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A CHARACTERIZATION OF VARIATIONALLY MCSHANE INTEGRABLE BANACH-SPACE VALUED FUNCTIONS

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0. Introduction

In [5] D.H. Fremlin studies, in a σ-finite outer regular quasi-Radon measure space, a method of integration of vector-valued functions which is an essential generalization of the McShane process of integration [8]. The method involves infinite McShane partitions by disjoint families of measurable sets of finite measure. For this integral Henstock's lemma no longer holds. We therefore consider the variational McShane integral (see [7]), which in general integrates a family of vector valued functions that is a proper subset of the family of McShane integrable functions. For a Banach-space valued function defined on a closed interval endowed with the Lebesgue measure, the variational McShane integral has been investigated in [1] by W. Congxin and Y. Xiabo (who called the integral the strong McShane integral). Congxin and Xiabo showed that a Banach-space valued function is variationally McShane integrable if and only if it is Bochner integrable. Their proof is based on the Frechet differentiability of the Bochner integral, and it cannot be directly generalized to an arbitrary measure space. In this paper we prove the surprising result (Theorem 1 below) that, in case of an arbitrary (even finite) quasi-Radon measure space, the class of variationally McShane integrable functions can be significantly larger. Our proof is rather elementary, but it gives a complete characterization of variationally McShane integrable functions. In particular, it also yields a simple proof of the result of Congxin and Xiaobo [1] (Theorem 2). Using this characterization, we generalize a result of V. Skvortsov and V. Solodov [10] showing that the McShane integral and the variational McShane integral are equivalent, when considered on a closed interval equipped with the Lebesgue measure, if and only if the range space is of finite dimension (Theorem 3).

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1. The variational McShane integral

Throughout this paper, X is a *Banach space* and $(\Omega, \mathcal{T}, \Sigma, \mu)$ is a *σ-finite and outer regular quasi-Radon measure space* (see [4]); i.e., we have:

(i) (Ω, Σ, μ) is a σ-finite, complete and outer regular measure space;
(ii) $\mathcal{T}$ is a topology such that $\mathcal{T} \subset \Sigma$;
(iii) $\mu(E) = \sup\{\mu(F) : F \subseteq E\,, F \text{ is closed}\}$ for every $E \in \Sigma$;
(iv) μ is $\tau-$*additive*, i.e., if $\mathcal{G} \subseteq \mathcal{T}$ is non–empty and upwards directed by inclusion, then $\mu(\cup_{G\in\mathcal{G}} G) = \sup\{\mu(G) : G \in \mathcal{G}\}$.

If $f : \Omega \to X$ is Pettis integrable with respect to μ, then we denote by $\nu_f(E)$ the value of Pettis–$\int_E f\,d\mu$. As is well known (see [9]), the variation $|\nu_f|$ of ν_f is a σ-finite measure that is absolutely continuous with respect to μ (in the sense that if $\mu(E) = 0$ then also $|\nu_f|(E) = 0$). If $|\nu_f|$ is finite, then it is also continuous in the $\varepsilon - \delta$ sense.

Henceforth, all integrals that are not explicitly described as Bochner integrals are to be understood as Pettis integrals,

A *generalized partial McShane partition in* Ω is a sequence $\langle (E_i, \omega_i)\rangle$ such that $\langle E_i\rangle$ is a family of pairwise disjoint sets of positive finite measure and $\omega_i \in \Omega$ for each i. If $\mu(\Omega \setminus \cup_i E_i) = 0$ we call $\langle (E_i, \omega_i)\rangle$ a *generalized McShane partition of* Ω. A function $\Delta : \Omega \to \mathcal{T}$ such that $\omega \in \Delta(\omega)$ for each $\omega \in \Omega$ is called a *gauge*. Given a gauge Δ we say that a generalized (partial) McShane partition $\langle (E_i, \omega_i)\rangle$ in Ω is Δ-*fine* if $E_i \subset \Delta(\omega_i)$ for each i.

Definition 1. A function $f : \Omega \to X$ is said to be variationally McShane integrable (in short, VM-integrable) if it is Pettis integrable and for each $\varepsilon > 0$ there exists a gauge $\Delta : \Omega \to \mathcal{T}$ such that

$$\sum_i \left\| f(\omega_i)\mu(E_i) - \int_{E_i} f\,d\mu \right\| < \varepsilon \tag{1}$$

for each Δ-fine generalized McShane partition $\langle (E_i, \omega_i)\rangle$ of Ω. The number $\nu_f(\Omega)$ is called the value of the variational McShane integral.

Note that (1) can be also written as

$$\sum_i \mu(E_i) \left\| f(\omega_i) - \int_{E_i} f\,d\mu/\mu(E_i) \right\| < \varepsilon\,.$$

We begin by stating a few basic properties of the variational McShane integral. The proofs are easy, and we do not present them here.

Proposition 1. *Let $(\Omega, \mathcal{T}, \Sigma, \mu)$ be a σ-finite and outer regular quasi-Radon measure space, and let X be a Banach space.*

(a) *If $f : \Omega \to X$ is VM-integrable, then f is also McShane integrable (see* [5] *for the definition), and the two integrals coincide.*

(b) *If* $f : \Omega \to X$ *is VM-integrable and* $A \subseteq \Omega$ *is nonempty, then* $f|A$ *is VM-integrable with respect to* $\mu|A$.
(c) *If* $f, g : \Omega \to X$ *are VM-integrable, then* $f + g$ *is also VM-integrable and* $\int(f+g) = \int f + \int g$.
(d) *If* Y *is a Banach space and* $T : X \to Y$ *is a bounded linear operator, then for each VM-integrable function* $f : \Omega \to X$, *the function* $Tf : \Omega \to Y$ *is also VM-integrable with* $\int Tf = T \int f$.
(e) *If* $f = 0$ μ*-a.e., then* f *is VM-integrable with its integral equal to zero.*

The proof of the main result will be broken into several lemmas. We start with a result stating that Bochner integrable functions are variationally McShane integrable. The proof is similar to that of Fremlin for McShane integrability [5], so we do not present it here.

LEMMA 1. *If* $f : \Omega \to X$ *is Bochner integrable, then* f *is also variationally McShane integrable.*

DEFINITION 2. We say that a non-negative measure ν in Ω is moderated (see [6]) if there exists a sequence $\langle W_n \rangle$ of open sets of finite ν-measure covering Ω.

LEMMA 2. *If* $f : \Omega \to X$ *is strongly measurable, Pettis integrable, and if* $|\nu_f|$ *is moderated, then* f *is variationally McShane integrable.*

Proof. It is well known that a strongly measurable and Pettis integrable function f can be represented in the form $f = g + h$, where g is Bochner integrable, $h = \sum_{n=1}^{\infty} x_n \chi_{A_n}$, and the equality is understood to hold μ-a.e. Moreover, the sets A_n can be taken to be pairwise disjoint and of finite measure. Since, by Lemma 1, g is variationally McShane integrable and its variation is moderated (because it is finite), we may assume for simplicity that $f = h = \sum_{n=1}^{\infty} x_n \chi_{A_n}$. Moreover, in view of Proposition 1, we may assume that f has countably many values and that we have $f = \sum_{n=1}^{\infty} x_n \chi_{A_n}$ everywhere, with $\Omega = \bigcup_n A_n$. In particular, this means that the x_n need not be all distinct. In fact, if f equals zero on a set of infinite measure, then zero has to occur infinitely often among the values x_n. Since $|\nu_f|$ is moderated, we may assume that each of the sets $A_n, n \in \mathbf{N}$, can be embedded into an open set W_n of finite $|\nu_f|$-measure. (Otherwise we take a sequence of open sets making $|\nu_f|$ moderated and then take all possible intersections of these sets with the sets $A_n, n \in \mathbf{N}$. We denote these new sets also by A_n. It may be necessary to modify the sets $A_n, n \in \mathbf{N}$, on a set of μ-measure zero.)

Let $\varepsilon > 0$ be fixed. Since the measure $|\nu_f|$ is absolutely continuous with respect to μ on each W_n, for each $n \in \mathbf{N}$ there exists $\eta_n > 0$ such that if $\mu(E) < \eta_n$ and $E \subseteq W_n$, then $|\nu_f|(E) < \varepsilon/2^{n+1}$. We may assume here that $\eta_n < \varepsilon/2$.

Since μ is outer regular, for each $n \in \mathbf{N}$ there exists an open set $G_n \supset A_n$ such that $G_n \subseteq W_n$ and

$$\mu(G_n \setminus A_n) < \frac{\eta_n}{2^n(\|x_n\|+1)} \,.$$

Then we have also

$$|\nu_f|(G_n \setminus A_n) < \frac{\varepsilon}{2^{n+1}} \,.$$

Define a gauge $\Delta : \Omega \to \mathcal{T}$ such that

$$\Delta(\omega) \subset G_n \qquad \text{if} \quad \omega \in A_n \,.$$

Let $\langle (E_i, \omega_i) : \; i \in \mathbf{N}\rangle$ be a Δ-fine generalized McShane partition. Note that, while for each i there exists exactly one n_i such that $\omega_i \in A_{n_i}$, it is possible that $n_i = n_j$ for some $i \neq j$. Since the partition is $\Delta-$fine, we have $E_i \subseteq G_{n_i}$.

Let

$$C_i = E_i \cap A_{n_i} \qquad \text{and} \qquad D_i = E_i \setminus A_{n_i} \,.$$

Since $f(\omega) = x_{n_i}$ whenever $\omega \in A_{n_i}$, we have

$$\begin{aligned}
\sum_i \left\| f(\omega_i)\mu(E_i) - \int_{E_i} f\,d\mu \right\| &= \sum_i \left\| \int_{E_i} [f(\omega_i) - f(\omega)]\,d\mu(\omega) \right\| \\
&= \sum_i \left\| \int_{C_i} [f(\omega_i) - f(\omega)]\,d\mu(\omega) + \int_{D_i} [f(\omega_i) - f(\omega)]\,d\mu(\omega) \right\| \\
&= \sum_i \left\| \int_{D_i} [f(\omega_i) - f(\omega)]\,d\mu(\omega) \right\| \\
&\le \sum_i \|f(\omega_i)\|\mu(D_i) + \sum_i \left\| \int_{D_i} f(\omega)\,d\mu(\omega) \right\| \\
&= \sum_n \sum_{\omega_i \in A_n} \|f(\omega_i)\|\mu(D_i) + \sum_i \left\| \int_{D_i} f(\omega)\,d\mu(\omega) \right\| \\
&= \sum_n \|x_n\|\mu\left(\bigcup_{\omega_i \in A_n} D_i \right) + \sum_n \sum_{\omega_i \in A_n} \left\| \int_{D_i} f(\omega)\,d\mu(\omega) \right\| \,.
\end{aligned}$$

Now, since $\bigcup_{\omega_i \in A_n} D_i \subseteq G_n \setminus A_n$, we have

$$\mu(\bigcup_{\omega_i \in A_n} D_i) < \frac{\eta_n}{2^n(\|x_n\|+1)}$$

and so

$$\sum_n \|x_n\|\mu\left(\bigcup_{\omega_i \in A_n} D_i \right) < \sum_n \frac{\eta_n}{2^n} < \frac{\varepsilon}{2} \,.$$

Similarly, if $\omega_i \in A_n$, then $D_i \subseteq G_n \setminus A_n$ and so

$$\sum_n \sum_{\omega_i \in A_n} \left\| \int_{D_i} f(\omega)\,d\mu(\omega) \right\| \le \sum_n |\nu_f|(G_n \setminus A_n) < \sum_n \frac{\varepsilon}{2^{n+1}} = \frac{\varepsilon}{2} \,.$$

Consequently,

$$\sum_i \left\| f(\omega_i)\mu(E_i) - \int_{E_i} f\, d\mu \right\| < \varepsilon,$$

which proves the variational McShane integrability of f. □

We next show the following result.

LEMMA 3. *If $f : \Omega \to X$ is variationally McShane integrable then f is strongly measurable.*

Proof. If μ is purely atomic, then the strong measurability is a direct consequence of the Pettis measurability theorem (see [3]). Thus we may assume that μ is non-atomic. Suppose that $f : \Omega \to X$ is variationally McShane integrable, but it is not strongly measurable. According to Corollary 3E of [5] the closed linear space H generated by $\nu_f(\Sigma)$ is separable. For each positive ε denote the set $\{\omega \in \Omega : \inf_{x\in H} \|f(\omega) - x\| > \varepsilon\}$ by Ω_ε. Notice that, since f is not strongly measurable, there exists ε_0 with $\mu^*(\Omega_{\varepsilon_0}) > 0$. (Otherwise we would have $f(\omega) \in H$ for almost all ω, and by the Pettis measurability theorem this would imply the strong measurability of f.) The space $(\Omega_{\varepsilon_0}, \Sigma \cap \Omega_{\varepsilon_0}, T \cap \Omega_{\varepsilon_0}, \mu|\Omega_{\varepsilon_0})$ is still an outer regular quasi-Radon measure space and $f|\Omega_{\varepsilon_0}$ is VM-integrable. For simplicity of notations we will assume that $\Omega = \Omega_{\varepsilon_0}$ and $\mu(\Omega) = 1$. Thus, for each $E \in \Sigma$ of positive measure and each $\omega \in \Omega$, we have

$$\left\| f(\omega) - \int_E f\, d\mu/\mu(E) \right\| > \varepsilon_0\,. \tag{2}$$

On the other hand, since f is variationally McShane integrable, there exists a gauge $\Delta : \Omega \to \mathcal{T}$ such that

$$\sum_i \mu(E_i) \left\| f(\omega_i) - \int_{E_i} f\, d\mu/\mu(E_i) \right\| < \varepsilon_0 \tag{3}$$

for each Δ-fine generalized McShane partition $\langle (E_i, \omega_i) \rangle$ of Ω. Since the formulae (2) and (3) cannot be valid simultaneously, we get a contradiction, which proves that f is strongly measurable. □

LEMMA 4. *If $f : \Omega \to X$ is variationally McShane integrable, then $|\nu_f|$ is moderated.*

Proof. Suppose that $|\nu_f|$ is not moderated and that μ is positive on nonempty open sets. As in the proof of Lemma 2, we may assume that $f = \sum_{n=1}^\infty x_n \chi_{A_n}$, where the sets A_n are pairwise disjoint and of finite measure, $\Omega = \bigcup_n A_n$, and f is Pettis integrable, but not Bochner integrable. Then there exists at least one set A_{n_0} such that if $U \supset A_{n_0}$ is open then

$\int_{U\setminus A_{n_0}} \|f\|\,d\mu = \infty$. Let $\varepsilon > 0$ be a fixed real number. For each $n \in \mathbf{N}$ let $G_n \supset A_n$ be an open set such that

$$\mu(G_n \setminus A_n) < \frac{\varepsilon}{2^{n+1}(\|x_n\|+1)}\,.$$

Let now $\Delta : \Omega \to \mathcal{T}$ be any gauge such that

$$\Delta(\omega) \subset G_n \qquad \text{if} \quad \omega \in A_n\,,$$

and let $\langle (E_i,\omega_i)\rangle_{i\in I}$ be an arbitrary Δ-fine partial McShane partition of Ω with a finite index set I. Note that for each i there exists exactly one n_i such that $\omega_i \in A_{n_i}$. Since the partition is $\Delta-$fine, we obtain in this case $E_i \subseteq G_{n_i}$.

Let

$$C_i = E_i \cap A_{n_i} \qquad \text{and} \qquad D_i = E_i \setminus A_{n_i}\,.$$

If $\omega \in A_{n_i}$, then $f(\omega) = x_{n_i}$, and so

$$\begin{aligned}
&\sum_{i\in I}\left\| f(\omega_i)\mu(E_i) - \int_{E_i} f\,d\mu \right\| \\
&\quad= \sum_{i\in I}\left\| \int_{C_i} [f(\omega_i)-f(\omega)]\,d\mu(\omega) + \int_{D_i} [f(\omega_i)-f(\omega)]\,d\mu(\omega)\right\| \\
&\quad= \sum_{i\in I}\left\| \int_{D_i} [f(\omega_i)-f(\omega)]\,d\mu(\omega)\right\| \\
&\quad\geq \sum_{i\in I}\left\| \int_{D_i} f(\omega)\,d\mu(\omega)\right\| - \sum_{i\in I}\|f(\omega_i)\|\mu(D_i) \\
&\quad= \sum_{i\in I}\left\| \int_{D_i} f(\omega)\,d\mu(\omega)\right\| - \sum_{n}\sum_{\omega_i\in A_n}\|f(\omega_i)\|\mu(D_i) \\
&\quad= \sum_{i\in I}\left\| \int_{D_i} f(\omega)\,d\mu(\omega)\right\| - \sum_{n}\|x_n\|\mu\left(\bigcup_{\omega_i\in A_n} D_i\right) \\
&\quad> \sum_{i\in I}\left\| \int_{D_i} f(\omega)\,d\mu(\omega)\right\| - \frac{\varepsilon}{2}\,,
\end{aligned}$$

where the last inequality follows from the relation $\bigcup_{\omega_i\in A_n} D_i \subseteq G_n \setminus A_n$.

Thus we obtain

$$\sum_{i\in\mathbf{N}}\left\| f(\omega_i)\mu(E_i) - \int_{E_i} f\,d\mu \right\| \geq \sum_{i\in\mathbf{N}}\left\| \int_{D_i} f(\omega)\,d\mu(\omega)\right\| - \frac{\varepsilon}{2} \tag{4}$$

for an arbitrary Δ-fine generalized McShane partition $\langle (E_i,\omega_i)\rangle_{i=1}^{\infty}$ of Ω.

Take now an arbitrary gauge $\widetilde{\Delta}$. We will show that, for this gauge and with the value of ε that was fixed at the beginning of the proof, condition (1) of the variational integrability of f does not hold, for a suitably chosen partition. Without loss of generality, we may assume that $\widetilde{\Delta}(\omega) \subseteq \Delta(\omega)$ for every $\omega \in \Omega$. We then have

$$A_{n_0} \subseteq \bigcup_{\zeta \in A_{n_0}} \widetilde{\Delta}(\zeta)\,.$$

Set $U_{n_0} = G_{n_0} \cap \bigcup_{\zeta \in A_{n_0}} \widetilde{\Delta}(\zeta)$ and let $\langle \zeta_i \rangle$ be a sequence of points from A_{n_0} satisfying the equality

$$\mu\left(U_{n_0} \setminus \bigcup_i \widetilde{\Delta}(\zeta_i)\right) = 0\,.$$

Let $W_1 = \widetilde{\Delta}(\zeta_1) \cap U_{n_0}$ and $W_i = \widetilde{\Delta}(\zeta_i) \cap U_{n_0} \setminus \bigcup_{j<i} W_j$, whenever $i \geq 2$. Without loss of generality we may assume that $\mu(W_i) > 0$ for all i. Then $\langle (W_i, \zeta_i) \rangle$ is a $\widetilde{\Delta}$-fine generalized partition of U_{n_0}. Let $\langle (F_p, \xi_p) \rangle$ be any $\widetilde{\Delta}$-fine generalized partition of $\Omega \setminus U_{n_0}$. Combining these partitions we obtain a generalized $\widetilde{\Delta}$-fine partition $\langle (V_r, \upsilon_r) \rangle$ of Ω. By assumption, f is not Bochner integrable on $U_{n_0} \setminus A_{n_0}$, so there exists a sequence $\langle (H_j) \rangle$ of pairwise disjoint subsets of $U_{n_0} \setminus A_{n_0}$ such that $U_{n_0} \setminus A_{n_0} = \bigcup_j H_j$ and (see [3])

$$\sum_j \left\| \int_{H_j} f \, d\mu \right\| = \infty\,. \tag{5}$$

We now modify the partition $\langle (V_r, \upsilon_r) \rangle$ as follows. We leave the pairs (F_p, ξ_p) unchanged, but replace each pair $\langle (W_i, \zeta_i) \rangle$ by the family of pairs

$$\{(W_i \cap H_j, \zeta_i) : \mu(W_i \cap H_j) > 0\} \cup \{(W_i \cap A_{n_0}, \zeta_i) : \mu(W_i \cap A_{n_0}) > 0\}.$$

Rearranging this family in a single sequence $\langle (E_i, \omega_i) \rangle$, we obtain a $\widetilde{\Delta}$-fine generalized partition of Ω. Note that, for each $\zeta_i \in A_{n_0}$, we have $D_i \subseteq U_{n_0} \setminus A_{n_0}$ and so $\mu(D_i) = 0$ or $D_i \subseteq H_j$ for some j. Moreover, we have

$$\mu\left((U_{n_0} \setminus A_{n_0}) \setminus \bigcup_{\zeta_i \in A_{n_0}} D_i\right) = 0.$$

Applying (4) and (5) we obtain

$$\begin{aligned}
&\sum_{i \in \mathbf{N}} \left\| f(\omega_i)\mu(E_i) - \int_{E_i} f \, d\mu \right\| \\
&\qquad \geq \sum_{\zeta_i \in A_{n_0}} \left\| \int_{D_i} f \, d\mu \right\| - \frac{\varepsilon}{2} \geq \sum_j \left\| \int_{H_j} f \, d\mu \right\| - \frac{\varepsilon}{2} = \infty\,.
\end{aligned}$$

It follows that f is not variationally McShane integrable. This completes the proof. $\square$

Combining the above lemmas, we obtain the main result of this paper:

Theorem 1. *A function $f : \Omega \to X$ is variationally McShane integrable if and only if it is strongly measurable and Pettis integrable, and $|\nu_f|$ is a moderated measure.*

As a corollary we obtain a generalization of the result proved in [1] for the case when $\Omega = [0, 1]$, with the Lebesgue measure.

Theorem 2. *Let $(\Omega, \mathcal{T}, \Sigma, \mu)$ be a compact finite Radon measure space. Then a function $f : \Omega \to X$ is variationally McShane integrable if and only if f is Bochner integrable.*

Proof. The "only if" part follows from Theorem 1 and from the observation that on a compact space, each moderated measure is finite. □

2. Examples

It was shown in [10] that if $[0, 1]$ is considered with the Lebesgue measure and the ordinary topology, then McShane integrability is equivalent to variational McShane integrability if and only if the Banach space is finite dimensional. Since for a σ-finite outer regular quasi-Radon space and a separable Banach space McShane integrability is equivalent to Pettis integrability (see [5]), Theorem 2 gives the following generalization of the result in [10]:

Theorem 3. *Let $(\Omega, \mathcal{T}, \Sigma, \mu)$ be a σ-finite quasi-Radon measure space, and let X be a Banach space. If there is an uncountable compact set $K \subseteq \Omega$ of positive measure such that μ is Radon on K, then McShane integrability is equivalent to variational McShane integrability if and only if the Banach space is finite dimensional.*

We do not know whether such a characterization holds in the case of an arbitrary non-atomic measure space. The topology of Ω certainly plays an important part here. In view of several results of [6] it is quite possible that the answer depends on some special axioms. As far as atomic measures are concerned, the following example shows that for some particular measures these two types of integration may coincide in case of an arbitrary separable Banach space.

Example 1. Consider the interval $[0, 1]$ endowed with the discrete topology and with a measure μ that is concentrated on the rationals of $[0, 1]$ and defined on the family of all subsets of $[0, 1]$. It is clear that the measure is quasi-Radon and, for any Banach space X, every strongly measurable X-valued μ-Pettis integrable function is variationally McShane integrable. Fremlin [5] showed that in the case of separable Banach spaces the McShane integral and the Pettis integral coincide. It follows that in the case of the above example the variational McShane integral coincides with the McShane integral for any separable Banach space.

Example 2. Consider the same measure space as above, but with the natural topology of $[0,1]$. This is also a quasi-Radon measure space. We will show, however, that for some particular measures μ not all Pettis integrable functions are variationally McShane integrable. By Lemma 4, it is enough to construct a measure μ and a function $f:[0,1]\to X$ such that $|\nu_f|$ is not moderated. In fact, we will construct a more complicated example of a function f such that $|\nu_f|$ is infinite on each non-empty open interval. Thus, in particular, the function f will not be VM-integrable on any interval.

To begin the construction, let $\{r_n : n \in \mathbf{N}\}$ be a fixed enumeration of all rationals from $[0,1]$, and let $(B_n)_{n=1}^{\infty}$ be a fixed enumeration of all open subintervals of $[0,1]$ with rational endpoints, such that each of the intervals occurs infinitely often. Given an infinite dimensional space X, let $\sum_{n=1}^{\infty} x_n$ be an unconditionally convergent series that is not absolutely convergent (see [2]). Our construction is by induction.

In the first step, let $\{r_{k_i} : i \in \mathbf{N}\}$ be the subsequence of $\{r_n : n \in \mathbf{N}\}$ (ordered in the same way as the entire sequence) consisting of all rationals from B_1. Let m_1 be the first index such that

$$\sum_{i=1}^{m_1} \|x_i\| \geq 1\,.$$

We set

$$\mu(\{r_{k_1}\}) = 1/2\,,\ \ldots, \mu(\{r_{k_{m_1}}\}) = 1/2^{m_1}$$

and

$$f_1 = \sum_{i \leq m_1} 2^i x_i \chi_{\{r_{k_i}\}}\,.$$

We denote the set $\{r_{k_1}, \ldots, r_{k_{m_1}}\}$ by W_1 and set $V_1 = \emptyset$. We then define a function $\varphi_1 : W_1 \to \mathbf{N}$ by setting $\varphi(r_{k_i}) = i$.

We now describe the general inductive step. Assume that the construction has been completed for $B_1, \ldots, B_p$. Thus, we have sets $V_1, \ldots, V_p$ of rationals, pairwise disjoint sets $W_1, \ldots, W_p$ of rationals, a sequence $m_1 \leq \ldots \leq m_p$ of integers and, for $i \leq p$, functions $\varphi_i : W_1 \cup \cdots \cup W_i \to \mathbf{N}$ and $f_i : W_1 \cup \cdots \cup W_i \to X$ with the following properties:

(a) for all $i \leq p$, $V_i = \bigcup_{j<i} W_j \cap B_i$ and $W_i \cap V_i = \emptyset$;
(b) for all $i \leq p$, $\sum_{r \in V_i \cup W_i} \|x_{\varphi_i(r)}\| \geq i$;
(c) for all $i \leq p$, $\varphi_i(W_1 \cup \ldots \cup W_i) = \{1, \ldots, m_i\}$;
(d) for all $r \in W_1 \cup \ldots W_p$, $\mu(\{r\}) = 2^{-\varphi_p(r)}$;
(e) for all $i < p$, $\varphi_{i+1}|(W_1 \cup \ldots W_i) = \varphi_i$;
(f) $f_p = \sum_{r \in W_1 \cup \cdots \cup W_p} 2^{\varphi_p(r)} x_{\varphi_p(r)} \chi_{\{r\}}$.

Let $\{r_{q_i} : i \in \mathbf{N}\}$ be the subsequence of those rationals from B_{p+1} that were not involved in the construction at any previous step, and let $V_{p+1} =$

$\bigcup_{j\leq p} W_j \cap B_{p+1}$. If $\sum_{r\in V_{p+1}} \|x_{\varphi_p(r)}\| \geq p+1$, then we continue the construction with B_{p+2}. Otherwise, let k be the first number such that

$$\sum_{r\in V_{p+1}} \left\|x_{\varphi(r)}\right\| + \sum_{i\leq k} \|x_{m_p+i}\| \geq p+1,$$

set

$$m_{p+1} = m_p + k\,, \quad W_{p+1} = \{r_{q_1}, \dots, r_{q_k}\}$$

and

$$\varphi_{p+1}(r_{q_i}) = m_p + i\,, \;\; \forall\, i \leq k \quad \text{and} \quad \varphi_{p+1}(r) = \varphi_p(r) \text{ if } r \in W_1 \cup \ldots \cup W_p\,.$$

Next, set for each $i \leq k$

$$\mu(\{r_{q_i}\}) = 2^{-\varphi_{p+1}(r_{q_i})}$$

and

$$f_{p+1} = \sum_{i\leq k} 2^{\varphi_{p+1}(r_{q_i})} x_{\varphi_{p+1}(r_{q_i})} \chi_{\{r_{q_i}\}}\,.$$

Now define

$$\varphi(r) = \varphi_p(r) \text{ if } r \in W_p \quad \text{and} \quad f(t) = \sum_{n=1}^{\infty} f_n(t) \quad \text{for all} \quad t \in [0,1]\,.$$

Since the series $\sum_{n=1}^{\infty} x_{\varphi(r_n)}$ is unconditionally convergent, the function f is Pettis integrable (see [3]). If B is an arbitrary open subinterval of $[0,1]$ with rational endpoints, then $|\nu_f(B)| = \sum_{r\in B} 2^{\varphi(r)} \|x_{\varphi(r)}\| \mu(\{r\})$ (see [3]). But since there exists an increasing sequence $k_1 < k_2 < \ldots$ such that $B = B_{k_1} = B_{k_2} = \ldots$, we then have, for each $i \in \mathbf{N}$,

$$|\nu_f|(B) = |\nu_f|(B_{k_i}) \geq \sum_{r\in V_{k_i}\cup W_{k_i}} \left\|x_{\varphi_{k_i}(r)}\right\| \geq k_i\,.$$

Hence $|\nu_f|(B) = \infty$, and consequently the measure $|\nu_f|$ is not moderated.

These two examples show that the class of variationally McShane integrable functions depends not only on the measure space, but on the topology making the measure quasi-Radon.

Example 3. We present here an example of a quasi-Radon measure space and a Banach-space valued function that generates a vector measure with moderated variation, but which itself is not Bochner integrable. This will show that Theorem 1 in general cannot be reduced to Bochner integrability.

Let $\Omega = [0,\infty)$, let Σ be the family of all Lebesgue measurable sets and let μ be given by

$$\mu(E) = \sum_{n=0}^{\infty} \frac{\lambda[E \cap (2n, 2n+1)]}{2^{n+1}}\,,$$

where λ is the Lebesgue measure. Let X be an infinite dimensional Banach space, and let $\sum_n x_n$ be an X-valued unconditionally convergent series that is not absolutely convergent. By a standard argument one can split the series into countably many absolutely convergent subseries with pairwise disjoint sets of terms, say

$$\sum_{i=1}^{\infty} x_{k_i}^n \,, \quad n = 1, 2, ... \text{ and } \{x_n : n \in \mathbf{N}\} = \bigcup_n \bigcup_i \{x_{k_i}^n\} \,.$$

We define functions f_n on the interval $[0, \infty)$ by

$$f_n(t) = 2^{n+1} \sum_{i=1}^{\infty} 2^{i+1} x_{k_i}^n \chi_{(2n+1/2^{i+1}, 2n+1/2^i)}(t)$$

and put $f(t) = \sum_{n=1}^{\infty} f_n(t)$. The function f is not Bochner integrable, but it has a moderated variation.

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References

[1] W. Congxin and Y. Xiaobo, *A Riemann-type definition of the Bochner integral*, J. Math. Study, **27** (1994), 32–36.

[2] J. Diestel, *Sequences and Series in Banach Spaces*, Graduate Texts in Math., vol. 92, Springer-Verlag, New York, 1984.

[3] J. Diestel and J. J. Uhl, *Vector Measures*, Mathematical Surveys, no. 15, Amer. Math. Soc., Providence, 1977.

[4] D. H. Fremlin, *Topological Riesz Spaces and Measure Theory*, Cambridge Univ. Press, Cambridge, 1974.

[5] ———, *The generalized McShane integral*, Illinois J. Math. **39** (1995), 39–67.

[6] R. J. Gardner and W. F. Pfeffer, *Borel measures*, in: Handbook of set-theoretic topology, North-Holland, Amsterdam, 1984, Chapter 22.

[7] R. Henstock, *The general theory of integration*, Oxford University Press, New York, 1991.

[8] E. J. McShane, *Unified integration*, Academic Press, San Diego, 1983.

[9] K. Musial, *Topics in the theory of Pettis integration*, Rend. Istit. Mat. Univ. Trieste **23** (1991), 177–262.

[10] V. A. Skvortsov and A. P. Solovov, *Variational integral for Banach-valued functions*, Real Analysis Exch. **24** (1998–1999), 799–805.

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