \mathcal{M}_p$ by Lemma 2.2(b), which implies (a) by Theorem 3.1(b).

When (a) is satisfied, our proof is similar to the proof of Theorem 4.3 of Chapter 11 in [19]. Set $Q = (\lambda - P)^k$, where $k = n_p(m - r)/r$ and $\lambda \notin P(\mathbf{R}^n)$. Then $Q^{-1} \in \mathcal{M}_p$, and so $0 \in \rho(Q_p)$. Also, $(\lambda - P_p)^k\phi = Q_p\phi$ for $\phi \in \mathcal{S}_c$.

Since (a) implies the closedness of $(\lambda - P_p)^k$, $(\lambda - P_p)^k$ is an extension of Q_p by Theorem 2.3(c). Thus

$$R(\lambda - P_p) \supset R((\lambda - P_p)^k) \supset R(Q_p) = H^p.$$

On the other hand, if $(\lambda - P_p)^k f = 0$, then

$$0 = \langle (\lambda - P_p)^k f, \phi \rangle = \langle f, (\lambda - P(-D))^k \phi \rangle \quad \text{for } \phi \in \mathcal{S}.$$

Since $\lambda \notin P(\mathbf{R}^n) = P(-\mathbf{R}^n)$, for every $\psi \in \mathcal{S}$ there exists $\phi \in \mathcal{S}$ such that $\psi = (\lambda - P(-D))^k \phi$. Consequently $f = 0$, and thus

$$\ker(\lambda - P_p) \subset \ker((\lambda - P_p)^k) = \{0\}.$$

So we obtain that $\lambda \in \rho(P_p)$. Thus $\sigma(P_p) \subset P(\mathbf{R}^n)$. The claim follows now from Theorem 3.1(a). $\square$

Condition (3.1) can be relaxed to

$$\left| \frac{D_j^k P(\xi)}{P(\xi)} \right| = O(|\xi|^{-sk}) \ (|\xi| \to \infty) \text{ for } 1 \le j \le n \text{ and } 0 < k \le [n_p] + 1, \tag{3.2}$$

since the same is true for the corresponding condition in Lemma 2.2(b) (see [14, p. 314]). Moreover, for $s \in (0, 1]$, a polynomial P is called *s-hypoelliptic* if it satisfies (3.1) for all $\alpha \in \mathbf{N}_0^n$.

We now turn to the essential spectrum of P_p, which is defined by

$$\sigma_e(P_p) = \cap \{\sigma(P_p + Q_p);\ Q_p \text{ is a compact linear operator on } H^p\}.$$

Clearly $\sigma_e(P_p)$ is a closed set. Similarly to Theorem 4.4 of Chapter 1 in [19] we obtain the following result.

Lemma 3.3. *If there exists a sequence $\{f_k\} \subset D(P_p)$ such that $\|f_k\|_{H^p} \to \delta > 0$, $(\lambda - P_p) f_k \to 0$, and $\{f_k\}$ has no convergent subsequence, then $\lambda \in \sigma_e(P_p)$.*

The following lemma is motivated by Lemma 2 in [12]. We now choose $\varphi \in \mathcal{S}$ in the definition of H^p such that $\hat{\varphi}(0) \neq 0$ and $\operatorname{supp} \varphi \subset \{x \in \mathbf{R}^n;\ |x| \le 1\}$.

Lemma 3.4. *Let $0 \neq \psi \in \mathcal{S}_c$, $\xi \in \mathbf{R}^n$, and $f_{k,\xi} = k^{-n/p} \psi(\frac{\cdot}{k}) e^{i\xi \cdot}$ for $k \in \mathbf{N}$. Then:*

(a) *$f_{k,\xi} \in \mathcal{S}_c$ for sufficiently large k.*
(b) *$\lim_{k \to \infty} \|f_{k,\xi}\|_{H^p} = \delta > 0$, where*

$$\delta = \begin{cases} \|\psi\|_{L^p} \sup_{t > 0} |\hat{\varphi}(t\xi)| & \text{if } \xi \neq 0, \\ \|\psi\|_{H^p} & \text{if } \xi = 0. \end{cases}$$

(c) *$\{f_{k,\xi}\}_{k \in \mathbf{N}}$ has no convergent subsequence in H^p.*

Proof. (a) Clearly $f_{k,\xi} \in \mathcal{S}$. Since $\hat{f}_{k,\xi} = k^{n-n/p}\hat{\psi}(k \cdot -k\xi)$ and $-k\xi \notin \operatorname{supp}\hat{\psi}$ for sufficiently large k, it follows that $f_{k,\xi} \in \mathcal{S}_c$.

(b) If $\xi = 0$, it is obvious that $\|f_{k,0}\|_{H^p} = \|\psi\|_{H^p} > 0$ for $k \in \mathbf{N}$, as desired. If $\xi \neq 0$, we have by (a) and the definition of H^p that for sufficiently large k,

$$0 \leq \|f_{k,\xi}\|_{H^p}^p - \int_{\mathbf{R}^n} \sup_{t \leq \sqrt{k}} |(\varphi_t * f_{k,\xi})(x)|^p dx \leq \int_{\mathbf{R}^n} \sup_{t > \sqrt{k}} |(\varphi_t * f_{k,\xi})(x)|^p dx.$$

Copying the proof of (3) in [12] yields that

$$\lim_{k\to\infty} \int_{\mathbf{R}^n} \sup_{t>\sqrt{k}} |(\varphi_t * f_{k,\xi})(x)|^p dx = 0,$$

so it suffices to show

$$\lim_{k\to\infty} \int_{\mathbf{R}^n} \sup_{t\leq\sqrt{k}} |(\varphi_t * f_{k,\xi})(x)|^p dx = \{\|\psi\|_{L^p} \sup_{t>0} |\hat{\varphi}(t\xi)|\}^p.$$

Again, copying the proof of (5) in [12] we obtain

$$\lim_{k\to\infty} \int_{\mathbf{R}^n} \sup_{t\leq\sqrt{k}} |(\varphi_t * f_{k,\xi})(x) - k^{-n/p}\psi(\tfrac{x}{k})(\varphi_t * e^{i\xi\cdot})(x)|^p dx = 0.$$

But $\varphi_t * e^{i\xi\cdot} = e^{i\xi\cdot}\hat{\varphi}(t\xi)$, so we obtain

$$\begin{aligned}\int_{\mathbf{R}^n} \sup_{t\leq\sqrt{k}} |k^{-n/p}\psi(\tfrac{x}{k})(\varphi_t * e^{i\xi\cdot})(x)|^p dx &= \int_{\mathbf{R}^n} k^{-n}|\psi(\tfrac{x}{k})|^p dx \{\sup_{t\leq\sqrt{k}} |\hat{\varphi}(t\xi)|\}^p \\ &= \{\|\psi\|_{L^p} \sup_{t\leq\sqrt{k}} |\hat{\varphi}(t\xi)|\}^p.\end{aligned}$$

The claim thus follows. Moreover, since $\hat{\varphi}(0) \neq 0$, we have $\sup_{t>0} |\hat{\varphi}(t\xi)| > 0$.

(c) Since $L^\infty \hookrightarrow \mathcal{S}'$ and $\|f_{k,\xi}\|_{L^\infty} = k^{-n/p}\|\psi\|_{L^\infty}$, we have $f_{k,\xi} \to 0$ in $\mathcal{S}'$. If $\{f_{k,\xi}\}_{k\in\mathbf{N}}$ has convergent subsequence in H^p, then we may assume without loss of generality that $f_{k,\xi} \to f$ in H^p. This implies that $f_{k,\xi} \to f$ in $\mathcal{S}'$, and thus $f = 0$. But by (b) $\|f\|_{H^p} = \delta > 0$, which yields a contradiction. □

We are now in a position to prove the result on the essential spectrum of P_p.

THEOREM 3.5. $\sigma_e(P_p) = \sigma(P_p)$.

Proof. If $\lambda \in P(\mathbf{R}^n)$, then $\lambda = P(\xi)$ for some $\xi \in \mathbf{R}^n$. Let $f_{k,\xi}$ be given as in Lemma 3.4. For sufficiently large k, we have by Leibniz's formula

$$(\lambda - P_p)f_{k,\xi} = (P(\xi) - P(D))f_{k,\xi} = \sum_{0<|\alpha|\leq m} \frac{1}{\alpha!}k^{-|\alpha|}P^{(\alpha)}(\xi)f_{k,\xi}^{\alpha},$$

where $m = \deg(P)$ and $f_{k,\xi}^\alpha = k^{-n/p}(D^\alpha\psi)(\frac{\cdot}{k})e^{i\xi\cdot}$. Since $\psi \in \mathcal{S}_c$ implies $D^\beta\psi \in \mathcal{S}_c$, $\lim_{k\to\infty} \|f_{k,\xi}^\alpha\|_{H^p}$ exists for $|\alpha| \le m$. Thus

$$\|(\lambda - P_p)f_{k,\xi}\|_{H^p}^p \le \sum_{0<|\alpha|\le m} (|P^{(\alpha)}(\xi)|/\alpha!)^p k^{-|\alpha|/p} \|f_{k,\xi}^\alpha\|_{H^p}^p \to 0 \ (k \to \infty).$$

It follows now from Lemma 3.3 that $\lambda \in \sigma_e(P_p)$, and therefore $\overline{P(\mathbf{R}^n)} \subset \sigma_e(P_p)$ by the closedness of $\sigma_e(P_p)$.

If $\lambda \in \sigma(P_p)\setminus\overline{P(\mathbf{R}^n)}$, then for each $\psi \in \mathcal{S}_c$, $f := \mathcal{F}^{-1}((\lambda - P)^{-1}\hat{\psi}) \in D(P_p)$ and $(\lambda - P_p)f = \psi$. This means that $\mathcal{S}_c \subset R(\lambda - P_p)$. Since $\lambda \in \sigma(P_p)$, the inequality

$$\|f\|_{H^p} \le M\|(\lambda - P_p)f\|_{H^p} \quad \text{for } f \in D(P_p) \tag{3.3}$$

cannot hold. Therefore there exists a sequence $\{f_k\} \subset D(P_p)$ such that $\|f_k\|_{H^p} = 1$ and $(\lambda - P_p)f_k \to 0$ in H^p. We note that $\{f_k\}$ has no convergent subsequence. Otherwise, the closedness of P_p would imply that λ is an eigenvalue of P_p, which contradicts Theorem 3.6(a) below. The desired result follows now from Lemma 3.3. □

For the point spectrum (i.e., the set of eigenvalues) and the resolvent of P_p we have the following result:

Theorem 3.6.
(a) *P_p has no eigenvalues.*
(b) *P_p has no compact resolvents.*

Proof. (a) If $(\lambda - P_p)f = 0$ for some $\lambda \in \mathbf{C}$ and $f \in D(P_p)$, then $(\lambda - P)\hat{f} = 0$. Since $\hat{f} \in C(\mathbf{R}^n)$ (cf. [21, p. 128]) and since $\{x \in \mathbf{R}^n;\ P(x) = \lambda\}$ has zero Lebesgue measure (cf. [3, p. 429]), we get $\hat{f} = 0$, and thus $f = 0$, as desired.

(b) Let $\lambda \in \rho(P_p)$. Choose $\phi \in C_c^\infty$ such that $\phi(\xi) = 1$ for $|\xi| \le 1$. Then

$$T_p f := \mathcal{F}^{-1}(\phi\hat{f}) = (\lambda - P_p)^{-1}\mathcal{F}^{-1}((\lambda - P)\phi\hat{f}) \quad \text{for } f \in H^p.$$

If $(\lambda - P_p)^{-1}$ is a compact operator on H^p, so is T_p. By Lemma 3.4, $\{f_{k,0}\}$ is a bounded subset in H^p and has no convergent subsequence. But a simple computation leads to $T_p f_{k,0} = f_{k,0}$ for sufficiently large k, which yields a contradiction. □

We let $\sigma_a(P_p)$ denote the approximate point spectrum of P_p, $\sigma_c(P_p)$ the continuous spectrum of P_p, and $\sigma_r(P_p)$ the residual spectrum of P_p. In view of Theorem 3.6(a), these are defined as follows:

$$\sigma_a(P_p) = \{\lambda \in \mathbf{C};\ R(\lambda - P_p) \text{ is not closed in } H^p\},$$

$$\sigma_c(P_p) = \left\{\lambda \in \mathbf{C};\ R(\lambda - P_p) \ne H^p \text{ and } \overline{R(\lambda - P_p)} = H^p\right\},$$

$$\sigma_r(P_p) = \left\{\lambda \in \mathbf{C};\ \overline{R(\lambda - P_p)} \ne H^p\right\}.$$

THEOREM 3.7.

(a) $\sigma_a(P_p) = \sigma(P_p)$.
(b) $\sigma_c(P_p) = \sigma(P_p) \setminus P(\mathbf{R}^n \setminus \{0\})$.
(c) $\sigma_r(P_p) = P(\mathbf{R}^n \setminus \{0\})$.

Proof. We divide the proof of this theorem into several steps.

Step 1: $P(\mathbf{R}^n) \subset \sigma_a(P_p)$. If $\lambda \in P(\mathbf{R}^n)$, then $\lambda = P(\xi)$ for some $\xi \in \mathbf{R}^n$. From the proof of Theorem 3.5 there exists a sequence $\{f_{k,\xi}\} \subset \mathcal{S}_c$ such that $\|f_{k,\xi}\|_{H^p} \to \delta(>0)$ and $(\lambda - P_p)f_{k,\xi} \to 0$. This shows that (3.3) cannot hold for $f \in \mathcal{S}_c$. By Theorem 2.3(c), $R(\lambda - P_p)$ is not a closed subspace in H^p, i.e., $\lambda \in \sigma_a(P_p)$.

Step 2: $\sigma(P_p) \setminus P(\mathbf{R}^n \setminus \{0\}) \subset \sigma_a(P_p) \cap \sigma_c(P_p)$. If $\lambda \in \sigma(P_p) \setminus P(\mathbf{R}^n)$, then $\mathcal{S}_c \subset R(\lambda - P_p)$ (see the proof of Theorem 3.5), and so $\lambda \in \sigma_a(P_p) \cap \sigma_c(P_p)$. If $\lambda \in P(\mathbf{R}^n) \setminus P(\mathbf{R}^n \setminus \{0\})$, then $P(0) = \lambda \neq P(\xi)$ for all $\xi \neq 0$. Following the proof of Theorem 3.5 we find that the assertion $\mathcal{S}_c \subset R(\lambda - P_p)$ is still true, and so $\lambda \in \sigma_c(P_p)$.

Step 3: $P(\mathbf{R}^n \setminus \{0\}) \subset \sigma_r(P_p)$. If $\lambda \in P(\mathbf{R}^n \setminus \{0\})$, then $\lambda = P(\xi)$ for some $\xi \neq 0$. We first consider the case $p = 1$. Let $e_\xi(x) = e^{-i\xi \cdot x}$ for $x \in \mathbf{R}^n$. Then a simple computation yields that $P(-D)e_\xi = \lambda e_\xi$. Also, $e_\xi \in \mathrm{BMO} = (H^1)^*$, where BMO denotes the space of functions of bounded mean oscillation on $\mathbf{R}^n$. Consequently,

$$0 = \langle (\lambda - P(-D))e_\xi, \phi \rangle = \langle e_\xi, (\lambda - P_1)\phi \rangle \quad \text{for } \phi \in C_c^\infty \cap H^1. \tag{3.4}$$

Since $C_c^\infty \cap H^1$ is a core of P_1 (see Theorem 2.3(c)), and since e_ξ ($\xi \neq 0$) is not a null element in BMO, it follows that $\overline{R(\lambda - P_1)} \neq H^1$, i.e., $\lambda \in \sigma_r(P_1)$. In the case $0 < p < 1$, we note that $e_\xi \in \tilde{\Lambda}_{n/p-n} = (H^p)^*$, where $\tilde{\Lambda}_s$ ($s > 0$) is the homogeneous Hölder space on $\mathbf{R}^n$ (see [14, p. 271] for the definition), and that (3.4) (with 1 replaced by p) still holds. Hence the claim follows from Theorem 2.3(c). □

4. The spectrum of PDOs in L^p $(p > 1)$

We first note that Lemma 2.2(b) and (c) also holds for $p > 1$ (cf. [14], [7]). In [6] Theorem 2.3(b) is shown to hold for $p > 1$ aside from the critical case. Theorems 3.1 and 3.2 also remain true for $p > 1$ (cf. [19]), but the critical case of Theorem 3.2 is not treated in [19] (see also [1]).

The following theorem collects some results for the case $p > 1$,

THEOREM 4.1. *Let $p > 1$ in assertions* (a)–(d).

(a) *If P is r-coercive and satisfies* (3.2) *with $1 \geq s \geq 1 - r/n_p$, then $\sigma(P_p) = P(\mathbf{R}^n)$.*
(b) $\sigma_e(P_p) = \sigma_a(P_p) = \sigma(P_p)$.
(c) *P_p has no compact resolvents.*

(d) $\sigma(P_p) \setminus P(\mathbf{R}^n) \subset \sigma_c(P_p)$. *Consequently,* $\sigma_r(P_p) \subset P(\mathbf{R}^n)$.
(e) *If* $p \in (1, 2]$, *then* P_p *has no eigenvalues. Consequently,* $\sigma_c(P_q) = \sigma(P_q)$ *and* $\sigma_r(P_q) = \emptyset$, *where* $\frac{1}{q} + \frac{1}{p} = 1$.

Proof. (a) Since Lemma 2.2(b) and (c) are true for $p > 1$, copying the proof of Theorem 3.2(c) leads to the claim.

(b) The assertion that $\overline{P(\mathbf{R}^n)} \subset \sigma_e(P_p) \cap \sigma_a(P_p)$ can be found in [19, p. 63,293], while the assertion that $\sigma(P_p) \setminus \overline{P(\mathbf{R}^n)} \subset \sigma_e(P_p) \cap \sigma_a(P_p)$ can be proved in the same way as in the proofs of Theorems 3.5 and 3.7(a) since every eigenvalue must be in $P(\mathbf{R}^n)$.

(c) and (d) can be shown by copying the proofs of Theorems 3.6(b) and Theorem 3.7, respectively.

(e) If $p \in (1, 2]$, the assertion that P_p has no eigenvalues is established in the proof of Theorem 4.1 of Chapter 11 in [19]. Since $\sigma_r(P_q)$ is exactly the set of all eigenvalues of P_p, the remaining assertions follow. $\square$

In the sequel, we will study the eigenvalues of P_p ($p > 2$). Denote by $m(\cdot)$ the Lebesgue measure on $\mathbf{R}^n$, and by K_δ the set $\{x \in \mathbf{R}^n;\ \operatorname{dist}(x, K) \le \delta\}$, where $K \subset \mathbf{R}^n$.

Lemma 4.2. *Let* K *be a compact subset in* $\mathbf{R}^n$, *and let* $m(K) = 0$. *Suppose there exists a constant* $M > 0$ *such that* $m(K_\delta) \le M\delta$ *for* $\delta \in (0, 1]$. *If* $\operatorname{supp} \hat{f} \subset K$ *for some* $f \in L^p$ $(2 < p < \frac{2n}{n-1})$, *then* $f = 0$.

Proof. Choose $\varphi \in C_c^\infty$ such that $\hat{\varphi}(0) = 1$ and $\operatorname{supp} \varphi \subset B_1$. Set $\psi_\delta = \chi_{K_\delta} * \varphi_{\delta/2}$ for $\delta \in (0, 1]$, where χ_{K_δ} is the characteristic function of K_δ. By Bernstein's theorem (see [20]) it follows that for $\delta \in (0, 1]$

$$\begin{aligned}
\|\hat{\psi}_\delta\|_{L^1} &\le M \|\psi_\delta\|_{L^2}^{1-n/2j} \sum_{|\alpha|=j} \|D^\alpha \psi_\delta\|_{L^2}^{n/2j} \\
&\le M(\|\chi_{K_\delta}\|_{L^2} \cdot \|\varphi\|_{L^1})^{1-n/2j} \sum_{|\alpha|=j} (\|\chi_{K_\delta}\|_{L^2} \cdot (\tfrac{2}{\delta})^j \|D^\alpha \varphi\|_{L^1})^{n/2j} \\
&\le M\delta^{(1-n)/2},
\end{aligned}$$

where $j = [n/2] + 1$ and M denotes a generic constant independent of δ. Consequently, for fixed $\phi \in \mathcal{S}$,

$$\|\mathcal{F}(\psi_\delta \mathcal{F}^{-1}\phi)\|_{L^1} \le \|\hat{\psi}_\delta\|_{L^1} \|\phi\|_{L^1} \le M\delta^{(1-n)/2}.$$

Also,

$$\|\mathcal{F}(\psi_\delta \mathcal{F}^{-1}\phi)\|_{L^2} \le \|\hat{\psi}_\delta\|_{L^2} \|\phi\|_{L^1} \le M\|\psi_\delta\|_{L^2} \le M\delta^{1/2}.$$

Thus by Hölder's inequality

$$\begin{aligned}\|\mathcal{F}(\psi_\delta \mathcal{F}^{-1}\phi)\|_{L^q}^q &\le \|\mathcal{F}(\psi_\delta \mathcal{F}^{-1}\phi)\|_{L^1}^{2-q} \|\mathcal{F}(\psi_\delta \mathcal{F}^{-1}\phi)\|_{L^2}^{2q-2} \\ &\le M\delta^{(qn+q-2n)/2},\end{aligned}$$

where $\frac{1}{q}+\frac{1}{p}=1$, and so $qn+q-2n>0$. From this we obtain that $\mathcal{F}((1-\psi_\delta)\mathcal{F}^{-1}\phi) \to \phi$ as $\delta \to 0$ in L^q. Since $(1-\psi_\delta)\mathcal{F}^{-1}\phi \in \mathcal{S}_K := \{\psi \in \mathcal{S};\ \operatorname{supp}\psi \subset \mathbf{R}^n \setminus K\}$, the set $\{\hat{\psi};\ \psi \in \mathcal{S}_K\}$ is dense in L^q. Now, by the assumption on f, $\langle f, \hat{\psi}\rangle = \langle \hat{f}, \psi\rangle = 0$ for all $\psi \in \mathcal{S}_K$, and therefore $f = 0$. $\square$

We are now in a position to treat the eigenvalues of P_p for $p \in (2, \frac{2n}{n-1})$.

THEOREM 4.3. *Let $2 < p < \frac{2n}{n-1}$. Then P_p has no eigenvalues.*

Proof. Suppose λ is an eigenvalue of P_p associated with eigenfunction $f \neq 0$; without loss of generality we may assume that $\lambda = 0$. Since $P\hat{f} = \mathcal{F}(P_p f) = 0$, $\operatorname{supp}\hat{f} \subset N := \{\xi \in \mathbf{R}^n;\ P(\xi) = 0\}$. Choose $\varphi \in C_c^\infty$ such that $\varphi\hat{f} \neq 0$. Then $g := \mathcal{F}^{-1}(\varphi\hat{f}) \in L^p$, $g \neq 0$, and $\operatorname{supp}\hat{g} \subset N \cap \operatorname{supp}\varphi$. Also, in view of the fact that

$$N = \{\xi \in \mathbf{R}^n;\ \operatorname{Re} P(\xi) = 0\} \cap \{\xi \in \mathbf{R}^n;\ \operatorname{Im} P(\xi) = 0\},$$

we may assume that P is a polynomial with real coefficients, and thus N is covered by a finite number of $(n-1)$-dimensional submanifolds $\{\xi \in \mathbf{R}^n;\ D^\alpha P(\xi) = 0,\ \nabla D^\alpha P(\xi) \neq 0\}$. We claim that $m((N \cap \operatorname{supp}\varphi)_\delta) \le M\delta$ for $\delta \in (0,1]$. In fact, for fixed $x \in N \cap \operatorname{supp}\varphi$, there exists a neighborhood U of x and a diffeomorphism $\Phi : U \to \Phi(U)$ such that $\Phi(U) \subset \{0\} \times \mathbf{R}^{n-1}$. Choose a bounded neighborhood U^* of x and $\delta' \in (0,1]$ such that $U_\delta^* \subset U$ for $\delta \in (0,\delta']$, and put $J_\Phi = \sup\{|\mathrm{Jacobi}(\Phi(x))|;\ x \in U_{\delta'}^*\}$. Then

$$m(U_\delta^*) \le J_{\Phi^{-1}} m(\Phi(U_\delta^*)) \le J_{\Phi^{-1}} m((\Phi(U^*))_{J_\Phi \delta}) \le M_1\delta \quad \text{for } \delta \in (0,\delta'],$$

which implies that $m(U_\delta^*) \le M_2\delta$ for $\delta \in (0,1]$. The claim follows thus from the compactness of $N \cap \operatorname{supp}\varphi$. Finally, noting that $m(N) = 0$ we obtain by Lemma 4.2 that $g = 0$, which yields a contradiction. $\square$

The following example shows that the bound $\frac{2n}{n-1}$ in Theorem 4.3 is best possible.

EXAMPLE 4.4. For $n \ge 2$ and $s > 0$ define $\delta_s \in \mathcal{S}'$ by

$$\langle \delta_s, \phi\rangle = \int_{|x|=s} \phi(x)\,d\sigma(x) \quad \text{for } \phi \in \mathcal{S},$$

where $d\sigma$ denotes the measure on the sphere $|x| = s$. Then (cf. [5, p. 198])

$$(\mathcal{F}^{-1}\delta_s)(y) = (s/2\pi)^{n/2}|y|^{1-n/2} J_{-1+n/2}(s|y|) \quad \text{for } y \in \mathbf{R}^n,$$

where J_γ denotes the Bessel function of order γ. It is known (see, e.g., [21, p. 338]) that

$$|J_{-1+n/2}(s|y|)| \le M_s|y|^{-1/2} \quad \text{for } |y| \ge 1,$$

and thus $\mathcal{F}^{-1}\delta_s \in L^p$ for $p > \frac{2n}{n-1}$. Since

$$\langle \mathcal{F}^{-1}((|\cdot|^2 - s^2)\delta_s), \phi\rangle = \langle \delta_s, (|\cdot|^2 - s^2)\mathcal{F}^{-1}\phi\rangle = 0 \quad \text{for } \phi \in \mathcal{S},$$

we obtain

$$(-\Delta_p - s^2)\mathcal{F}^{-1}\delta_s = \mathcal{F}^{-1}((|\cdot|^2 - s^2)\delta_s) = 0 \quad \text{for } p > \frac{2n}{n-1},$$

where $-\Delta_p$ denotes the operator P_p with $P(\xi) = |\xi|^2$. This means that s^2 is an eigenvalue of $-\Delta_p$ ($p > \frac{2n}{n-1}$, $n \ge 2$), and $\mathcal{F}^{-1}\delta_s$ is the corresponding eigenfunction. Moreover, we remark that for every $p > 0$, $\sigma(-\Delta_p) = [0, \infty)$ by Theorem 3.2(b) and Theorem 4.1(a), and that $0 \in \sigma_c(-\Delta_p)$ by Theorems 3.7(b) and 4.1(d).

If K is an $(n-1)$-dimensional plane of $\mathbf{R}^n$, corresponding to Lemma 4.2 we have

Lemma 4.5. *Let K be an $(n-1)$-dimensional plane of $\mathbf{R}^n$. If* $\operatorname{supp}\hat{f} \subset K$ *for some $f \in L^p$ ($p > 2$), then $f = 0$.*

Proof. Since the assertion is invariant under affine transformations, we may assume without loss of generality that $\operatorname{supp}\hat{f} \subset \{0\} \times \mathbf{R}^{n-1}$. This implies that $\operatorname{supp}(\mathcal{F}^{-1}f) \subset \{0\} \times \mathbf{R}^{n-1}$. Choose $\psi \in C_c^\infty(\mathbf{R})$ such that $\operatorname{supp}\psi \subset [-1, 1]$ and $\psi(t) = 1$ for $t \in [-1/2, 1/2]$, and set

$$\psi_\varepsilon(x) = \psi(x_1/\varepsilon) \quad \text{for } x = (x_1, \cdots, x_n) \in \mathbf{R}^n \text{ and } \varepsilon > 0.$$

Let $\frac{1}{q} + \frac{1}{p} = 1$ and $r \in (1, q)$. Then $\psi_\varepsilon \in \mathcal{M}_r$ and for $\phi \in \mathcal{S}$,

$$\|\mathcal{F}^{-1}(\psi_\varepsilon\hat{\phi})\|_{L^r} \le \|\psi_\varepsilon\|_{\mathcal{M}_r}\|\phi\|_{L^r} = \|\psi_1\|_{\mathcal{M}_r}\|\phi\|_{L^r} \quad (\varepsilon > 0).$$

Also, by Lebesgue's dominated convergence theorem one sees

$$\|\mathcal{F}^{-1}(\psi_\varepsilon\hat{\phi})\|_{L^2} = (2\pi)^{-n/2}\|\psi_\varepsilon\hat{\phi}\|_{L^2} \to 0 \quad (\varepsilon \to 0).$$

It follows thus from Hölder's inequality that

$$\|\mathcal{F}^{-1}(\psi_\varepsilon\hat{\phi})\|_{L^q}^q \le \|\mathcal{F}^{-1}(\psi_\varepsilon\hat{\phi})\|_{L^r}^{r\frac{2-q}{2-r}}\|\mathcal{F}^{-1}(\psi_\varepsilon\hat{\phi})\|_{L^2}^{2\frac{q-r}{2-r}} \to 0 \quad (\varepsilon \to 0).$$

Now, the rest of the proof follows the proof of Lemma 4.2. □

Similarly to Theorem 4.3, by Lemma 4.5 we can deduce the following result, in which we note that if λ is an eigenvalue of P_p, then $\lambda \in P(\mathbf{R}^n)$.

Theorem 4.6. *Let $\lambda \in P(\mathbf{R}^n)$, and let $\{\xi \in \mathbf{R}^n;\ P(\xi) = \lambda\}$ be covered by a finite number of $(n-1)$-dimensional planes of $\mathbf{R}^n$. Then λ is not an eigenvalue of P_p for all $p > 2$.*

Proof. If λ is an eigenvalue of P_p associated with eigenfunction $f \neq 0$, then there exist $(n-1)$-dimensional planes $\{K_j\}_{j=1}^k$ such that

$$\operatorname{supp} \hat{f} \subset \{\xi \in \mathbf{R}^n;\ P(\xi) = \lambda\} \subset \bigcup_{1 \le j \le k} K_j.$$

When $\operatorname{supp} \hat{f} \cap K_j \neq \emptyset$ for some j, we may choose $\varphi \in C_c^\infty$ such that $\varphi \hat{f} \neq 0$ and $\operatorname{supp} \varphi \subset K_j$. Then $0 \neq \mathcal{F}^{-1}(\varphi \hat{f}) \in L^p$ and $\operatorname{supp}(\varphi \hat{f}) \subset K_j$, which contradicts Lemma 4.5. This means that $f = 0$, which contradicts the assumption again. $\square$

Consider the polynomial

$$P(\xi) = i\xi_1 + (\xi_2^2 + \cdots + \xi_n^2)^l \quad \text{for } \xi \in \mathbf{R}^n \text{ and } l \in \mathbf{N}.$$

In particular, when $l = 1$, $P(D)$ is the heat operator. Clearly, for every $\lambda \in P(\mathbf{R}^n)$, $\{\xi \in \mathbf{R}^n;\ P(\xi) = \lambda\}$ is contained in an $(n-1)$-dimensional plane. Thus, by Theorem 4.6, P_p has no eigenvalues for all $p > 2$. Again, by Theorem 4.6, all one-order PDOs have no eigenvalues in L^p $(p > 2)$. Moreover, by Theorem 4.3 all ordinary differential operators have no eigenvalues in $L^p(\mathbf{R})$ $(p > 2)$.

5. Examples

In this section we will give some examples in H^p $(p > 0)$.

Example 5.1. Consider the polynomial

$$P(\xi) = (1 + \xi_1^2)(1 + (\xi_1 - \xi_2^k)^2) \quad \text{for } k \in \mathbf{N}.$$

It is known that P is 2-coercive (see [1, p. 36]). We claim that P also satisfies (3.2) with $s = \frac{1-k}{k}$. Indeed, it is easy to check that $|D_1^j P(\xi)/P(\xi)| \le 24$ $(j \ge 1)$, and that

$$D_2^j P(\xi) = \begin{cases} (-i)^j (1 + \xi_1^2) \xi_2^{k-j} \left(\frac{(2k)!}{(2k-j)!} \xi_2^k - \frac{2k!}{(k-j)!} \xi_1 \right) & \text{for } 1 \le j \le k, \\ (-i)^j \frac{(2k)!}{(2k-j)!} (1 + \xi_1^2) \xi_2^{2k-j} & \text{for } k+1 \le j \le 2k. \end{cases}$$

If $|\xi_2^k| \ge 2|\xi_1|$, then for $j \ge 1$ we have

$$\left| \frac{D_2^j P(\xi)}{P(\xi)} \right| \le M,$$

where M denotes a generic constant independent of ξ. If $|\xi_2^k| < 2|\xi_1|$, then

$$\left| \frac{D_2 P(\xi)}{P(\xi)} \right| \le M|\xi|^{1-1/k} \text{ and } \left| \frac{D_2^j P(\xi)}{P(\xi)} \right| \le M|\xi|^{2-j/k} \quad (j \ge 2).$$

Combining these estimates yields the claim. Thus by Theorems 3.2(c) and 4.1(a) we obtain that $\sigma(P_p) = P(\mathbf{R}^2) = [1, \infty)$ for $p \ge \frac{4k-2}{4k-1}$ and $k \in \mathbf{N}$. In particular, this is true for $p > 1$ and $k \in \mathbf{N}$, which shows that the corresponding results in [1] (see also [15]) are incorrect. More precisely, if $p > 1$ and $|\frac{1}{2} - \frac{1}{p}| \geq \frac{1}{4} + \frac{1}{k}$, Albrecht and Ricker [1] claimed that $\sigma(P_p) = \mathbf{C}$ for $k \geq 5$, based on a result due to Ruiz (see [16], [17], [18]). This is an oversight. In fact, the polynomial considered by Ruiz is as follows:

$$Q(\xi) = (1 + \xi_1^2)(1 + (\xi_2 - \xi_1^k)^2) \quad \text{for } k \in \mathbf{N}.$$

We can show as above that Q is $\frac{2}{k}$-coercive and satisfies (3.2) with $s = \frac{1-k}{k}$. Again, by Theorems 3.2(c) and 4.1(a), Q_p satisfies the spectral mapping property if $|\frac{1}{2} - \frac{1}{p}| \leq \frac{1}{2k-1}$. Note that the bounds $\frac{2}{k}$ and $\frac{1-k}{k}$ are best possible for Q, but not for Q_p in the case $k \geq 4$. In fact, when $k \geq 4$, Q_p satisfies the spectral mapping property if and only if $|\frac{1}{2} - \frac{1}{p}| < \frac{1}{4} + \frac{1}{k}$ (see [16], [17], [18]).

We next give an example in which $\sigma(P_p) = \mathbf{C}$ for $0 < p \leq 1$. We first need the following proposition.

Proposition 5.2. *Let $0 < p < q \leq 2$. Then $\sigma(P_q) \subset \sigma(P_p)$.*

Proof. By Theorem 3.1(b) it is sufficient to show that $\mathcal{M}_p \subset \mathcal{M}_q$. Indeed, for $u \in \mathcal{M}_p$ we have $u \in \mathcal{M}_2$. It follows thus by Theorem B in [13] and by interpolation between p and 2 (cf. [2], [22]) that $u \in \mathcal{M}_q$. $\square$

Example 5.3. For the polynomial

$$P(\xi) = \xi_1^2 - (\xi_2^2 + \xi_3^2 + \xi_4^2 + i)^2$$

it is known that P is 1-coercive and satisfies (3.1) with $s = -1/2$ (see [19, p. 295]). Then, by Theorems 3.2(c) and 4.1(a), $\sigma(P_p) = P(\mathbf{R}^4)$ for $p \in [\frac{3}{2}, 3]$. On the other hand, Iha and Schubert [9] proved that $\sigma(P_p) = \mathbf{C}$ for $p \in (1, \frac{8}{7})$. By Proposition 5.2 we obtain that $\sigma(P_p) = \mathbf{C}$ for $p \in (0, 1]$. Note that, in view of Iha and Schubert's result (see [9, p. 224]), P cannot be chosen in the form $\xi_1^2 - (\xi_2^2 + \xi_3^2 + i)^2$ as in [19, p. 295], [6, p. 620,622,625], [1, p. 35], and [15, p. 243]. We now turn to the factors of P, for example,

$$R(\xi) := \xi_1 - \xi_2^2 - \xi_3^2 - \xi_4^2 - i.$$

Set

$$u_z(\xi) = (\xi_1 - \xi_2^2 - \xi_3^2 - \xi_4^2 - z)^{-1} \quad \text{for } z \in \mathbf{C} \setminus \mathbf{R}.$$

For $p > 1$ and $p \neq 2$, Kenig and Tomas [10], [11] showed that $u_{-i} \notin \mathcal{M}_p$. Thus it is not hard to deduce by Theorem 1.13 in [7] that $u_z \notin \mathcal{M}_p$ for $z \in \mathbf{C} \setminus \mathbf{R}$. By Theorem 4.1 of Chapter 4 in [19] it follows that $\{\lambda \in \mathbf{C};\ \operatorname{Im} \lambda \neq -1\} \subset \sigma(R_p)$. Since $\sigma(R_p)$ is closed, $\sigma(R_p) = \mathbf{C}$. Combining this with Proposition 5.2 yields that $\sigma(R_p) = \mathbf{C}$ for $p > 0$ ($p \neq 2$). Noting that $R(\mathbf{R}^4) = \{\lambda \in \mathbf{C};\ \operatorname{Im} \lambda = -1\}$, one sees that R_p does not satisfy the spectral mapping property for $p > 0$ ($p \neq 2$).

EXAMPLE 5.4. Let $Q = PP_*$, where P is given as in Example 5.3 and $P_*(\xi) = 1 + |\xi|^2$ for $\xi \in \mathbf{R}^4$. Since P is 1-coercive and satisfies (3.1) with $s = -1/2$, it is not hard to see that Q is 3-coercive and also satisfies (3.1) with $s = -1/2$. It follows thus from Theorems 3.2(c) and 4.1(a) that $\sigma(Q_p) = Q(\mathbf{R}^4)$ for $p \geq 1$. We claim now that $\sigma(Q_p) = \mathbf{C}$ for $p \in (\frac{1}{2}, \frac{8}{13})$. If $\sigma(Q_q) \neq \mathbf{C}$ for some $q \in (\frac{1}{2}, \frac{8}{13})$, Theorem 3.2(a) leads to $\sigma(Q_q) = Q(\mathbf{R}^4)$. In view of $0 \notin Q(\mathbf{R}^4)$ we obtain by Theorem 3.1(b) that $Q^{-1} \in \mathcal{M}_q$. Define $u_z = P^{-1}P_*^{(1-3z)/2}$ for $0 \leq \operatorname{Re} z \leq 1$. Since P is 1-coercive, we have $u_{it} \in \mathcal{M}_2$ and

$$\|u_{it}\|_{\mathcal{M}_2} = \|u_{it}\|_{L^\infty} \leq \|P^{-1}P_*^{1/2}\|_{L^\infty} < \infty \quad \text{for } t \in \mathbf{R}.$$

On the other hand, by induction on $|\alpha|$ we get

$$|D^\alpha P_*(\xi)^{it}| \leq M_\alpha (1 + |t|)^{|\alpha|} |\xi|^{-|\alpha|} \quad \text{for } \xi \neq 0,\ \alpha \in \mathbf{N}_0^n \text{ and } t \in \mathbf{R}.$$

It follows thus from a generalization of Mihlin's multiplier theorem (see [2]) that $u_{1+it} \in \mathcal{M}_q$ and

$$\|u_{1+it}\|_{\mathcal{M}_q} \leq \|Q^{-1}\|_{\mathcal{M}_q} \|P_*^{-3it/2}\|_{\mathcal{M}_q} \leq M(1 + |t|)^{4(\frac{1}{q} - \frac{1}{2})} \quad \text{for } t \in \mathbf{R}.$$

Using the complex interpolation theorem (cf. [2]) we obtain that $P^{-1} = u_{1/3} \in \mathcal{M}_{q'}$, where $\frac{1}{q'} = \frac{1-\theta}{2} + \frac{\theta}{q}$ with $\theta = \frac{1}{3}$. Since $\frac{1}{2} < q < \frac{8}{13}$, we have $1 < q' < \frac{8}{7}$, which contradicts a result of Iha and Schubert [9]. By the claim and Proposition 5.2 we see that $\sigma(Q_p) = \mathbf{C}$ for $p \in (0, \frac{8}{13})$. But $Q(\mathbf{R}^4) \neq \mathbf{C}$, and so Q_p satisfies the spectral mapping property for $p \geq 1$, but not for $p \in (0, \frac{8}{13})$.

EXAMPLE 5.5. For given $e = (e_1, \cdots, e_n) \in \mathbf{N}^n$, set $|\alpha/e| = \sum_{k=1}^n \alpha_k/e_k$ for $\alpha = (\alpha_1, \cdots, \alpha_n) \in \mathbf{N}_0^n$. If $P(\xi) := \sum_{|\alpha/e| \leq 1} a_\alpha \xi^\alpha$ ($\xi \in \mathbf{R}^n$) is semi-elliptic, i.e., $\sum_{|\alpha/e|=1} a_\alpha \xi^\alpha \neq 0$ for $\xi \neq 0$, then P is an r-coercive polynomial of degree m, where $r = \min\{e_k\}$ and $m = \max\{e_k\}$. Also, P is $\frac{r}{m}$-hypoelliptic. Thus, by Theorems 3.2(c) and 4.1(a), $\sigma(P_p) = P(\mathbf{R}^n)$ if $n_p \leq rm/(m - r)$. When $n_p < rm/(m - r)$ and $p > 1$, the assertion is shown in [19, p. 71].

We conclude the paper with two questions. It is well known that all PDOs satisfy the spectral mapping property in L^2 (cf., e.g., [19]). Based on Examples 5.3 and 5.4, we ask the following

QUESTION 1. *Suppose P_p does not satisfy the spectral mapping property for some $p > 0$ ($p \neq 2$). Is it true that $\sigma(P_p) = \mathbf{C}$?*

In Example 5, one sees that for any $p_0 > 0$ there exists a semi-elliptic polynomial P such that P_p satisfies the spectral mapping property for all $p > p_0$ ($p \neq 2$). Combining Theorem 3.2(b) with this leads naturally to the following

QUESTION 2. *Suppose P_p satisfies the spectral mapping property for all $p > 0$. Is P elliptic?*

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