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DUNFORD-PETTIS AND DIEUDONNÉ POLYNOMIALS ON BANACH SPACES

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Abstract. We introduce two classes of m-homogeneous polynomials defined on Banach spaces, which extend the classes of Dunford-Pettis and Dieudonné linear operators. These extensions allow us to prove that several characterization theorems related to the Dunford-Pettis, Schur, and reciprocal Dunford-Pettis properties, are also valid in the more general case of homogeneous polynomials of any degree $m \in \mathbb{N}$.

1. Introduction

There are several generalizations of Dunford-Pettis and Dieudonné operators to the polynomial context in the literature (see, e.g., [2], [5], [10], [13], [17], [20], [21]). Even though linear operators are homogeneous polynomials, these extensions show some important differences between the linear and nonlinear case, such as the fact that not every compact polynomial is completely continuous.

The generalization that we introduce here is based on the general behavior of m-homogeneous polynomials with respect to the polynomial and weak topologies. We describe this behavior in Theorem 2.2. A consequence of this result is that every m-homogeneous polynomial transforms sequences which are Cauchy in the m-polynomial topology (the τ_m-Cauchy sequences) into weak Cauchy sequences. Thus we define Dunford-Pettis and Dieudonné polynomials as those polynomials which transform τ_m-Cauchy sequences into norm and weak convergent sequences, respectively. This natural character of the extension allows us to prove that some well known results concerning the classes of Dunford-Pettis and Dieudonné operators are special cases of more general results about homogeneous polynomials. We prove in Theorem 2.10 a diagram of inclusion relations that is valid for all Banach spaces and which, in particular, shows that a compact polynomial is always Dunford-Pettis. We also give several characterizations of the *polynomial Dunford-Pettis property*

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(Theorem 3.2), the *Schur property* (Theorem 3.5), the *polynomial reciprocal Dunford-Pettis property* (Proposition 3.13), and we prove some related results.

Throughout this work, E and F will denote Banach spaces over $\mathbb{K} = \mathbb{C}$ or $\mathbb{K} = \mathbb{R}$, E^* the topological dual space of E, and B_E the unit ball. For $m \in \mathbb{N}$, we will denote by $\mathcal{L}(^m E; F)$ the Banach space of all continuous m-linear operators from $E^m = E \times \overset{m}{\cdots} \times E$ into F. This space is canonically isomorphic to the space of linear operators $\mathcal{L}(\hat{\otimes}_\pi^m E; F)$, where $\hat{\otimes}_\pi^m E$ is the m-fold tensor product endowed with the projective topology.

A map $P : E \to F$ is an m-homogeneous polynomial if it is the restriction to the diagonal elements $(x, \dots, x) \in E \times \overset{m}{\cdots} \times E$ of a continuous m-linear map $T \in \mathcal{L}(^m E; F)$; the map is unique, if it is required to be symmetric. The Banach space of m-homogeneous polynomials from E to F (resp. $\mathbb{K}$), with the sup norm on the unit ball of E will be denoted by $\mathcal{P}(^m E; F)$ (resp. $\mathcal{P}(^m E)$).

The closed linear span of the set $\{x \otimes \overset{m}{\cdots} \otimes x,\ x \in E\}$ is a complemented subspace of the projective m-fold tensor product $\hat{\otimes}_\pi^m E$. We denote this subspace by $\hat{\otimes}_{s,\pi}^m E$, and the polynomial $\theta_m \in \mathcal{P}(^m E, \hat{\otimes}_{s,\pi}^m E)$ defined by $\theta_m(x) = x \otimes \overset{m}{\cdots} \otimes x$ will be called *the canonical m-homogeneous polynomial on E*.

For $P \in \mathcal{P}(^m E, F)$, $\hat{P} \in \mathcal{L}(\hat{\otimes}_{s,\pi}^m E, F)$ will denote the linear operator such that $P(x) = \hat{P} \circ \theta_m(x)$ for every $x \in E$. The *adjoint operator* $P^* \in \mathcal{L}(F^*, \mathcal{P}(^m E))$ of P is the continuous linear operator such that $P^*(f^*) = f^* \circ P$, for every $f^* \in F^*$.

2. Dunford-Pettis and Dieudonné polynomials

Let E be a Banach space and let $m \in \mathbb{N}$. We denote by E_{τ_m} the space E endowed with the initial topology induced by the family of m-homogeneous polynomials $\mathcal{P}(^m E)$. The weak topology corresponds to the case $m = 1$ and will be denoted, as usual, by E_w. The collection of sets of the form

$$\mathcal{U}_{x_0} = \{x \in E : |p_i(x) - p_i(x_0)| < \epsilon_i,\ 0 < \epsilon_i,\ p_i \in \mathcal{P}(^m E),\ i = 1, \dots, n\}$$

is a basis of neighborhoods of the point $x_0 \in E_{\tau_m}$. We will say that the sequence $(x_n) \subset E$ is τ_m-Cauchy, if for every increasing sequence of natural numbers (n_j) and all $p \in \mathcal{P}(^m E)$, $\lim_{j,k} |p(x_{n_j}) - p(x_{n_k})| = 0$.

Remark 2.1. A sequence $(x_n) \subset E$ is τ_m-Cauchy (resp. τ_m-convergent to $x \in E$) if and only if $(\theta_m(x_n))_n$ is weak Cauchy (resp. weakly convergent to $\theta_m(x)$) in $\hat{\otimes}_{s,\pi}^m E$.

Theorem 2.2. *Let E and F be Banach spaces, let $m \in \mathbb{N}$ and let $P : E \to F$ be a transformation such that there exists an m-linear map T from $E \times \overset{m}{\cdots} \times E$ to F with $P(x) = T(x, \dots, x)$ for $x \in E$. Then we have:*

(1) *P is continuous (i.e., $P \in \mathcal{P}(^m E, F)$) if and only if P is a continuous map from E_{τ_m} to F_w.*

(2) *Every bounded linear operator* $T \in \mathcal{L}(E,F)$ *is a continuous map from* E_{τ_m} *to* F_{τ_m}.

Proof. We first prove (1). If the polynomial P is continuous, its associated linear operator $\hat{P} \in \mathcal{L}(\hat{\otimes}^m_{s,\pi} E, F)$ is also continuous. Hence $\hat{P}$ is continuous from $(\hat{\otimes}^m_{s,\pi} E)_w$ to F_w. Now, the polynomial $\theta_m : E_{\tau_m} \to (\hat{\otimes}^m_{s,\pi} E)_w$ is continuous because the τ_m topology on E is the topology induced by the family $\{f^* \circ \theta_m : f^* \in (\hat{\otimes}^m_{s,\pi} E)^*\}$. Therefore $P = \hat{P} \circ \theta_m$ is continuous from E_{τ_m} to F_w. To obtain the converse direction, note that the norm on E defines a topology that is stronger than the τ_m-topology. Thus, whenever an m-homogeneous map P is $(\tau_m - w)$ continuous, it is also $(\|\cdot\|$-$w)$ continuous. Hence it is enough to prove that the set $P(B_E)$ is bounded in F_w. Consider a weak neighborhood $\mathcal{U}$ of 0 in F. By the $\|\cdot\| - w$ continuity of P, there exists $\delta > 0$ for which $P(\delta B_E) \subset \mathcal{U}$. Then, $P(B_E) = \delta^{-m} P(\delta B_E) \subset \delta^{-m}\mathcal{U}$. This proves that $P(B_E)$ is a bounded set in F_w and, consequently, also a bounded set in $F_{\|\cdot\|}$.

To prove (2) we must construct, for each $x_0 \in E$ and each neighborhood V of $T(x_0)$ of the form

$$V = \{y \in F : |p_i(y) - p_i(T(x_0))| < \epsilon_i, p_i \in \mathcal{P}(^m F), i = 1, \dots, n\},$$

a τ_m-neighborhood U of x_0 such that $T(U) \subset V$. In order to do so, consider the bounded linear operator $T^{(m)} := T \otimes \overset{m}{\cdots} \otimes T \in \mathcal{L}(\hat{\otimes}^m_{s,\pi} E, \hat{\otimes}^m_{s,\pi} F)$ and the weak neighborhood

$$\tilde{V} = \{z \in \hat{\otimes}^m_{s,\pi} F : |\hat{p}_i(z) - \hat{p}_i(\theta_m(T(x_0)))| < \epsilon_i, i = 1, \dots, n\}$$

of $T^{(m)}(\theta_m(x_0)) = \theta_m(T(x_0))$, where $\hat{p_i}$ denotes the linear operator associated with the polynomial p_i. Observe that $y \in V$ if and only if $\theta_m(y) \in \tilde{V}$. Since $T^{(m)}$ is continuous, the set

$$\{w \in \hat{\otimes}^m_{s,\pi} E : T^{(m)}(w) \in \tilde{V}\}$$

is a weak open subset of $\hat{\otimes}^m_{s,\pi} E$, which contains $\theta_m(x_0)$. Thus, there exists a basic weak neighborhood of $\theta_m(x_0)$,

$$\tilde{U} = \{w \in \hat{\otimes}^m_{s,\pi} E : |\hat{q}_j(w) - \hat{q}_j(\theta_m(x_0))| < \delta_j, \hat{q}_j \in (\hat{\otimes}^m_{s,\pi} E)^*, j = 1, \dots, k\},$$

such that $T^{(m)}(\tilde{U}) \subset \tilde{V}$. To complete the proof, it is enough to observe that the subset

$$U = \{x \in E; |q_j(x) - q_j(x_0)| < \delta_j, j = 1, \dots, k\},$$

where q_j is the polynomial determined by $\hat{q}_j$, is a τ_m neighborhood of x_0 satisfying $T(U) \subset V$. □

Corollary 2.3. *Let E and F be Banach spaces and let $m \in \mathbb{N}$. Then, every m-homogeneous polynomial $P \in \mathcal{P}(^m E, F)$ transforms τ_m-Cauchy sequences in E into weakly Cauchy sequences in F.*

By virtue of this corollary, we define the extension of Dunford-Pettis and Dieudonné linear operators to the m-homogeneous case as follows:

Definition 2.4. Let E and F be Banach spaces and let $m \in \mathbb{N}$. An m-homogeneous polynomial $P \in \mathcal{P}(^mE, F)$ is a *Dunford-Pettis polynomial* (resp. a *Dieudonné polynomial*) if it transforms τ_m-Cauchy sequences in E into norm convergent (resp. weakly convergent) sequences in F. We denote this class by $\mathcal{P}_{DP}(^mE, F)$ (resp. $\mathcal{P}_D(^mE, F)$).

Remark 2.5. Equivalently, one may define Dunford-Pettis (resp. Dieudonné) polynomials as follows: Consider a polynomial $P \in \mathcal{P}(^mE, F)$; it is Dunford-Pettis (resp. Dieudonné) if for every subset $A \subset E$ for which every sequence $(x_n)_n \subset A$ contains a τ_m-Cauchy subsequence, $P(A)$ is a relatively compact (resp. weakly compact) set in F.

Remark 2.6. Every polynomial $P \in \mathcal{P}(^mE, F)$ for which the associated linear operator $\hat{P} \in \mathcal{L}(\hat{\otimes}^m_{s,\pi} E, F)$ is Dunford-Pettis, is a Dunford-Pettis polynomial. On the other hand, there are examples showing that in general the two classes are different: every polynomial in $\mathcal{P}(^m\ell_m, F)$ is Dunford-Pettis (see Section 3.9 below), but clearly not every linear operator $T \in \mathcal{L}(\hat{\otimes}^m_{s,\pi}\ell_m, F)$ is Dunford-Pettis. An analogous remark holds for Dieudonné polynomials; in this case we take the James' space constructed from the ℓ_p norm, J_p, as an example of a space for which every m-homogeneous polynomial is Dieudonné (for $m \geq p$), but not every operator in $\mathcal{L}(\hat{\otimes}^m_{s,\pi} J_p, F)$ is Dieudonné (see the Appendix).

Dunford-Pettis polynomials also have a characterization in terms of the adjoint operator. We shall say that a subset $A \subset \mathcal{P}(^mE)$ is in the class $\mathcal{L}_m(\mathcal{P}(^mE))$ if every τ_m-Cauchy sequence (x_n) in E satisfies

$$\lim_{k,n\to\infty} \sup\{|q(x_k) - q(x_n)|; q \in A\} = 0. \tag{1}$$

Lemma 2.7. *A polynomial $P \in \mathcal{P}(^mE, F)$ is a Dunford-Pettis polynomial if and only if its adjoint operator $P^* \in \mathcal{L}(F^*, \mathcal{P}(^mE))$, defined by $P^*(f^*) = f^* \circ P$, has the property that the subset $P^*(B_{F^*})$ is in the class $\mathcal{L}_m(\mathcal{P}(^mE))$.*

Proof. Consider a τ_m-Cauchy sequence $(x_n)_n \subset E$. The result follows easily from the chain of equations

$$\begin{aligned}\lim_{k,n\to\infty} \|P(x_k) - P(x_n)\| &= \lim_{k,n\to\infty} \sup_{f^*\in B_{F^*}} \{|f^*(P(x_k)) - f^*(P(x_n))|\} \\ &= \lim_{k,n\to\infty} \sup_{q\in P^*(B_{F^*})} \{|q(x_k) - q(x_n)|\}\end{aligned}$$

□

We now consider the relation between the class of Dunford-Pettis polynomials introduced above and the class of polynomials called *completely continuous* in [6] and [13] (or *weakly sequentially continuous* in [2] and [16]). For E, F and m as before, we denote this class by $\mathcal{P}_{cc}(^mE,F)$; we will refer to it as the class of *completely continuous polynomials*. Recall that the elements in this class are those polynomials that transform weak Cauchy sequences of E into norm convergent sequences of F (or, equivalently, weakly convergent sequences into norm convergent sequences; see [2, Theorem 2.3]) .

THEOREM 2.8. *Let E be a Banach space and let $m \in \mathbb{N}$. For every Banach space F, the relation $\mathcal{P}_{cc}(^mE,F) \subset \mathcal{P}_{DP}(^mE,F)$ holds. Moreover, both classes coincide for every Banach space F if and only if every polynomial $p \in \mathcal{P}(^mE)$ is completely continuous.*

Proof. Let $(x_n)_n$ be a τ_m-Cauchy sequence in E. By [5, Proposition 3.6], there exists a weak Cauchy subsequence $(x_{n_j})_j$. By its very definition, for every polynomial $P \in \mathcal{P}_{cc}(^mE,F)$ the sequence $(P(x_{n_j}))_j$ is norm convergent. In view of Remark 2.5, this shows that P is Dunford-Pettis. If both classes coincide for all Banach spaces F, then, in particular, $\mathcal{P}_{cc}(^mE) = \mathcal{P}_{DP}(^mE)$. Since we always have $\mathcal{P}(^mE) = \mathcal{P}_{DP}(^mE)$ (see 2.12 below), it follows that every scalar polynomial $p \in \mathcal{P}(^mE)$ is completely continuous. Finally, suppose that every m-homogeneous scalar polynomial is completely continuous. In [6] it was shown that this is equivalent to saying that every weak Cauchy sequence in E is a τ_m-Cauchy sequence. Hence every Dunford-Pettis polynomial is completely continuous. □

REMARK 2.9. In [13] the authors defined the class of *weakly completely continuous* polynomials ($\mathcal{P}_{wcc}$), i.e., those polynomials that take the weak Cauchy sequences into weakly convergent sequences, as a generalization of the class of Dieudonné operators. It is possible to prove that the relation $\mathcal{P}_{wcc}(^mE,F) \subset \mathcal{P}_D(^mE,F)$ always holds, and that both classes coincide for all F if and only if every m-homogeneous scalar polynomial is (weakly) completely continuous.

Recall that a polynomial $P \in \mathcal{P}(^mE,F)$ is compact (resp. weakly compact) if it transforms every bounded subset of E into a relatively compact (resp. relatively weakly compact) subset of F. We denote the class of such polynomials by $\mathcal{P}_{co}$ (resp. $\mathcal{P}_{wc}$). We say that a polynomial P is unconditionally converging if it transforms the sequences of partial sums of weakly unconditionally Cauchy series in E into norm convergent series (see [11]), and we denote the class of such polynomials by $\mathcal{P}_{uc}$. We now establish relationships among these classes of polynomials that hold for all Banach spaces. Observe that these are analogous to the relations that hold in the linear case.

Theorem 2.10. *Let E and F be Banach spaces and let $m \in \mathbb{N}$ be fixed. Then the following inclusion relations hold:*

$$\mathcal{P}_{co}(^mE,F) \begin{array}{c} \subset \quad \mathcal{P}_{wc}(^mE,F) \quad \subset \\ \subset \quad \mathcal{P}_{\mathcal{DP}}(^mE,F) \quad \subset \end{array} \mathcal{P}_{\mathcal{D}}(^mE,F) \subset \mathcal{P}_{uc}(^mE,F) \subset \mathcal{P}(^mE,F)$$

Proof. The only relation that it is not immediate from the definitions is the one that asserts that every Dieudonné polynomial is unconditionally converging. Fix a Dieudonné m-homogeneous polynomial and consider a weakly unconditionally Cauchy series in E. The corresponding sequence of partial sums is a τ_m-Cauchy sequence (see [11]), so its image under the Dieudonné polynomial is a weakly convergent sequence in F. This is enough for a polynomial to be unconditionally converging (see [11] again). □

Proposition 2.11. *For fixed E, F and m, each class in Theorem 2.10 defines a closed subspace of $\mathcal{P}(^mE,F)$, and each class is preserved by composition (on both sides) with bounded linear operators.*

Proof. The proof of this result is clear. It suffices to remark that linear operators transform τ_m-Cauchy sequences into τ_m-Cauchy sequences (see Theorem 2.2). □

2.12. Remarks and examples. (1) The classes in Theorem 2.10 are different, in the sense that for each inclusion it is possible to find E and F for which the inclusion is strict. Even more, the example can be taken as the canonical polynomial defined on a suitable space. Consider $m \in \mathbb{N}$ and $\theta_m \in \mathcal{P}(^mE, \hat{\otimes}^m_{s,\pi}E)$. If $E = \ell_p$, $p > m$, then the polynomial θ_m is weakly compact but not compact. This polynomial is a Dieudonné polynomial, but not a Dunford-Pettis polynomial. If $E = \ell_p$, $p \leq m$, then θ_m is a Dunford-Pettis polynomial (see 3.9 below), but clearly is not compact. The case $E = \ell_1$ provides an example of a Dieudonné non-weakly compact polynomial (see 3.17 below). Just as the identity map on the James' space J is unconditionally converging but not Dieudonné, the canonical polynomial defined on J_p, $p > m$, is unconditionally converging (because the space contains no copies of c_0; see [11]) but it is not Dieudonné. The summing basis is a τ_m-Cauchy sequence whose image does not converge in the weak topology of $\hat{\otimes}^m_{s,\pi}J_p$ (see the Appendix). Finally, the canonical polynomial defined on $E = c_0$ is not unconditionally converging.

(2) If F is finite dimensional, then for every Banach space E, $\mathcal{P}_{co}(^mE,F) = \mathcal{P}(^mE,F)$. In particular, all of the classes in Theorem 2.10 coincide. Observe that this is not the case for the *completely continuous polynomials* (see Theorem 2.8 above) as the example $p \in \mathcal{P}(^m\ell_m)$, $p((a_i)) = \sum_i^\infty a_i^m$ shows. The polynomial p is compact (its range is contained in the finite dimensional space $\mathbb{K}$), but it is not completely continuous.

(3) If F is a Schur space (i.e., the weak Cauchy and the norm convergent sequences coincide on F), then for every Banach space E one has $\mathcal{P}_{DP}(^mE, F) = \mathcal{P}(^mE, F)$.

(4) The space E has the property that for every Banach space F, $\mathcal{P}_{co}(^mE, F) = \mathcal{P}(^mE, F)$ holds if and only if E is finite dimensional.

3. Polynomial properties

In this section, we prove that some characterizations of linear properties involving Dunford-Pettis or Dieudonné operators remain valid in the more general case of m-homogeneous polynomials, regardless of the value of $m \in \mathbb{N}$. Proofs of the corresponding results in the linear case (which will be used freely) can be found in [4]. The first result deals with the so-called *Dunford-Pettis property*:

Definition 3.1. A Banach space E has the *m-Dunford-Pettis* (or m-DP) property if, for every Banach space F, every weakly compact polynomial in $\mathcal{P}(^mE, F)$ is Dunford-Pettis.

Several different forms of the polynomial Dunford-Pettis property, depending on the definition of the class of *Dunford-Pettis polynomials* one works with, have been studied in the literature (see, for instance, [3], [7], [10], [17] [20], [21]). In [17] there is a unifying approach depending on different types of sequential convergence. The m-DP property involves Cauchy sequences instead of convergent sequences in a non-linear topology. Nevertheless, the proof of the following result is similar to those in [7] or [17], so we will only sketch the argument.

Recall that a subset $A \subset E$ is a Dunford-Pettis set if for every weakly null sequence $(x_n^*)_n \subset E^*$, $\lim_{n\to\infty} \sup\{|x_n^*(a)| : a \in A\} = 0$. Also recall that a subset of a Dunford-Pettis set is Dunford-Pettis, and that every Dunford-Pettis set on a Banach space E contains a τ_m-Cauchy sequence. (This is a direct consequence of Proposition 2.2 and Theorem 3.1 of [7].)

Theorem 3.2. *Let E be a Banach space and let $k, m \in \mathbb{N}$. The following conditions are equivalent:*

(a) *E has the m-Dunford-Pettis property.*
(b) *For every Banach space F, every weakly compact polynomial $P \in \mathcal{P}(^kE, F)$ transforms τ_m-Cauchy sequences of E into relatively compact sets of F.*
(c) *Every weakly compact polynomial $P \in \mathcal{P}(^kE, c_0)$ transforms τ_m-Cauchy sequences of E into relatively compact sets of c_0.*
(d) *Every τ_m-Cauchy sequence in E defines a Dunford-Pettis set on E.*

Proof. We will use the following characterization of Dunford-Pettis sets, which is a consequence of [5, Proposition 3.1] and [7, Theorem 3.1]:

$A \subset E$ is a Dunford-Pettis set if and only if, for some (all) $k \in \mathbb{N}$, and every weakly compact polynomial $P \in \mathcal{P}({}^kE, F)$ (resp. $F = c_0$), $P(A)$ is a relatively compact set in F (resp. c_0.)

To prove the implication (d) $\Rightarrow$ (b), let (x_n) be a τ_m-Cauchy sequence in E. By hypothesis, (x_n) defines a Dunford-Pettis set, so every weakly compact polynomial $P \in \mathcal{P}({}^kE, F)$ takes (x_n) into a relatively compact set. Conversely, assume that (b) holds and consider a τ_m-Cauchy sequence in E. For every F and every weakly compact polynomial $P \in \mathcal{P}({}^mE, F)$, $(P(x_n))_n$ is a relatively compact set in F. The above characterization of Dunford-Pettis sets then ensures that (x_n) is a Dunford-Pettis set. Hence (b) and (d) are equivalent. The same argument shows the equivalence of (d) with (a) or (c). □

Remark 3.3. In the case $k = m$, by Corollary 2.3, conditions (b) and (c) say that a weakly compact polynomial transforms a τ_m-Cauchy sequence into a norm convergent sequence. If $k \neq m$ it might be necessary to take subsequences, as the following example shows. Consider the identity operator on ℓ_2 (a space with the 2-DP property, see 3.9 below) and fix $x \neq 0$. Then the sequence $((-1)^n x)_n$ is τ_2-convergent to x (and hence τ_2-Cauchy), but it is not norm convergent.

Remark 3.4. A finite sum of spaces with the m-DP property also has the m-DP property. This follows from the characterization of the property in terms of linear operators (the second and third conditions of Theorem 3.2 for $k = 1$).

In the following theorem we characterize in several ways those Banach spaces on which all m-homogeneous polynomials are Dunford-Pettis.

Theorem 3.5. *For a Banach space E and $k, m \in \mathbb{N}$, the following conditions are equivalent:*

(a) *For every Banach space F, every polynomial $P \in \mathcal{P}({}^mE, F)$ is a Dunford-Pettis polynomial.*
(b) *Every τ_m-Cauchy sequence in E contains a norm convergent subsequence.*
(c) *For every Banach space F, every polynomial $P \in \mathcal{P}({}^kE, F)$ transforms the τ_m-Cauchy sequences of E into relatively compact sets in F.*
(d) *If A is a bounded subset of $\mathcal{P}({}^mE)$, then A is in the class $\mathcal{L}_m(\mathcal{P}({}^mE))$.*

Proof. We first prove that (c) implies (b). Consider the canonical polynomial θ_k from E to $\hat{\otimes}^k_{s,\pi} E$ and a τ_m-Cauchy sequence (x_n) in E. We are going to show that (x_n) has a norm convergent subsequence. By assumption (c), $(\theta_k(x_n))_n$ has a norm convergent subsequence. From [7, Theorem 3.1] it follows that $(x_n)_n$ has a norm convergent subsequence. The implication (a)

$\Rightarrow$ (b) can be proved in the same manner, with θ_m in place of θ_k. That (b) implies both (a) and (c) follows from the norm continuity of $P \in \mathcal{P}(^kE, F)$.

Assume now that E satisfies condition (d) and consider an element $P \in \mathcal{P}(^mE, F)$. By hypothesis, the subset $P^*(B_{F^*})$ is in the class $\mathcal{L}_m(\mathcal{P}(^mE))$, so P is a Dunford-Pettis polynomial (by Lemma 2.7), and we obtain (a). To prove the converse implication, (a) $\Rightarrow$ (d), we construct, for each bounded subset of $\mathcal{P}(^mE)$, an m-homogeneous polynomial as follows. For a bounded subset $A \subset \mathcal{P}(^mE)$, consider the Banach space $B(A)$ of bounded continuous functions defined on A, with the sup norm, and the polynomial

$$\begin{array}{rcl} P : E & \longrightarrow & B(A) \\ x & \longmapsto & P(x)(q) := q(x). \end{array}$$

Then P is in $\mathcal{P}(^mE, B(A))$. Condition (a) ensures that P is Dunford-Pettis, so $P^*(B_{B(A)^*})$ is a set in the class $\mathcal{L}_m(\mathcal{P}(^mE))$. For every $q \in A$, define $x_q^* \in (B(A))^*$ by $x_q^*(f) = f(q)$, for all $f \in B(A)$. Then $P^*(x_q^*) = q$, so A is contained in the image of a bounded subset of $(B(A))^*$, and thus is in the class $\mathcal{L}_m(\mathcal{P}(^mE))$. $\square$

Definition 3.6. A Banach space E which satisfies any of the conditions in Theorem 3.5 is called an *m-Schur space*.

From Theorem 2.10 and part (a) of Theorem 3.5, it is clear that every m-Schur space has the m-Dunford-Pettis property. However, the converse is not true. For each $m \in \mathbb{N}$, the space $c_0 \oplus \ell_m$ is an example of a non-Dunford-Pettis space, which has the m-Dunford-Pettis property but is not m-Schur because the canonical basis of c_0 on the space is polynomially null (and thus, in particular, τ_m-Cauchy) but is not norm null. In fact, the condition on a space under which both properties are equivalent is the same as in the linear case; it is not a '*polynomial type*' property:

Proposition 3.7. *Let $m \in \mathbb{N}$ and let E be a Banach space with the m-Dunford-Pettis property. The following conditions are equivalent:*

(1) *E is an m-Schur space.*
(2) *Every Dunford-Pettis set in E is relatively compact.*

Proof. We mentioned above that every Dunford-Pettis set contains a τ_m-Cauchy sequence. Hence, if the Banach space is m-Schur, every Dunford-Pettis set contains a norm convergent sequence. This proves the implication (1) $\Rightarrow$ (2). Conversely, let $(x_n)_n$ be a τ_m-Cauchy sequence in E. By part (d) of Theorem 3.2, (x_n) defines a Dunford-Pettis set which, under assumption (2), is a relatively compact set, i.e., E satisfies condition (b) in Theorem 3.5. $\square$

In [14] the authors studied conditions under which, for a Banach space E, every polynomial $P \in \mathcal{P}(^mE, c_0)$ is completely continuous for some $m \in \mathbb{N}$. They proved that this is the case if and only if every linear bounded operator

$T \in \mathcal{L}(E, c_0)$ is completely continuous. In particular, this property does not depend on m. We will see that the situation is quite different if the above condition on the Banach space is replaced by requiring that every polynomial $P \in \mathcal{P}(^m E, c_0)$ is Dunford-Pettis. Indeed, we shall show that $E = \ell_m$ is an example of a space which satisfies this property for polynomials of degree m, but not for polynomials of degree $k < m$. Recall that a set $A \subset E$ is said to be limited if for every weak*-null sequence $(x_n^*)_n$ in E^*, $\lim_{n\to\infty} \sup\{|x_n^*(a)| : a \in A\} = 0$ (see, e.g., [8]). We then have the following result:

PROPOSITION 3.8. *Let E be a Banach space and let $k, m \in \mathbb{N}$. The following conditions are equivalent:*

(a) *Every polynomial $P \in \mathcal{P}(^m E, c_0)$ is a Dunford-Pettis polynomial.*
(b) *Every polynomial $P \in \mathcal{P}(^k E, c_0)$ transforms τ_m-Cauchy sequences in E into relatively compact sets in c_0.*
(c) *Every τ_m-Cauchy sequence in E defines a limited set on E.*

The proof is similar to that of Theorem 3.2, once one observes that, as a consequence of [5, Proposition 3.1] and [7, Theorem 3.1], the limited sets may be characterized as follows:

$A \subset E$ is limited if and only if for some (all) $k \in \mathbb{N}$ and every polynomial $P \in \mathcal{P}(^k E, c_0)$, $P(A)$ is a relatively compact set in c_0.

It is clear that an m-Schur space satisfies $\mathcal{P}(^m E, c_o) = \mathcal{P}_{DP}(^m E, c_0)$; in particular, this holds for the space $E = \ell_m$. However, $P((a_i)_i) = (a_i^k)_i$ defines an element in $\mathcal{P}(^k \ell_m, c_o)$ which is not Dunford-Pettis if $k < m$. Hence the property characterized in Proposition 3.8 certainly depends on m.

The spaces E for which $\mathcal{P}(^m E, c_o) = \mathcal{P}_{DP}(^m E, c_0)$ include the Grothendieck spaces with the m-DP property, such as the space $\ell_m \oplus \ell_\infty$.

REMARKS AND EXAMPLES 3.9. (1) For $1 < p < \infty$, the space ℓ_p has the m-Schur (or m-DP) property if and only if $m \geq p$ (see [9, Theorem 6.3]). Since in the non-reflexive James' space J_p every normalized weakly null sequence contains a subsequence that generates a complemented subspace isomorphic to ℓ_p, this space inherits the above properties from ℓ_p; namely, it is m-Schur (and has the m-DP property) if and only if $p \leq m$ (see the Appendix).

As mentioned above, there exist several different polynomial extensions of the Dunford-Pettis and Schur properties in the literature. Nevertheless, it is sometimes possible to prove, mutatis mutandis, the same kind of results for these extensions. This is the case in the following examples.

(2) A Banach space with the Dunford-Pettis property has the m-DP property for all $m \in \mathbb{N}$. To prove this result, we use the characterization (d) in Theorem 3.2. Consider a τ_m-Cauchy sequence (x_n) in E. By [5, Proposition 3.6], this sequence has a weak Cauchy subsequence. Now, the Dunford-Pettis

property on E asserts that this subsequence defines a Dunford-Pettis set. Arguing similarly with each subsequence of (x_n), one concludes that the sequence itself is a Dunford-Pettis set.

If a Banach space E has the Dunford-Pettis property, every scalar polynomial is weakly sequentially continuous (see [20, Corollary 3]). By Proposition 2.8, the completely continuous and Dunford-Pettis classes of polynomials coincide for all $m \in \mathbb{N}$. Thus, in this case, the result coincides with that in [21, Corollary 2.2].

(3) As a consequence of Theorem 3.2 we saw that the m-DP property is preserved by finite sums. From condition (c) of Theorem 3.5 it follows that the m-Schur property is also preserved by finite sums. In fact, using an argument similar to that of [3, Proposition 2.5], one can show that the m-DP and m-Schur properties are preserved by ℓ_p sums, for $1 \leq p \leq m$, and that the m-DP property is also preserved by c_0-sums.

(4) If K is a compact set and E is an m-Schur space, the space $\mathcal{C}(K, E)$ has the m-DP property. The proof is analogous to that of [3, Proposition 2.6].

The *reciprocal Dunford-Pettis property* and the *Dieudonné property* were introduced by A. Grothendieck in [15]. We now introduce an extension to the m-homogeneous case, following the general scheme of Theorem 2.10.

DEFINITION 3.10. Let E be a Banach space and let $m \in \mathbb{N}$. We say that E has the *m-reciprocal Dunford-Pettis property* (resp. *m-Dieudonné property*) if for every Banach space F, every Dunford-Pettis polynomial (resp. every Dieudonné polynomial) $P \in \mathcal{P}(^m E, F)$ is weakly compact. We shall denote these properties by m-RDP and m-D, respectively.

From Theorem 2.10, it follows that a Banach space with the m-D property also has the m-RDP property.

PROPOSITION 3.11. *Let E be a Banach space such that $\hat{\otimes}^m_{s,\pi} E$ contains no copies of ℓ_1. Then E has the m-D and m-RDP properties.*

Proof. By the remark above, it is enough to prove that E has the m-D property. Consider a Dieudonné polynomial $P \in \mathcal{P}(^m E, F)$ and a bounded subset $A \subset E$. By Rosenthal's ℓ_1-Theorem, if $\hat{\otimes}^m_{s,\pi} E$ contains no copies of ℓ_1, each sequence $(x_n) \subset A$ is such that $(\theta_m(x_n)) \subset \hat{\otimes}^m_{s,\pi} E$ contains a weak Cauchy subsequence. This means that (x_n) contains a τ_m-Cauchy subsequence (x_{n_k}) (see Remark 2.1). Since P is Dieudonné, $(P(x_{n_k}))$ is a weakly convergent sequence in F. Thus P is a weakly compact polynomial. $\square$

PROPOSITION 3.12. *The m-reciprocal Dunford-Pettis property (resp. the m-Dieudonné property) on a Banach space E is preserved in every quotient space of E.*

Proof. Let $\pi : E \to F$ be a quotient map and $P \in \mathcal{P}(^m F, G)$ be a Dunford-Pettis polynomial. As we saw in Proposition 2.11, $P \circ \pi \in \mathcal{P}(^m E, G)$ defines a Dunford-Pettis polynomial and thus a weakly compact polynomial. The set $P(\pi(B_E))$ is a relatively weakly compact set in G. From the surjectivity of π, it follows that $P(B_F)$ is also a relatively weakly compact set of G, so P is a weakly compact polynomial. An analogous proof can be given in the m-Dieudonné case. $\square$

Proposition 3.13. *For a Banach space E and $m \in \mathbb{N}$, the following conditions are equivalent:*

(1) *E has the m-reciprocal Dunford-Pettis property.*
(2) *Every subset A in the class $\mathcal{L}_m(\mathcal{P}(^m E))$ is relatively weakly compact.*

Proof. Let $P \in \mathcal{P}_{DP}(^m E, F)$. By Lemma 2.7, the subset $P^*(B_{F^*})$ is in the class $\mathcal{L}_m(\mathcal{P}(^m E))$. Thus, if one assumes (2), this set is a relatively weakly compact set. This proves that P is a weakly compact polynomial. For the proof of the converse implication we consider, for each subset A in the class $\mathcal{L}_m(\mathcal{P}(^m E))$, the polynomial $P \in \mathcal{P}(^m E, B(A))$ constructed in the proof of Theorem 3.5. The polynomial P is a Dunford-Pettis polynomial. To see this, note that for a τ_m-Cauchy sequence (x_n) in E one has

$$\begin{aligned} \|P(x_n) - P(x_k)\| &= \sup\{|(P(x_n) - P(x_k))(q)|;\, q \in A\} \\ &= \sup\{|q(x_n) - q(x_k)|;\, q \in A\}. \end{aligned}$$

Since A belongs to the class $\mathcal{L}_m(\mathcal{P}(^m E))$, this set satisfies equation (1) (preceding Lemma 2.7). This means that $(P(x_n))$ is a norm convergent sequence. Hence, P is Dunford-Pettis. The hypothesis on the space asserts that P is also a weakly compact polynomial, so its adjoint operator P^* transforms the bounded subsets of F^* into relatively weakly compact subsets of $\mathcal{P}(^m E)$. To complete the proof, it suffices to note that the subset A is the image under P^* of a bounded set in F^*. $\square$

Proposition 3.14. *Let $m > 1$ and let E be a Banach space with the m-reciprocal Dunford-Pettis property (resp. the m-Dieudonné property) for $m > 1$. Then we have:*

(1) *E does not contain a subspace isomorphic to ℓ_1.*
(2) *For $1 \leq k \leq m$ the Banach space $\hat{\otimes}^k_{s,\pi} E$ has the reciprocal Dunford-Pettis property (resp. the Dieudonné property).*

Proof. To prove (1), note that $\hat{\otimes}^m_{s,\pi} E$ contains a complemented copy of ℓ_1 whenever E contains a copy of ℓ_1 (see [7, Corollary 3.11]). If this is the case, the projection operator determines, by its values on the diagonal set, a non-compact polynomial $P \in \mathcal{P}(^m E, \ell_1)$. Recall that, since ℓ_1 is a Schur space, every weakly compact polynomial with values in ℓ_1 is compact and every

m-homogeneous polynomial is Dunford-Pettis (see 2.12 above). Then P is a Dunford-Pettis polynomial which is not weakly compact. Thus, the space cannot have the m-RDP property, and hence cannot have the m-D property either.

To prove (2) it is enough to establish the result for $k = m$, because $\hat{\otimes}^k_{s,\pi} E$ is a complemented subspace of $\hat{\otimes}^m_{s,\pi} E$. Let $T \in \mathcal{L}(\hat{\otimes}^m_{s,\pi} E, F)$ be a Dunford-Pettis operator. By Remark 2.6, the associated polynomial is Dunford-Pettis. The m-RDP property implies that it is weakly compact, which means that T is a weakly compact operator. The m-D case is analogous. □

The following theorem generalizes an important and non-trivial result (see [4]), which corresponds to the case $m = 1$ of our result. The theorem characterizes the containment of ℓ_1 in a space in terms of the Dunford-Pettis operators defined on it. The presence of ℓ_1 in the symmetric tensor product of a space can also be obtained from its Dunford-Pettis polynomials. Except for the equivalence with the fifth condition, the following result has been proved in [7, Theorem 3.8]. The equivalence (3)⇔(5) can be proved in the same way as Proposition 3.13, by changing *'weakly compact polynomial'* to *'compact polynomial'* and *'relatively weakly compact set'* to *'relatively compact set'*.

THEOREM 3.15. *Let E be a Banach space and let $m \in \mathbb{N}$. The following conditions are equivalent:*

(1) *The space $\hat{\otimes}^m_{s,\pi} E$ contains no copy of ℓ_1.*
(2) *Every bounded sequence in E contains a τ_m-Cauchy subsequence.*
(3) *For every Banach space F, every Dunford-Pettis polynomial in $P \in \mathcal{P}(^m E, F)$ is compact.*
(4) *Every Dunford-Pettis polynomial $P \in \mathcal{P}(^m E, c_0)$ is compact.*
(5) *Every set in the class $\mathcal{L}_m(\mathcal{P}(^m E))$ is a relatively compact set of $\mathcal{P}(^m E)$.*

It follows from this result that in the case when $\hat{\otimes}^m_{s,\pi} E$ does not contain a copy of ℓ_1, a polynomial $P \in \mathcal{P}(^m E, F)$ is Dunford-Pettis if and only if its associated linear operator $\hat{P} \in \mathcal{L}(\hat{\otimes}^m_{s,\pi} E, F)$ is Dunford-Pettis, since both classes coincide with the corresponding compact subclasses. (Recall that in Remark 2.6 we proved that this is not generally true).

We next observe that whenever an infinite dimensional Banach space is m-Schur, the space $\hat{\otimes}^m_{s,\pi} E$ contains a copy of ℓ_1. Otherwise, by the previous theorem, every Dunford-Pettis polynomial $P \in \mathcal{P}_{DP}(^m E, F)$ would be compact. Since, by the m-Schur property, every m-homogeneous polynomial is Dunford-Pettis (Theorem 3.5), it follows that every m-homogeneous polynomial is compact. Thus, E must be finite dimensional (see 2.12 above).

COROLLARY 3.16. *A Banach space E with the m-Schur property and the m-RDP property is a finite dimensional space.*

Proof. Let E be an infinite dimensional Banach space. If E has the m-Schur property then the space $\hat{\otimes}^m_{s,\pi} E$ contains a copy of ℓ_1 and every m-homogeneous polynomial is Dunford-Pettis. If E also has the m-RDP property, then every m-homogeneous polynomial is weakly compact, which means that $\hat{\otimes}^m_{s,\pi} E$ is a reflexive space (see [22]). But this contradicts the fact that $\hat{\otimes}^m_{s,\pi} E$ contains a copy of ℓ_1. □

3.17. FINAL REMARKS. (1) We have seen in several examples, that many linear properties can be viewed as the case $m = 1$ of more general properties for m-homogeneous polynomials. Indeed, several of our proofs are suitable adaptations of the classical proofs in the linear case. Reasoning in the same way, it is not difficult to prove the following results.

(a) If E is a Banach space with the m-RDP property and F is a Schur space, then $\mathcal{P}(^m E, F) = \mathcal{P}_{co}(^m E, F)$.
(b) The dual E^* of a Banach space E is a Schur space if and only if for some (all) $m \in \mathbb{N}$, the space $\hat{\otimes}^m_{s,\pi} E$ contains no copies of ℓ_1 and E has the m-DP property (see [7]).
(c) Consider a Banach space E with the Approximation Property (see [18]). Every m-homogeneous polynomial defined on this space is Dieudonné if and only if τ_m defines a complete topology on E. In particular, for every space F one has $\mathcal{P}(^m \ell_1, F) = \mathcal{P}_D(^m \ell_1, F)$.
(d) If every unconditionally converging m-homogeneous polynomial defined on a Banach space E is weakly compact, then E has the m-D and m-RDP properties. In particular, the space c_0 has the m-D and m-RDP properties for all $m \in \mathbb{N}$ (see [11]).

(2) The properties m-DP, m-Schur, m-RDP and m-D studied here, are preserved in complemented subspaces. This result follows from the fact that those classes of polynomials which determine such properties are stable under composition with bounded operators (see Proposition 2.11).

(3) Since the τ_m topology on a space is not linear if $m > 1$, it is not easy to determine whether a Dunford-Pettis polynomial (resp. Dieudonné) can be defined as a polynomial which transforms the τ_m-convergent sequences into norm (resp. weak) convergent sequences. This is true, for example, for the m-homogeneous polynomials in ℓ_p for $p > m$, where the τ_m-Cauchy and the τ_m-convergent sequences coincide.

A Λ-space was defined in [9] as a Banach space in which the sequences that are τ_m-null for all $m \in \mathbb{N}$, are norm null. Accordingly, we define a Λ-Schur space as a Banach space in which the sequences that are τ_m-Cauchy for all $m \in \mathbb{N}$, are norm convergent. A Λ-Schur space is always a Λ-space. The following property (P), introduced in [1], makes these two notions equivalent:

A Banach space E is said to have property (P) if, whenever (u_j) and (v_j) are two bounded sequences in E such that for every $n \geq 1$ and every $p \in \mathcal{P}(^n E)$,

$|p(u_j) - p(v_j)| \to 0$, *it follows that* $|q(u_j - v_j)| \to 0$ *for every* $m \geq 1$ *and every* $q \in \mathcal{P}(^m E)$.

Let E be a Λ-space with Property (P). To show that E is also Λ-Schur, consider a sequence (x_n) which is τ_m-Cauchy for all $m \in \mathbb{N}$. Property (P) ensures that, for every pair of subsequences (x_{n_j}) and (x_{k_j}), the sequence $(x_{n_j} - x_{k_j})$ is τ_m-null, for all m. Thus, this sequence is a norm null sequence (because E is a Λ-space), and hence (x_n) is norm convergent.

Appendix. On the J_p space

We give here the definition and main properties of the space J_p that we have used above. For $1 < p < \infty$, J_p is the Banach space of real sequences $x = (a_i)_i$ such that $\lim_{i\to\infty} a_i = 0$ and

$$\|x\| = \sup \Big(\frac{1}{2} \sum_{i=0}^{n} |a_{p_{i+1}} - a_{p_i}|^p \Big)^{1/p} < \infty,$$

where $a_0 = 0$ and the sup is taken over all choices of n and all positive integers $0 = p_0 < p_1 < \ldots < p_n$.

The proofs of the following properties are, almost word for word, identical to the proofs for the corresponding properties on the James space J (i.e., the case $p = 2$ of J_p).

(1) The unit vector basis $(e_i)_i$ of J_p is shrinking and the summing basis $(\xi_n)_n$ (where $\xi_n = \sum_{i=1}^{n} e_i$) defines a weak Cauchy sequence which is not weakly convergent (see [19]; for $p = 2$, see [12, 2.a.2]).

(2) For $j = 1, 2, \ldots$ let $y_j = \sum_{n=p_j}^{q_j} \alpha_n e_n$ be a block basic sequence of $(e_i)_i$. If $p_{j+1} - q_j > 1$ then $[y_j]_j$ is complemented in J_p. If, in addition, $(y_j)_j$ is seminormalized, then it is equivalent to the unit vector basis of ℓ_p (for $p = 2$ see [12, 2.d.2]).

(3) Every seminormalized weakly null sequence on J_p contains a subsequence that generates a complemented subspace and which is equivalent to the unit basis of ℓ_p.

Property (3) follows from (2) and from the fact that, in a space with basis, every such sequence contains a subsequence equivalent to a block basic sequence (see [18, 1.a.12]). One can take the block basis to satisfy the condition $p_{j+1} - q_j > 1$ of (2), and such that if it generates a complemented subspace, so does the subsequence.

Proposition. *All polynomials* $p \in \mathcal{P}(^m J_p)$ *are weakly sequentially continuous if and only if* $m < p$. *If* $m \geq p$ *the space* J_p *has the* m*-Schur (and* m*-DP) properties.*

Proof. Consider first the case when $m < p$ and let $(x_n)_n$ be a sequence that converges weakly but not in norm to x. By property (3) above, there exists a subsequence such that $(x_{n_j} - x)_j$ is equivalent to the unit basis of ℓ_p and such that this sequence generates a complemented subspace. Then, for every $p \in \mathcal{P}({}^k J_p)$, $1 \leq k \leq m$, we have $\lim_j p(x_{n_j} - x) = 0$, and hence $\lim_j p(x_{n_j}) = p(x)$. Since this argument can be applied to each subsequence of $(x_n)_n$, the entire sequence converges to x in E_{τ_m}.

Now consider the case when $m \geq p$ and let (x_n) be an τ_m-Cauchy sequence. The sequence (x_n) has a weak Cauchy subsequence. If it does not converge in norm, there exist two subsequences such that $(x_{n_j} - x_{k_j})_j$ is equivalent to the unit basis of ℓ_p and generates a complemented subspace F. But this yields a contradiction to the fact that in F the sequence $(x_{n_j} - x_{k_j})_j$ must be norm null, since $F \simeq \ell_p$ is an m-Schur space. □

In Remark (1) of 2.12 we used the fact that the sequence $(\xi_n)_n$ is τ_m-Cauchy in J_p if $m < p$, but $(\theta_m(\xi_n))_n$ has no weakly convergent subsequences: The first of these two assertions follows from the proposition above, since $(\xi_n)_n$ is weak Cauchy. To prove the second, assume that $(\theta_m(\xi_n))_n$ has a weakly convergent subsequence $(\theta_m(\xi_{n_j}))_j$. Then the subsequence $(\xi_{n_j})_j$ has itself a weakly convergent sequence in J_p (see [5, Proposition 3.6]). However, this is false.

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