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ON THE GLOBAL STRUCTURE OF HOPF HYPERSURFACES IN A COMPLEX SPACE FORM

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ABSTRACT. It is known that a tube over a Kähler submanifold in a complex space form is a Hopf hypersurface. In some sense the reverse statement is true: a connected compact generic immersed C^{2n-1} regular Hopf hypersurface in the complex projective space is a tube over an irreducible algebraic variety. In the complex hyperbolic space a connected compact generic immersed C^{2n-1} regular Hopf hypersurface is a geodesic hypersphere.

Introduction

A natural class of real hypersurfaces in a complex space form $\overline{M}(c)$ of constant holomorphic curvature 4c is the class of Hopf hypersurfaces. For a unit normal vector ξ of a hypersurface M the vector $J\xi$ is a tangent vector to M, where J is the complex structure of the complex space form $\overline{M}(c)$.

DEFINITION. A hypersurface $M \subset \overline{M}(c)$ is called a Hopf hypersurface if the vector $J\xi$ is a principal direction at every point of M.

Y. Maeda [11] proved that for Hopf hypersurfaces in the n-dimensional complex projective space $\mathbf{C}P^n$ the corresponding principal curvature in the direction $J\xi$ is constant. It is known that a tube over a Kähler submanifold in a complex projective space is a Hopf hypersurface. T.E. Cecil and P.J. Ryan studied the local and global structure of Hopf hypersurfaces with constant rank of the focal map Φ_r.

Let M be an embedded hypersurface of $\overline{M}(c)$ of the regularity class C^2. Let NM be the normal bundle of M with projection $p:\ NM \to M$ and let BM be the unit normal bundle. For $\xi \in NM$ let $F(\xi)$ be the point in $\overline{M}(c)$ reached by traversing a distance $|\xi|$ along the geodesic in $\overline{M}(c)$ originating at $x = p(\xi)$ with the initial tangent vector ξ.

A point $P \in \overline{M}(c)$ is called a focal point of multiplicity $\nu > 0$ of (M, x) if $P = F(\xi)$ and the Jacobian of the map F has nullity ν at ξ.

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Definition. The tube of radius r over M is the image of the map $\Phi_r : BM \to \overline{M}(c)$ given by $\Phi_r(\xi) = F(r\xi),\ \ \xi \in BM$.

T.E. Cecil and P.J. Ryan proved the following result.

Lemma 1 [1]. *Let M be a connected, orientable Hopf hypersurface of $\mathbf{C}P^n$ with corresponding constant principal curvature $\mu = 2\cot 2r$. Suppose the map Φ_r has constant rank q on M. Then q is even and every point $x_0 \in M$ has a neighborhood U such that $\Phi_r(U)$ is an embedded complex $q/2$-dimensional submanifold of $\mathbf{C}P^n$.*

We remark that, in Lemma 1 and in Lemma 13 below, C^3 regularity is enough. From Lemmas 1 and 13 we obtain that a Hopf hypersurface with Φ_r of constant rank is an analytical hypersurface. It follows from this fact that $\Phi_r(U)$ is a complex submanifold and parametrization functions of $\Phi_r(U)$ satisfy an elliptic system of the PDE's with analytical coefficients. From C^2 regularity of $\Phi_r(U)$ we obtain that $\Phi_r(U)$ is analytic.

The global version of Lemma 1 has the following form [1] :

Let M be a connected compact embedded real Hopf hypersurface in $\mathbf{C}P^n$ with corresponding constant principal curvature $\mu = 2\cot 2r$. Suppose the map Φ_r has constant rank q on M. Then Φ_r factors through a holomorphic immersion of the complex $q/2$-dimensional manifold M/T_0 into $\mathbf{C}P^n$, where T_0 are $(2n - q - 1)$-dimensional spheres, the leaves of the distribution

$$T_0(x) = \{y \in T_xM,\ (\Phi_r)_*(y) = 0\}.$$

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1. The main results

The following theorem gives a complete description of the global structure of Hopf hypersurfaces in complex space forms.

Let M be an immersed regular hypersurface in a regular manifold N. Suppose that for a point $P \in N$ of self-intersection the linear span of the tangent hyperplanes to the branches of M coincides with the tangent space T_PN of the ambient manifold. This point is called a generic point of self-intersection. If every point of self-intersection of the hypersurface M is a generic point of self-intersection then the hypersurface M is called a generic immersed hypersurface.

THEOREM 1. *Let M be a C^{2n-1} regular compact generic immersed orientable Hopf hypersurface in the complex projective space $\mathbf{C}P^n$ $(n \geqslant 2)$. Then M is a tube over an irreducible algebraic variety.*

COROLLARY. *Let M be a C^{2n-1} regular connected compact embedded Hopf hypersurface in the complex projective space $\mathbf{C}P^n$ $(n \geqslant 2)$. Then M is a tube over an irreducible algebraic variety.*

The following are some standard examples of Hopf hypersurfaces in $\mathbf{C}P^n$ of constant holomorphic curvature 4.

1. A geodesic hypersphere M is the set of points at a fixed distance $r < \frac{\pi}{2}$ from a point $P \in \mathbf{C}P^n$. It is obvious that M is also the tube of radius $\frac{\pi}{2} - r$ over the hyperplane $\mathbf{C}P^{n-1} \subset \mathbf{C}P^n$ dual to the point P.

2. A tube over a totally geodesic space $\mathbf{C}P^k$ $(1 \leqslant k \leqslant n-1)$.

3. A tube over a totally geodesic real projective space RP^n and over a complex quadric $Q^{n-1} = \{(z_0, \dots, z_n\} \subset \mathbf{C}P^n : \ z_0^2 + z_1^2 + \dots + z_n^2 = 0\}$.

A tube of small radius r over a closed irreducible algebraic manifold in $\mathbf{C}P^n$ is an analytic Hopf hypersurface. But let $f = x_0^6 x_3^2 + x_1^3 x_2^5 = 0$ be the algebraic variety M in $\mathbf{C}P^3$. Then the point $P(1, 0, 0, 0)$ is a singular point (since $\operatorname{grad} f/P = 0$). In any neighborhood of the point P the normal curvatures at smooth points vary from $-\infty$ to $+\infty$. From Lemma 12 below it follows that normal curvatures of the tube of any radius r tend to $+\infty$. It follows that the tube of any radius r has regularity less then $C^{1,1}$.

V. Miquel proved the following theorem:

THEOREM (V. MIQUEL, [13]). *Let M be a connected compact embedded Hopf hypersurface in $\mathbf{C}P^n$ contained in a geodesic ball of radius $R < \frac{\pi}{2}$. Suppose that*

(1) *M has constant mean curvature H.*
(2) *The principal curvature μ in the direction $J\xi$ satisfies the inequality*

$$\mu \geqslant 2 \cot \left(2 \operatorname{arc} \cot \left[\frac{(2n-1)H - \mu}{2n-2} \right] \right).$$

Then M is a geodesic hypersphere.

We prove the following theorem.

THEOREM 2. *Let M be a C^{2n-1} regular connected compact generic immersed orientable Hopf hypersurface in the complex projective space $\mathbf{C}P^n$ $(n \geqslant 2)$ contained in a geodesic ball of radius $R < \frac{\pi}{2}$. Then M is a geodesic hypersphere.*

Let $\mathbf{C}H^n$ be the complex hyperbolic space of constant holomorphic curvature -4. We prove the following theorem.

THEOREM 3. *Let M be a connected compact generic immersed orientable C^{2n-1} regular Hopf hypersurface in the complex hyperbolic space $\mathbf{C}H^n$ ($n \geqslant 2$). Then the Hopf hypersurface M is a geodesic hypersphere.*

2. Lemmas

LEMMA 2 (Y. MAEDA, [11]). *Let M be a connected Hopf hypersurface in the complex projective space $\mathbf{C}P^n$. Then the principal curvature μ of M in the direction $J\xi$ is constant.*

Let A_ξ be the shape operator of M.

LEMMA 3 (T.E. CECIL, P.J. RYAN [1]). *Suppose $J\xi$ is an eigenvector of A_ξ with an eigenvalue μ. Then we have:*

(a) $(F_*)_{r\xi}(X, 0) = 0$ *if $\lambda = \cot r$ is an eigenvalue of A_ξ and X is a vector in the eigenspace T_λ corresponding to the eigenvalue λ.*
(b) $(F_*)_{r\xi}(J\xi, 0) = 0$ *if $\mu = 2\cot 2r$.*
(c) $(F_*)_{r\xi}(X, V) \neq 0$ *except as determined by (a) and (b).*

Now, let M be a real hypersurface of a complex space form $\overline{M}^n(c)$ of constant holomorphic curvature $4c$ and let ξ be a unit normal field on M. If $X \in T_P M$, $P \in M$, then one has a decomposition

$$JX = \phi X + f(X)\xi$$

into the tangent and normal components, respectively. So, ϕ is a (1, 1)-tensor field and f is a 1-form. These satisfy

$$\phi^2 X = -X + f(X)U, \quad \phi U = 0, \quad f(\phi X) = 0$$

for any vector field X tangent to M, where $U = -J\xi$. Moreover, we have

$$g(\phi X, Y) + g(X, \phi Y) = 0, \quad f(X) = g(X, U),$$
$$g(\phi X, \phi Y) = g(X, Y) - f(X)f(Y)$$

with g the metric tensor in $\overline{M}^n(c)$. We denote by A the shape operator on $T_P M$ associated with ξ.

LEMMA 4.
1. ([9]) *Let M be a Hopf hypersurface in $\overline{M}^n(c)$. Then we have:*
(a) $-2c\phi = \mu(\phi A + A\phi) - 2A\phi A$,
(b) $X\mu = (U\mu)f(X)$,

and

$$(U\mu)\, g\left((\phi A + A\phi)X, Y\right) = 0,$$

where μ is the principal curvature in the direction $U = -J\xi$, X, Y are vectors tangent to M, and $U\mu$ is the derivative of the function μ in the direction U.

Moreover, if $\phi A + A\phi = 0$ *then*

$$cg(X,\, \phi Y) = -g(\phi AX,\, AY) = g(A\phi X,\, AY),$$
$$cg(\phi X,\, \phi X) = -g(A\phi X,\, A\phi X)$$

and so $c \leqslant 0$.

2. ([11]) *Let* M *be a Hopf hypersurface in* $\mathbf{C}P^n$. *If* $X \in T_\alpha \subset T_P M$, *then*

$$JX \in T_{\mu\alpha+2/2\alpha-\mu} \subset T_P M,$$

where T_α *is an eigenspace corresponding to a principal curvature* α.

It follows from equation (a) of the first part of the lemma that α cannot be equal to μ or $\mu/2$.

Definition. Let A be a subset of a metric space X. Let $\delta(A)$ denote the diameter of A, and let

$$\delta^p(A) = [\delta(A)]^p \text{ for } p > 0,$$
$$\delta^0(A) = \begin{cases} 1 & \text{if } A \neq \varnothing, \\ 0 & \text{if } A = \varnothing. \end{cases}$$

For $p \geqslant 0$ and $\varepsilon > 0$ define

$$H^p_\varepsilon(A) = \inf\left\{\sum_{i=1}^{\infty} \delta^p(A_n):\ A \subset \cup A_n \text{ and } \delta(A_n) < \varepsilon\right\},$$
$$H^p(A) = \lim_{\varepsilon\to 0^+} H^p_\varepsilon(A) = \sup H^p_\varepsilon(A).$$

We call H^p the Hausdorff p-measure.

Lemma 5 (H. Federer, [4]). *If* $m > \nu \geqslant 0$ *and* $k \geqslant 1$ *are integers,* A *is an open subset of* R^m, $B \subset A$, Y *is a normed vector space and* $f:\ A \to Y$ *is a map of class* C^k *such that*

$$\operatorname{Dim}\operatorname{im} f_*(x) \leqslant \nu \quad \textit{for } x \in B,$$

then

$$H^{\nu+(m-\nu)/k}[f(B)] = 0.$$

Definition. Let Ω be a complex manifold. A set $A \subset \Omega$ is called an analytic set in Ω if for each point $a \in \Omega$ there exists a neighborhood U of a and functions $f_1, \ldots, f_N$ holomorphic in U such that $A \cap U = Z_{f_1} \cap \cdots \cap Z_{f_k} \cap U$, where Z_f is the set of zeros of a holomorphic function f.

A point a of an analytic set A is called a regular point if there exists a neighborhood U of a in Ω such that $A \cap U$ is a complex submanifold of U. The complex dimension of $A \cap U$ is then called the dimension of A at the point a and is denoted by $\dim_a A$. The set of all regular points of an analytic set is denoted by $\operatorname{reg} A$. Its complement $A \setminus \operatorname{reg} A$ is denoted by $\operatorname{sng} A$. The set $\operatorname{sng} A$ is called the set of singular points of the set A. It can be shown by induction on the dimension of the manifold Ω that $\operatorname{sng} A$ is nowhere dense and closed. This allows us to define the dimension of A at any point a of A as

$$\dim_a A = \lim_{z \to a} \dim_z A \ (z \in \operatorname{reg} A).$$

The set A is called purely p-dimensional if $\dim_z A = p$ for all $z \in A$ (see [2], [3]).

LEMMA 6 (B. SHIFFMAN, [16]). *Let E be a closed subset of a complex manifold Ω and let A be a purely q-dimensional analytic subset of $\Omega \setminus E$. If $H^{2q-1}(E) = 0$ then the closure $\overline{A}$ of the set A in Ω is a purely q-dimensional analytic subset of Ω.*

DEFINITION (D. MUMFORD, [14]). Let $U \subset \mathbf{C}^n$ be an open set. A closed subset $X \subset U$ is a $*$-analytic subset of U if X can be decomposed as

$$X = X^{(r)} \cup X^{(r-1)} \cup \dots \cup X^{(0)},$$

where for all i, $X^{(i)}$ is an i-dimensional complex submanifold of U and $\overline{X}^{(i)} \subset X^{(i)} \cup X^{(i-1)} \dots \cup X^{(0)}$. If $X^{(r)} \neq \varnothing$, then r is called the dimension of X.

An analytic set is always $*$-analytic [14].

LEMMA 7 (CHOW'S THEOREM, [14]). *If $X \subset \mathbf{C}P^n$ is a closed $*$-analytic subset, then X is a finite union of algebraic varieties.*

LEMMA 8 [3] . *An analytic set A in a complex manifold Σ is irreducible if and only if the set* $\operatorname{reg} A$ *is connected.*

Let $X \subset \mathbf{C}P^n$ denote a closed irreducible algebraic variety of dimension l (which may have singularities), and let $X_e \subset X$ denote the (non-empty) open subset of its smooth points. (For the definitions of irreducible, singular and smooth points see [14].) Define

$$V_X' = \left\{ (x,\, y) \in \mathbf{C}P^n \times \mathbf{C}\check{P}^n \,|\, x \in X_e \text{ and } y \text{ is tangent hyperplane at } x \right\},$$

where $\mathbf{C}\check{P}^n$ is the dual complex projective space.

The closure V_X of V_X' in the Zariski topology on $\mathbf{C}P^n \times \mathbf{C}\check{P}^n$ is called the tangent hyperplane bundle of X. It is a closed irreducible algebraic variety of dimension $(n-1)$. The first projection maps V_X onto X:

$$\pi_1 : \ V_X \to X, \quad (x,\, y) \to x.$$

Consider now the second projection

$$\pi_2 : \ V_X \to \mathbf{C}\check{P}^n, \quad (x,\, y) \to y.$$

Its image $\check{X} = \pi_2(V_X)$ is a closed irreducible variety of $\mathbf{C}\check{P}^n$ of dimension at most $(n-1)$, the dual variety of X [9].

Lemma 9 (Duality Theorem) [6, 10]. *The tangent hyperplane bundles of a closed irreducible algebraic variety X and its dual variety $\check{X}$ coincide: We have $V_{\check{X}} = V_X$ and hence $\check{\check{X}} = X$.*

Let $\mathbf{C}P^n$ be the complex projective space with a standard Fubini-Study metric. To a hyperplane $L \subset \mathbf{C}P^n$ passing through a point $x \in \mathbf{C}P^n$ we associate the point $y \in \mathbf{C}P^n$ representing the complex line in $\mathbf{C}^{n+1}$ orthogonal to L. Then the distance $\rho(x,y)$ is equal to $\pi/2$. One can identify $\mathbf{C}\check{P}^n$ with $\mathbf{C}P^n$ in this way and consider $\check{X}$ as a subset in $\mathbf{C}P^n$.

It is possible to define a tube over a closed irreducible algebraic variety $X \subset \mathbf{C}P^n$ which may have singularities. Let $(x,\, y) \in V_X \subset \mathbf{C}P^n \times \mathbf{C}\check{P}^n = \mathbf{C}P^n \times \mathbf{C}P^n$, $x \in X$, $y \in \check{X}$, and let $L(x,\, y)$ be a complex projective line through $x,\ y \in \mathbf{C}P^n$. Then $L(x,y)$ is a totally geodesic two-dimensional sphere in $\mathbf{C}P^n$ of curvature 4, the distance $\rho(x,\, y)$ is equal to $\pi/2$, and x and y are poles of the sphere $L(x,\, y)$. The set of points of $L(x,\, y)$ at a distance r from the point x is a circle $S_r(x,\, y)$ with the center x. The union

$$S_r = \bigcup_{(x,y)\in V_X} S_r(x,\, y)$$

is called the tube of radius r over X. The set S_r is also the tube of radius $\frac{\pi}{2} - r$ over the dual variety $\check{X}$.

If all the points of X are regular, this definition coincides with one given above.

The set of points $\operatorname{sng} V_X \subset V_X$ such that $(x,\, y) \in \operatorname{sng} V_X$ if $x \in \operatorname{sng} X$ or $y \in \operatorname{sng} \check{X}$ is a closed algebraic subvariety of V_X, $\operatorname{reg} V_X = V_X \setminus \operatorname{sng} V_X$ is an open set of V_X in the Zariski topology.

Let $X \subset \mathbf{C}P^n$ be a closed irreducible algebraic variety and let x_0 be a Zariski open set in X. Then the closure of x_0 in the classical topology is X [14].

Consider the Segre map

$$\sigma :\ \mathbf{C}P^n \times \mathbf{C}\check{P}^n \to \mathbf{C}P^{(n+1)^2-1}.$$

Then $\sigma(V_X)$ is a closed irreducible algebraic variety in $\mathbf{C}P^{(n+1)^2-1}$, and the set $\operatorname{reg} V_X$ is an open set of V_X in the Zariski topology.

As corollary we obtain the following result.

Lemma 10. *The closure of the set $\operatorname{reg} V_X \subset \mathbf{C}P^n \times \mathbf{C}P^n$ in the standard topology coincides with the tangent bundle V_X.*

Therefore the tube over X is the closure of the set

$$\bigcup_{(x,y)\in \mathrm{Reg} V_X} S_r(x,\, y).$$

Lemma 11 [5]. *Let X be a compact topological space. Suppose A is a closed subset such that $X \setminus A$ is a smooth n-dimensional orientable manifold without boundary. Then*

$$H_q(X, A) \backsimeq H^{n-q}(X \setminus A),$$

where H_i and H^i are homology and cohomology groups.

Lemma 12 [1]. *Suppose $J\xi$ is an eigenvector of the shape operator A_ξ of a Hopf hypersurface M in the complex projective space, with the corresponding eigenvalue $2\cot 2\Theta$, $0 < \Theta < \frac{\pi}{2}$. Suppose $J\xi, X_2, \ldots, X_n$ is a basis of principal vectors of A_ξ with $A_\xi X_j = \cot \Theta_j X_j$, $2 \leqslant j \leqslant n$, $0 < \Theta_j < \pi$; $\frac{\partial}{\partial t_j}$ $(2 \leqslant j \leqslant k)$ are normal vectors. Then the shape operator A_r of the tube Φ_r is given in terms of its principal vectors by*

(a) $A_r\left(\frac{\partial}{\partial t_j}\right) = -\cot r\left(\frac{\partial}{\partial t_j}\right), \quad 2 \leqslant j \leqslant k;$
(b) $A_r\,(X_j,\,0) = \cot\,(\Theta_j - r)\,(X_j,\,0)\,, \quad 2 \leqslant j \leqslant n;$
(c) $A_r(J\xi,\,0) = \cot\,(2(\Theta - r))\,(J\xi,\,0).$

For a complex hyperbolic space $\mathbf{C}H^n$ the following analog of Lemma 1 holds:

Lemma 13 [13]. *Let M be an orientable Hopf hypersurface of $\mathbf{C}H^n$ such that the principal curvature μ in the direction $J\xi$ is constant and equal to $\mu = 2\coth 2r$. Suppose that Φ_r has constant rank q on M. Then for every point $x_0 \in M$ there exists an open neighborhood U of x_0 such that $\Phi_r U$ is a $q/2$-dimensional complex submanifold embedded in $\mathbf{C}H^n$.*

Lemma 14 [15]. *Let Ω be a Hermitian complex manifold with exact fundamental form $\omega = d\gamma$. Let A be an analytical q-dimensional set with boundary $\partial A \subset \Omega$ such that $A \cup \partial A$ is compact. Then*

$$H^{2q}(A) \leqslant \frac{1}{q}\,(max_{\partial A}|\gamma|)\; H^{2q-1}(\partial A),$$

where $H^{2q}(A)$, $H^{2q-1}(\partial A)$ are Hausdorff measures, and

$$|\gamma|(z) = \max\left\{|\gamma(\upsilon)| :\, \upsilon \in T_z\Omega,\ |\upsilon| = 1\right\}.$$

Lemma 15 [8]. *Let M be a Hopf hypersurface of a complex space form $\overline{M}^n(c)$ $(c \neq 0)$. If U is an eigenvector of A, then the principal curvature $\mu = g(AU, U)$ is constant.*

3. Proofs of the theorems

Let M_s be the set of points of M such that $\operatorname{rank}(\Phi_r)_*(M_s) = s$, $F_s = \Phi_r(M_s)$, $F = \Phi_r(M)$. From Lemma 4 we obtain that if $X \in T_\alpha \subset T_P M$, where T_α is the eigenspace corresponding to the principal curvature $\alpha = \cot r$, then $JX \in T_\alpha$. Hence s is even and if $s < 2q$ then $s \leqslant 2q - 2$.

Let

$$E = \bigcup_{s<2q} F_s \cup F_0,$$

$$F_0 = \Big\{ x \in F : x = \Phi_r(L_1) = \Phi_r(L_2),\ L_1 \neq L_2 \subset M, \\ \operatorname{rank}(\Phi_r)_*(P_1) = \operatorname{rank}(\Phi_r)_*(P_2) = 2q \Big\},$$

for $P_i \in L_i$, where the L_i are leaves of the distribution $\operatorname{Ker}(\Phi_r)_*$.

Proof of Theorem 1. Let M be a compact Hopf hypersurface in $\mathbf{C}P^n$. This means that the vector $J\xi$ is a principal direction of M, where ξ is the unit normal vector and J is the complex structure in $\mathbf{C}P^n$. From Lemma 2 it follows that the corresponding principal curvature μ is constant and $\mu = 2\cot 2r$. Let $2q$ be the maximal rank of $(\Phi_r)_*$ on M. Let $P \in M$ be a point such that $\operatorname{rank}(\Phi_r)_*(P) = 2q$ and let M_{2q} be the corresponding connected component of M such that $P \in M_{2q}$ and for $Q \in M_{2q}$, $\operatorname{rank}(\Phi)_*(Q) = 2q$. Set $F_{2q} = \Phi_r(M_{2q})$, $\widetilde{F} = F_{2q} \cap (\mathbf{C}P^n \setminus E)$. From Lemma 1 we obtain that $\widetilde{F}$ is a purely analytic set, $\dim_z \widetilde{F} = q$, $z \in \widetilde{F}$.

Locally, F_0 is a transversal intersection of two complex submanifolds of dimension q. Hence F_0 is an analytic set of real dimension $\leqslant 2q-2$ and Hausdorff measure

$$H^{2q-1}(F_0) = 0.$$

Now apply Lemma 5 to the set $E_1 = \bigcup_{s<2q} F_s$ and the map Φ_r. Then $\nu \leqslant 2q-2$. If the class of regularity of M is greater or equal to $2(n-q+1)$, then the class of regularity of Φ_r is $k \geqslant 2(n-q+1)-1$ and

$$\nu + \frac{2n-1-\nu}{k} \leqslant 2q-2+\frac{2n-1}{k} \leqslant 2q-1$$

for $k \geqslant 2n-1$. From Lemma 5 we have $H^{2q-1}(E_1) = 0$ and so $H^{2q-1}(E) = 0$. From Lemma 6 we obtain that the closure of $\widetilde{F}$ is a purely q-dimensional analytic subset of $\mathbf{C}P^n$. Since any analytic subset is $*$-analytic we obtain from Chow's Theorem (Lemma 7) that $\operatorname{cl}\widetilde{F} \subset \mathbf{C}P^n$ is a finite union of algebraic varieties. An analytic set A is irreducible if and only if the set $\operatorname{reg} A$ is connected. From Lemma 8 it follows that $\operatorname{cl}\widetilde{F}$ is irreducible as analytic set and we obtain that $\operatorname{cl}\widetilde{F} = X$ is an irreducible algebraic variety.

Let S_r be a tube over $X = \operatorname{cl}\widetilde{F}$. By Lemma 10 we have $S_r \subset M$ and $S_r = \operatorname{cl} M_{2q}$. We will prove that $\operatorname{cl} M_{2q} = M$. Suppose that $\operatorname{cl} M_{2q} \neq M$. Then in every neighborhood of a point $P \in \partial M_{2q}$ there exist points $Q \in M \setminus \operatorname{cl} M_{2q}$. Let $P \in \partial M_{2q}$. Then $P \in S_r(x, y)$ such that $x \in \operatorname{sng} X$, $y \in \operatorname{sng} \check{X}$. Then

$$\partial M_{2q} = \bigcup_{x \in \operatorname{sng} X, y \in \operatorname{sng} \check{X}} S_r(x, y).$$

Otherwise some neighborhood of P belongs to $\operatorname{cl} M_{2q}$ and $P \in \operatorname{int}\operatorname{cl} M_{2q}$. The set of points

$$\operatorname{sng}(X,\, \check{X}) = \operatorname{sng} X \times \mathbf{C}P^n \cap \mathbf{C}P^n \times \operatorname{sng} \check{X} \subset V_X \subset \mathbf{C}P^n \times \mathbf{C}P^n$$

is a closed algebraic subvariety of V_X. The dimension of $\operatorname{sng}(X,\, \check{X})$ is $\leqslant n-2$ because the dimension of V_X is equal to $n-1$. The set ∂M_{2q} is a fiber bundle over $\operatorname{sng}(X,\, \check{X})$ with the circle S^1 as a leaf. The real dimension of $\operatorname{sng}(X,\, \check{X})$ is $\leqslant 2(n-2)$, whence

$$H_{2n-3}\Big(\operatorname{sng}(X,\, \check{X}),\, \mathbf{Z}\Big) = 0.$$

For $E = \partial M_{2q}$, $B = \operatorname{sng}(X,\, \check{X})$, $F = S^1$ the exact Thom-Gysin sequence has the form [17]

$$H_{2n-1}\Big(\operatorname{sng}(X,\, \check{X}),\, \mathbf{Z}\Big) \to H_{2n-3}\Big(\operatorname{sng}(X,\, \check{X}),\, \mathbf{Z}\Big) \to$$
$$\to H_{2n-2}(\partial M_{2q},\; \mathbf{Z}) \to H_{2n-2}\Big(\operatorname{sng}(X,\, \check{X}),\, \mathbf{Z}\Big),$$
$$0 \to 0 \to H_{2n-2}(\partial M_{2q},\; \mathbf{Z}) \to 0.$$

We obtain

$$H_{2n-2}(\partial M_{2q},\; \mathbf{Z}) = 0.$$

Next, we apply Lemma 11 with $X = M$, $A = \partial M_{2q}$. Then

$$H_{2n-1}(M,\; \partial M_{2q}) = H^0(M \setminus \partial M_{2q}).$$

But $M \setminus \partial M_{2q}$ has $m > 1$ connected components and

$$H^0(M \setminus \partial\; M_{2q},\, \mathbf{Z}) = \bigoplus_{i=1}^{m} \mathbf{Z}$$

is the direct sum of m copies of $\mathbf{Z}$ [17].

For the pair $(M,\; \partial M_{2q})$ the exact homology sequence has the form

$$H_{2n-1}(\partial M_{2q},\, \mathbf{Z}) \to H_{2n-1}(M,\, \mathbf{Z}) \to H_{2n-1}(M,\; \partial M_{2q},\, \mathbf{Z}) \to$$
$$\to H_{2n-2}(\partial M_{2q},\; \mathbf{Z}),$$

$$H_{2n-1}(\partial M_{2q},\, \mathbf{Z}) = H_{2n-2}(\partial M_{2q},\, \mathbf{Z}) = 0; \quad H_{2n-1}(M,\, \mathbf{Z}) = \mathbf{Z}.$$

It follows that $H_{2n-1}(M,\; \partial M_{2q},\, \mathbf{Z}) = \mathbf{Z}$, contradicting the above result. Thus $\operatorname{cl} M_{2q} = M$ and M is a tube over the irreducible algebraic variety $\operatorname{cl} \widetilde{F} = X$. □

Proof of Theorem 2. Let S be the hypersphere of minimal radius r_0 such that the hypersurface M is contained in the ball D with boundary $\partial D = S$. Let P be a point of tangency of M and S. Let ξ be the inward unit normal vector at the point P. Then the principal curvature in the direction $J\xi$ is $\mu = 2\cot 2\rho \geqslant 2\cot 2r_0$, and so $\rho \leqslant r_0 < \pi/2$. Another principal curvature $k_i = \cot \Theta_i$ at the point P satisfies $\cot \Theta_i \geqslant \cot r_0$, where $2\cot 2r_0$, $\cot r_0$

are principal curvatures of the hypersphere S. Then $\Theta_i \leqslant r_0$. Let $r = \rho - \pi/2$. From Lemma 12 we obtain that the principal curvatures of the tube Φ_r over M are equal to

$$(k_i)_r = \operatorname{tg}(\rho - \Theta_i) \leqslant \operatorname{tg}(r_0 - \Theta_i) < \infty.$$

Hence $\operatorname{rank}(\Phi_r)_*(P) = 2(n-1)$, and from Theorem 1 we get that $\Phi_r(M) = \operatorname{cl}\widetilde{F} = X$ is an irreducible hypersurface of degree d. Let X_k be a sequence of smooth algebraic hypersurfaces such that $\lim X_k = X$, degree $X_k = d$ [7] , and let $\check{X}$ and $\check{X}_k$ be dual algebraic varieties. Then

$$M = \Phi_{\frac{\pi}{2}-r}(X) = \Phi_r(\check{X}),$$

and from Lemma 9 we obtain $\check{X} = \lim \check{X}_k$. From the above argument we have for $\Phi_{\frac{\pi}{2}-r}(X_k) = M_k$,

$$\lim M_k = M.$$

For large k, M_k is contained in the balls D_k of radius $R < \pi/2$, and M_k does not intersect complex projective space $x_0 = 0$.

Let $f = 0$ be the equation of the algebraic hypersurface X_n where f is a homogeneous polynomial, $\operatorname{grad} f \neq 0$. By Bezout's Theorem [15] the system of equations

$$x_0 = 0, \quad f = 0, \quad f_{x_0} = 0$$

has a nontrivial solution if $n \geqslant 3$ and the degree of the polynomial f is $\geqslant 2$. This means that M_k intersects the hyperplane $x_0 = 0$. It follows that f is a linear function and the X_k are all hyperplanes, and the M_k are hyperspheres. Then the hypersurface M is a geodesic hypersphere, too.

For $n = 2$ the equation of the tube has the parametric form

$$z_j = x_j \cos r + \sin r \, \frac{\overline{\frac{\partial f}{\partial x_j}}}{|\operatorname{grad} f|} \, e^{it},$$

where the x_j are coordinates of points of the algebraic variety, $0 \leqslant t \leqslant 2\pi$, $0 \leqslant r \leqslant \frac{\pi}{2}$, and r is radius of the tube Φ_r; $j = 0, 1, 2$.

From the real point of view X is a compact two-dimensional manifold.

Let

$$g_1 = |x_0 \cos r|, \quad g_2 = \left| \frac{\overline{\frac{\partial f}{\partial x_0}}}{|\operatorname{grad} f|} \sin r \right|.$$

If the degree of the polynomial f is $\geqslant 2$, the zero sets of these regular functions on the manifold X are non empty on the manifold X. Hence there exists a point $P \in X$ such that $g_1 = g_2 = \rho$. Then $z_0 = \rho \left(e^{i\alpha} + e^{i(\beta+t)}\right)$. Moreover, if $t = \alpha - \beta - \pi$ then $z_0 = 0$. This means that M_k intersects the hyperplane $x_0 = 0$. Thus f is a linear function and M_k and M are geodesic hyperspheres as in the case $n \geqslant 3$. □

Proof of Theorem 3. Let S be the hypersphere of minimal radius r_0 such that the hypersurface M is contained in the ball D with boundary S. Let P_0 be a point of tangency of M and S. Let ξ be the inward unit normal vector of M at the point P_0. From Lemma 15 it follows that the principal curvature μ in the direction $J\xi$ is constant. At the point P_0 this curvature satisfies the inequality $\mu \geqslant 2\coth 2r_0$ and $\mu = 2\coth 2r$. We now follow the proof of Theorem 1, using Lemma 13 instead of Lemma 1. Consider the map Φ_r. For a Hopf hypersurface, $\operatorname{rank}(\Phi_r)_*$ is always even. This follows from Lemma 4.

Suppose $2q$ is the maximal rank of $(\Phi_r)_*$ at the points of M. Let $P \in M$ be a point such that $\operatorname{rank}(\Phi_r)_*(P) = 2q$ and M_{2q} is the connected component of M such that for $Q \in M_{2q}$, $\operatorname{rank}(\Phi_r)_*(Q) = 2q$. As in the proof of Theorem 1, set

$$F = \Phi_r(M), \quad F_{2q} = \Phi_r(M_{2q}), \quad F_s = \Phi_r(M_s),$$

$$E = F_0 \bigcup_{s<2q} F_s, \quad \widetilde{F} = F_{2q} \cap \mathbf{C}H^n \setminus E.$$

We obtain that $\operatorname{cl}\widetilde{F} = X$ is a compact analytic set in $\mathbf{C}H^n$ with boundary $\partial X \subset E$. The Hausdorff measure $H^{2q-1}(\partial X)$ is equal to 0. From Lemma 14 it follows that $H^{2q}(X) = 0$. This is possible only if $q = 0$ and X is a point. Thus M is a tube over a point and M is a geodesic hypersphere. □

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