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ORBITS OF CONDITIONAL EXPECTATIONS

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ABSTRACT. Let $N \subseteq M$ be von Neumann algebras and let $E : M \to N$ be a faithful normal conditional expectation. In this work it is shown that the similarity orbit $\mathcal{S}(E)$ of E by the natural action of the invertible group of G_M of M has a natural complex analytic structure and that the map $G_M \to \mathcal{S}(E)$ given by this action is a smooth principal bundle. It is also shown that if N is finite then $\mathcal{S}(E)$ admits a Reductive Structure. These results were previously known under the additional assumptions that the index is finite and $N' \cap M \subseteq N$. Conversely, if the orbit $\mathcal{S}(E)$ has a Homogeneous Reductive Structure for every expectation defined on M, then M is finite. For every algebra M and every expectation E, a covering space of the unitary orbit $\mathcal{U}(E)$ is constructed in terms of the connected component of 1 in the normalizer of E. Moreover, this covering space is the universal covering in each of the following cases: (1) M is a finite factor and $\mathrm{Ind}(E) < \infty$; (2) M is properly infinite and E is any expectation; (3) E is the conditional expectation onto the centralizer of a state. Therefore, in these cases, the fundamental group of $\mathcal{U}(E)$ can be characterized as the Weyl group of E.

1. Introduction

Let M be a von Neumann algebra with group of invertible elements G_M and unitary group $\mathcal{U}_M$. Denote by $\mathcal{E}(M)$ the space of faithful normal conditional expectations defined on M and by $\mathcal{B}(M)$ the algebra of bounded linear operators on M. Consider the action

$$L : G_M \times \mathcal{B}(M) \to \mathcal{B}(M)$$

given by

$$L_g(T) = gT(g^{-1} \cdot g)g^{-1}, \quad g \in G_M,\ T \in \mathcal{B}(M).$$

Let $E \in \mathcal{E}(M)$ be a conditional expectation. Define the unitary orbit of E by

$$\mathcal{U}(E) = \{L_u(E) : u \in \mathcal{U}_M\} \subseteq \mathcal{E}(M), \tag{1}$$

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with the quotient topology induced by the norm topology of $\mathcal{U}_M$. Thus we have a natural fibration

$$\Pi_E : \mathcal{U}_M \to \mathcal{U}(E) \quad \text{given by} \quad \Pi_E(u) = L_u(E),\ u \in \mathcal{U}_M. \tag{2}$$

The goal of this work is the study of the homotopy groups and the differential geometry of the orbit $\mathcal{U}(E)$ or, more precisely, of the fibration Π_E. Results on this problem appear in [4], [3] and [21], mainly under two very restrictive hypotheses: the finite index condition for E and the condition $E(M)' \cap M \subseteq E(M)$. In addition, in [5] a detailed study of this problem is given for the case when E is a state.

In order to study the homotopy type of these orbits we construct a covering space over each orbit $\mathcal{U}(E)$ whose group of covering transformations is the so-called Weyl group of the expectation E. To describe this structure we need the following definitions:

At the level of the unitary group $\mathcal{U}_M$ of M, the isotropy group of the action L, i.e., $\Pi_E^{-1}(E)$, is a very well known group, usually called the *normalizer* of E. This group has been studied, among other authors, by A. Connes [10] and Kosaki [18] in connection with crossed product inclusions of algebras (see also [7]). We shall denote the normalizer of E by

$$\mathcal{N}_E = \{\ u \in \mathcal{U}_M \ :\ E(uxu^*) = uE(x)u^*, x \in M\ \}. \tag{3}$$

Let $N = E(M)$. Then N is a von Neumann subalgebra of M. We also consider the von Neumann algebra

$$M_E = \{x \in N' \cap M \ :\ E(xm) = E(mx) \ \text{ for all } \ m \in M\}, \tag{4}$$

usually called the centralizer of E (see [9] or [14]). In [7] it was shown that the connected component of 1 in $\mathcal{N}_E$ is the group

$$\mathcal{H}_E = \mathcal{U}_N \cdot \mathcal{U}_{M_E} = \{vw : v \in \mathcal{U}_N \ \text{ and } \ w \in \mathcal{U}_{M_E}\}, \tag{5}$$

which is a closed, open and invariant subgroup of $\mathcal{N}_E$. The set of connected components of $\mathcal{N}_E$ is a discrete group, called the Weyl group of E:

$$W(E) = \pi_0(\mathcal{N}_E) \simeq \mathcal{N}_E/\mathcal{H}_E. \tag{6}$$

This group has several characterizations in very different contexts (see [18], [7] and [8]).

We show that, for any von Neumann algebra M and any $E \in \mathcal{E}(M)$, the space $\mathcal{X}(E) = \mathcal{U}_M/\mathcal{H}_E$ and its natural projection onto $\mathcal{U}(E) \simeq \mathcal{U}_M/\mathcal{N}_E$ (see diagram (9) below) defines a covering map whose group of covering transformations can be identified with the Weyl group $W(E)$ (Theorem 2.3).

In all examples we know, $\mathcal{X}(E)$ is actually the *universal* covering for $\mathcal{U}(E)$, and the fundamental group $\pi_1(\mathcal{U}(E), E)$ therefore coincides with the Weyl group $W(E)$. We conjecture that this is true for all von Neumann algebras M and all conditional expectations $E \in \mathcal{E}(M)$. In Theorem 2.6 we show that if any of the conditions

(1) M is properly infinite,
(2) M is finite, $\dim \mathcal{Z}(M) < \infty$ and E has finite index,
(3) $E = E_\varphi \in \mathcal{E}(M, M_\varphi)$ is the canonical expectation associated with a faithful normal state φ of M, i.e., M_φ is the centralizer of φ and $\varphi \circ E = \varphi$

holds, then $\mathcal{X}(E)$ is simply connected and therefore is the universal covering for the orbit $\mathcal{U}(E)$. Consequently,

$$\pi_1(\mathcal{U}(E), E) \simeq W(E).$$

In order to study the differential geometry of the orbit of an expectation E we consider the entire similarity orbit

$$\mathcal{S}(E) = \{L_g(E) : g \in G_M\} \subseteq \mathcal{B}(M) \tag{7}$$

and the fibration (with the same notation as its restriction to $\mathcal{U}_M$)

$$\Pi_E : G_M \to \mathcal{S}(E) \quad \text{given by} \quad \Pi_E(g) = L_g(E),\ g \in G_M. \tag{8}$$

Note that $L_g(E)$ is not necessarily a conditional expectation for all $g \in G_M$. Nevertheless we prefer to use this setting, since the group G_M is a complex analytic Banach Lie group and the orbit $\mathcal{S}(E)$ can be given a complex analytic manifold structure. In any case, all geometrical results obtained for $\mathcal{S}(E)$ also hold for the unitary orbit $\mathcal{U}(E)$, if one replaces “complex analytic” by “real analytic”.

In order to study the differential geometry of similarity orbits we need to generalize several results of [7] mentioned above to the invertible groups setting. This is carried out in Section 3, where the connected component of the isotropy group I_E of the action L at E is characterized (Proposition 3.3) and the new Weyl group, which appears naturally, is shown to be the same group as the “old” one (Theorem 3.5).

In Section 4 we first show that $\mathcal{S}(E)$ can always be given a unique complex analytic differential structure such that the map Π_E defined in (8) becomes a submersion (Theorem 4.8). The key tool is the construction, in the style of [9], of a conditional expectation $F \in \mathcal{E}(M)$ onto the centralizer M_E which commutes with E. This allows us to obtain a complement in M for the subspace $N + M_E$, which can be naturally identified with the tangent space of $\mathcal{S}(E)$ at E.

We next show that if $N = E(M)$ is a finite von Neumann algebra, then $\mathcal{S}(E)$ has a unique structure of a Homogeneous Reductive Space (HRS) (see Definition 4.10 and Proposition 4.13). This family of HRS’s is of geometrical interest. Indeed, perhaps the most general families of examples of infinite dimensional HRS’s modeled in operator algebras are studied in [1] and [3]. All these examples can be represented as quotients of the unitary (or invertible) group of an algebra M by the unitary (or invertible) group of some subalgebra.

However, this is not possible in the case of the orbit of a conditional expectation. Indeed, the isotropy group I_E can be large enough to generate the entire algebra M, whereas $\mathcal{S}(E) \simeq G_M/I_E$. Moreover, the map $\Pi_0 : G_M \to G_M/\mathcal{Z}_E$ $(= \mathcal{Y}(E))$, where $\mathcal{Z}_E$ is the connected component of 1 in I_E, also defines an HRS if N is finite. Actually (see Theorem 4.8), this is how we exhibit the HRS structure of $\mathcal{S}(E)$, since $\mathcal{Y}(E)$ is a covering space for $\mathcal{S}(E)$ and these spaces therefore are locally homeomorphic (and diffeomorphic). But $\mathcal{Y}(E)$ cannot be represented as a quotient as before (if $N \not\subseteq M_E$ and $M_E \not\subseteq N$), since, by Proposition 4.6, $\mathcal{Z}_E = G_{M_E} \cdot G_N$, which is not the invertible group of any subalgebra of M.

At the end of Section 4 we show that the existence of HRS structures for any expectation $E \in \mathcal{E}(M)$ forces M to be a finite von Neumann algebra (Theorem 4.17).

2. The universal covering of $\mathcal{U}(E)$

Let $N \subseteq M$ be von Neumann algebras. From now on we shall denote by $\mathcal{E}(M,N)$ the space of faithful normal conditional expectations $E : M \to N$. Let $E \in \mathcal{E}(M,N)$. Recall the definitions of the sets $\mathcal{U}(E)$, $\mathcal{N}_E$, M_E and $\mathcal{H}_E$ associated with E, given in equations (1), (3), (4) and (5), respectively. Consider the space $\mathcal{X}(E) = \mathcal{U}_M/\mathcal{H}_E$, with the quotient topology of the norm topology of $\mathcal{U}_M$, and denote by Π_0 the projection from $\mathcal{U}_M$ onto $\mathcal{X}(E)$. The situation we shall study is the following: we have a commutative diagram

$$\begin{array}{ccc} \mathcal{U}_M & \xrightarrow{\ \Pi_0\ } & \mathcal{X}(E) = \mathcal{U}_M/\mathcal{H}_E \\ & \Pi_E \searrow & \Big\downarrow \Phi \\ & & \mathcal{U}(E) \simeq \mathcal{U}_M/\mathcal{N}_E \end{array} \tag{9}$$

where the map Φ is defined by $\Phi(u\mathcal{H}_E) = \Pi_E(u) \sim u\mathcal{N}_E$, $u \in \mathcal{U}_M$. In [4] it was shown that when $N' \cap M \subseteq N$, and the Jones index of E is finite, then the M-unitary orbit of the Jones projection e of E,

$$\mathcal{U}_M(e) = \{ueu^* : u \in \mathcal{U}_M\} \simeq \mathcal{U}_M/\mathcal{U}_N,$$

is a covering space for $\mathcal{U}(E)$. Note that, under the above assumptions, we have $\mathcal{U}_M(e) \simeq \mathcal{X}(E)$, since both spaces can be identified with $\mathcal{U}_M/\mathcal{U}_N$ and since (see [4]) the quotient topology and the norm topology coincide on $\mathcal{U}_M(e)$.

In this paper we will show that the map Φ is always a covering map, without the two hypotheses appearing in [4], and that $\mathcal{X}(E) = \mathcal{U}_M/\mathcal{H}_E$ is a covering space for $\mathcal{U}(E)$, with group of covering transformations $W(E)$. Moreover, in several cases (see 2.6) Φ is the universal covering of $\mathcal{U}(E)$ and, in particular, $\pi_1(\mathcal{U}(E)) \simeq W(E)$.

Note that the Weyl group $W(E) = \mathcal{N}_E/\mathcal{H}_E$, since it is contained in $\mathcal{X}(E)$, has a natural action on $\mathcal{X}(E)$ given by right multiplication. This action is well defined because $\mathcal{H}_E$ is a normal subgroup of $\mathcal{N}_E$.

Proposition 2.1. *Let $N \subseteq M$ be von Neumann algebras and let $E \in \mathcal{E}(M,N)$ be a faithful normal conditional expectation. Then, with the notations of diagram* (9)*, we have:*

(1) *The map Φ is continuous.*
(2) *For any $u \in \mathcal{U}_M$, the fibre by Φ of $L_uE \in \mathcal{U}(E)$ is precisely the orbit of $\Pi_0(u) \in \mathcal{X}(E)$ by the action of $W(E)$.*
(3) *The unitary orbit $\mathcal{U}(E)$ is homeomorphic to $\mathcal{X}(E)/W(E)$ (i.e., the space of orbits by the action of $W(E)$ in $\mathcal{X}(E)$), both considered with the quotient topology.*

Proof. Assertions (1) and (2) follow immediately from the commutative diagram (9) and the fact that $\Phi^{-1}(E) = W(E)$. Let $\rho : \mathcal{X}(E) \to \mathcal{X}(E)/W(E)$ be the canonical projection. To prove (3), consider the map $\bar\Phi : \mathcal{X}(E)/W(E) \to \mathcal{U}(E)$ given by $\bar\Phi(\rho(h)) = \Phi(h)$, $h \in \mathcal{X}(E)$. Then $\bar\Phi$ is the desired homeomorphism. Indeed, it is clear that $\bar\Phi$ is well defined and bijective. The map $\bar\Phi$ is also continuous, since $\Phi = \bar\Phi \circ \rho$ is continuous. On the other hand, let U be an open set in $\mathcal{X}(E)/W(E)$. By the full commutative diagram

$$\begin{array}{ccccc} \mathcal{U}_M & \xrightarrow{\ \Pi_0\ } & \mathcal{X}(E) = \mathcal{U}_M/\mathcal{H}_E & \xrightarrow{\ \rho\ } & \mathcal{X}(E)/W(E) \\ & \Pi_E \searrow & \Phi \Big\downarrow & \bar\Phi \swarrow & \\ & & \mathcal{U}(E) = \mathcal{U}_M/\mathcal{N}_E & & \end{array} \tag{10}$$

and the fact that Π_E is an open map, it is clear that $\bar\Phi(U)$ is open in $\mathcal{U}(E)$. □

Remark 2.2. In order to show that the map Φ defined in diagram (9) is a covering map we shall use the following well known result of algebraic topology (see, for instance, Chapter 1 of [13]):

Let X be a locally pathwise connected and connected topological space, and let G be a group of homeomorphisms of X that operates properly discontinuously (i.e., for each $x \in X$ there exists an open set V_x such that $V_x \cap g(V_x) = \emptyset$ for every $g \in G$, $g \neq e$). Consider the map $p : X \to X/G$. Then X is a covering space for X/G with covering map p and group of covering transformations G, and $p_*(\pi_1(X, x_0))$ is a normal subgroup of $\pi_1(X/G, p(x_0))$.

Theorem 2.3. *Let $N \subseteq M$ be von Neumann algebras and let $E \in \mathcal{E}(M,N)$ be a faithful normal conditional expectation. Then, with the notations of diagram* (9)*, the space $\mathcal{X}(E)$ is a covering space for $\mathcal{U}(E)$, with covering map Φ and group of covering transformations $W(E)$.*

Proof. By the previous remark, it suffices to show that $W(E)$ operates properly discontinuously on $\mathcal{X}(E)$. Fix $u \in \mathcal{U}_M$ and consider the open set

$$W_u = \{w \in \mathcal{U}_M : \|w - u\| < 1/2\}.$$

For each element $k \in W(E)$, we choose some $u_k \in \mathcal{N}_E$ such that $\Pi_0(u_k) = u_k\mathcal{H}_E = k$. Since the map Π_0 is open, we can consider the open set

$$V_u = \Pi_0(W_u) \subseteq \mathcal{X}(E).$$

Note that, for $k \in W(E)$, $V_u k = \Pi_0(W_u u_k)$. In order to prove that the action of $W(E)$ in $\mathcal{X}(E)$ is properly discontinuous, we only need to show that $V_u \cap V_u k = \emptyset$ for every $k \in W(E)$ with $k \neq 1$. Suppose that this is not true. Then, for some $k \in W(E)$, $k \neq 1$, there exist $w_1, w_2 \in W_u$ and $z \in \mathcal{H}_E$ such that $w_1 u_k = w_2 z$. Then

$$w_1^* w_2 = u_k z^* \in \mathcal{N}_E \setminus \mathcal{H}_E.$$

But, since $w_1, w_2 \in W_u$,

$$\|w_1^* w_2 - 1\| = \|w_2 - w_1\| < 1.$$

This implies that $w_1^* w_2 \in \mathcal{H}_E$ (see [7] or the proof of Proposition 3.3 below), which is a contradiction. □

Corollary 2.4. *The group $\Psi_*(\pi_1(\mathcal{X}(E)))$ is a normal subgroup of $\pi_1(\mathcal{U}(E))$, and we have the isomorphism*

$$\pi_1(\mathcal{U}(E))/\Psi_*(\pi_1(\mathcal{X}(E))) \simeq W(E).$$

Proof. By Proposition 2.1, the fibre $\Psi^{-1}(E)$ equals $W(E)$. The assertion follows from the homotopy exact sequence induced by the covering map Ψ. □

Remark 2.5. Let φ be a faithful normal state of the von Neumann algebra M. In [5] Andruchow and Varela show that the unitary orbit of φ,

$$\mathcal{U}(\varphi) = \{\varphi(u^* \cdot u) : u \in \mathcal{U}_M\},$$

is simply connected. Therefore the unitary group of the centralizer M_φ of φ coincides with the normalizer $\mathcal{N}_\varphi$ of φ, considered as a conditional expectation. Then the covering space is given by

$$\mathcal{X}(\varphi) = \mathcal{U}_M/\mathcal{U}_{M_\varphi} = \mathcal{U}_M/\mathcal{N}_\varphi \simeq \mathcal{U}(\varphi),$$

and $\mathcal{U}(\varphi)$ is its own universal covering.

Moreover, if $E_\varphi \in \mathcal{E}(M, M_\varphi)$ is the canonical expectation such that $\varphi \circ E_\varphi = \varphi$, then $\mathcal{U}(\varphi) \simeq \mathcal{X}(E_\varphi)$ and so $\mathcal{U}(\varphi)$ is the universal covering for $\mathcal{U}(E_\varphi)$. Indeed, since

$$\mathcal{X}(E_\varphi) = \mathcal{U}_M/\mathcal{U}_{M_\varphi}\mathcal{U}_{M_{E_\varphi}} \quad \text{and} \quad \mathcal{U}(\varphi) \simeq \mathcal{U}_M/\mathcal{U}_{M_\varphi},$$

it suffices to show that $M_{E_\varphi} \subseteq M_\varphi$. But this follows from the definition of M_{E_φ} (see (4)) and the fact that $\varphi \circ E_\varphi = \varphi$. This gives a large class of conditional expectations for which the covering space $\mathcal{X}(E)$ is the universal covering. We extend this class in the following theorem.

THEOREM 2.6. *Let M be a von Neumann algebra, let $E \in \mathcal{E}(M)$, and suppose that one of the following conditions holds:*

(1) *M is properly infinite.*
(2) *M is a II_1 factor and E has finite index.*
(3) *$E = E_\varphi \in \mathcal{E}(M, M_\varphi)$ is the canonical expectation associated with a faithful normal state φ of M as in Remark* 2.5.

Then $\mathcal{X}(E)$ is simply connected, and hence is the universal covering for the orbit $\mathcal{U}(E)$. Consequently,

$$\pi_1(\mathcal{U}(E), E) \simeq W(E).$$

Proof. Consider the fibre bundle

$$\Pi_0 : \mathcal{U}_M \to \mathcal{U}_M/\mathcal{H}_E = \mathcal{X}(E). \tag{11}$$

Recall that a fibre bundle gives rise to an exact sequence of homotopy groups. In our case, the bundle Π_0 yields the exact sequence

$$\ldots \to \pi_2(\mathcal{X}(E)) \to \pi_1(\mathcal{H}_E) \xrightarrow{i_*} \pi_1(\mathcal{U}_M) \to \pi_1(\mathcal{X}(E)) \to \pi_0(\mathcal{H}_E) = 0, \tag{12}$$

where 1 is taken as base point for the homotopy groups of the unitary groups and $[1]_{\mathcal{X}(E)} = \mathcal{H}_E$ is the base point for $\mathcal{X}(E)$. Here i_* denotes the homomorphism induced by the inclusion $i : \mathcal{H}_E \hookrightarrow \mathcal{U}_M$. We can then use results by Handelmann [15] and Schröeder [27] on computing the homotopy group of the unitary group of a von Neumann algebra.

If condition (1) holds, the result follows by appealing to the homotopy exact sequence (12), and the fact [15] that $\mathcal{U}_M$ has trivial π_1 group if M is properly infinite.

Suppose next that (2) holds. Since M is a II_1 factor and $\mathrm{Ind}(E) < \infty$ it is known (see [25]) that $N = E(M)$ is also of type II_1 and $\dim \mathcal{Z}(N) < \infty$. Let us recall the following results (see [5], [15] and [27]):

(1) If M is a von Neumann algebra of type II_1, then $\pi_1(\mathcal{U}_M)$ is isomorphic to the additive group $\mathcal{Z}(M)_{sa}$ of selfadjoint elements in $\mathcal{Z}(M)$.
(2) Let $j : \mathcal{U}_N \to \mathcal{U}_M$ be the inclusion map. Then the image of the homomorphism $j_* : \pi_1(\mathcal{U}_N) \to \pi_1(\mathcal{U}_M) \simeq \mathcal{Z}(M)_{sa}$ is equal to the additive group generated by the set $\{\mathrm{tr}(p) : p \text{ projection in } N\}$, where tr is the center valued trace of M.

In our case, $\pi_1(\mathcal{U}_M) \simeq \mathbb{R}$. Let $k : \mathcal{U}_N \to \mathcal{H}_E$ be the inclusion map. Clearly, $i_* \circ k_* = j_*$, where i_* is the map defined in (12). Then $j_*(\pi_1(\mathcal{U}_N)) \subseteq i_*(\pi_1(\mathcal{H}_E))$. Let $p \in \mathcal{Z}(N)$ be a minimal projection. Then pNp is a II_1 factor and the trace of projections in pNp generates the additive group $\mathbb{R}$. Hence i_* is surjective and $\pi_1(\mathcal{X}(E))$ must be trivial by the homotopy exact sequence (12).

If condition (3) holds, the result follows from [5] and Remark 2.5. □

Remark 2.7. Using the same techniques as in the proof of this theorem, it can be shown that $\mathcal{X}(E)$ is simply connected if $\operatorname{Ind}(E) < \infty$, M is finite and $\dim \mathcal{Z}(M) < \infty$.

Example 2.8. Let M be a von Neumann algebra and let $p \in M$ be a projection. Then p determines the conditional expectation $E_p : M \to N = \{p\}' \cap M$ given by

$$E_p(x) = pxp + (1-p)x(1-p), \quad x \in M.$$

Denote by $\mathcal{U}(p) = \{upu^* : u \in \mathcal{U}_M\}$ the unitary orbit of p, which is a connected component of the Grassmannians of M. Then

$$\mathcal{U}(p) \simeq \mathcal{X}(E_p)$$

in the sense that both spaces are homeomorphic to $\mathcal{U}_M/\mathcal{U}_N$, since $N' \cap M \subseteq N$ and so $\mathcal{H}_{E_p} = \mathcal{U}_N$. Note that on $\mathcal{U}(p)$ we consider the norm topology as a subset of M (see [12] or [26]). Using Theorem 2.6 it is not difficult to show that the Grassmannian $\mathcal{U}(p)$ is always simply connected. Indeed, $\pi_1(\mathcal{U}(p))$ splits in the finite and the properly infinite parts of M, and parts (1) and (3) of Theorem 2.6 can be applied (see also [2]).

The Weyl group of E_p is trivial if $1 - p \notin \mathcal{U}(p)$, and it has two elements if $1 - p \in \mathcal{U}(p)$, since in this case, for any $u \in \mathcal{U}_M$ satisfying $upu^* = 1 - p$ we have $L_u E_p = E_{1-p} = E_p$ and $\mathcal{N}_{E_p} = \mathcal{U}_N \cup u \cdot \mathcal{U}_N$.

A similar study can be made for systems of projections, i.e., n-tuples $P = (p_1, \ldots, p_n)$ of pairwise orthogonal projections such that $\sum p_i = 1$ (see [11]). Using Theorem 2.6, we again see that the joint unitary orbit of P is simply connected and is homeomorphic to the space $\mathcal{X}(E_P)$ associated with the conditional expectation $E_P(x) = \sum p_i x p_i$, $x \in M$. The Weyl group is a subgroup of the permutation group S_n, determined by those projections in P that are equivalent (and therefore unitary equivalent) in M.

3. The Weyl group, invertible case

In [7], the Weyl group was defined in terms of the unitary group of the von Neumann algebra M. Our aim in this section is to extend this work to the case when the action over the conditional expectations is given by the invertible (instead of unitary) elements.

Let us recall some definitions. Let M be a von Neumann algebra. We consider the action $L : G_M \times \mathcal{B}(M) \to \mathcal{B}(M)$ given by $L_g(T) = gT(g^{-1} \cdot g)g^{-1}$, $g \in G_M$, $T \in \mathcal{B}(M)$. Let $E \in \mathcal{E}(M)$ be a conditional expectation. Then, as we have already mentioned, $L_g(E)$ is not necessarily a conditional expectation for all $g \in G_M$, but we can still consider the orbit of the expectation $\mathcal{S}(E) = \{L_g(E) : g \in G_M\}$ and the fibration $\Pi_E : G_M \to \mathcal{S}(E)$ given by $\Pi_E(g) = L_g(E)$, $g \in G_M$. The role, played in the unitary case by the normalizer $\mathcal{N}_E = \{u \in \mathcal{U}_M : L_u(E) = E\}$ as the isotropy group of the action, is now played by

$$I_E = \{g \in G_M : L_g(E) = E\}. \tag{13}$$

Let $N = E(M) \subseteq M$. Then N is a von Neumann algebra. Recall that the centralizer of E is the von Neumann algebra $M_E = \{x \in N' \cap M \ : \ E(xm) = E(mx) \text{ for all } m \in M\}$. Define the group

$$\mathcal{Z}_E = G_{M_E} \cdot G_N \subseteq I_E. \tag{14}$$

Proposition 3.1. *Let M be a von Neumann algebra, let $E \in \mathcal{E}(M)$, and consider the groups I_E and $\mathcal{Z}_E$ defined by* (13) *and* (14). *Then we have:*

(1) *If $g \in I_E$, then $gE(g^{-1}) = E(g^{-1})g \in M_E$.*
(2) *If $g \in I_E$ and $E(g^{-1})$ is invertible, then $g \in \mathcal{Z}_E$.*
(3) *$I_E \cap M^+ = \mathcal{Z}_E^+$.*

Proof. If $g \in I_E$, then $gE(g^{-1}) = gE(g^{-1}g^{-1}g) = E(g^{-1})g$. Let $N = E(M)$ and $b \in N$. Then

$$gE(g^{-1})b = gE(g^{-1}b) = gE(g^{-1}bg^{-1}g) = E(bg^{-1})g = bE(g^{-1})g,$$

so that $gE(g^{-1}) \in N' \cap M$. If $x \in M$, then, since $gNg^{-1} = N$, we have

$$\begin{aligned} E(gE(g^{-1})x) &= E(E(g^{-1})gx) = E(g^{-1})E(gx) \\ &= E(g^{-1})gE(xg)g^{-1} = E(xgE(g^{-1})), \end{aligned}$$

thus proving that $gE(g^{-1}) \in M_E$.

If $E(g^{-1})$ is invertible, then $g = gE(g^{-1}) \cdot E(g^{-1})^{-1} \in G_{M_E}G_N = \mathcal{Z}_E$ by (1). Finally, if $g \in I_E$ and $g > 0$, then $g^{-1} > 0$, and, as E is faithful, $E(g^{-1}) > 0$. Hence $E(g^{-1}) \in G_N$, and by (2) we have $g \in \mathcal{Z}_E$. □

Lemma 3.2. *If $g \in G_M$ and $\|g - 1\| < \varepsilon < 1$, then*

$$\|g^{-1} - 1\| < \frac{\varepsilon}{1 - \varepsilon}.$$

In particular, if $\varepsilon \leq 1/2$, then

$$\|g^{-1} - 1\| < 2\varepsilon.$$

The proof of this result is straightforward.

Proposition 3.3. *Let M be a von Neumann algebra, let $E \in \mathcal{E}(M)$, and consider the groups $\mathcal{Z}_E \subseteq I_E$ defined by* (13) *and* (14). *The group $\mathcal{Z}_E$ is open, closed and connected in I_E. Moreover, the connected component of I_E at any $u \in I_E$ is exactly $u \cdot \mathcal{Z}_E$.*

Proof. We first show that $\mathcal{Z}_E$ is open at 1. Let $g \in I_E$ be such that $\|g - 1\| < 1/2$. Then, by Lemma 3.2, we have $\|g^{-1} - 1\| < 1$ and thus $\|E(g^{-1}) - 1\| < 1$, which implies that $E(g^{-1})$ is invertible. Thus, by Proposition 3.1, we have $g \in \mathcal{Z}_E$.

Now let $h \in \mathcal{Z}_E$, and let V be a neighborhood of 1 such that $V \cap I_E \subseteq \mathcal{Z}_E$. Then hV is a neighborhood of h and, if $g \in hV$, then $h^{-1}g \in V$, so that $h^{-1}g \in \mathcal{Z}_E$. Since $h \in \mathcal{Z}_E$, we have $g \in \mathcal{Z}_E$, so $\mathcal{Z}_E$ is open. Clearly $g\mathcal{Z}_E$ is open for every $g \in I_E$, and since we can obtain I_E as a disjoint union of open sets $g\mathcal{Z}_E$, these sets are also closed. It is easily seen that $\mathcal{Z}_E$ is connected, since $\mathcal{Z}_E$ is the product of two connected groups. The last assertion of the proposition is now clear. □

We deduce that the group $\mathcal{Z}_E$ is closed, open and invariant in I_E. Thus we have a new Weyl group defined by

$$W_1(E) = \pi_0(I_E) \simeq I_E/\mathcal{Z}_E \tag{15}$$

We next show that this new Weyl group agrees with the old (unitary) group. We first need a lemma.

Lemma 3.4. *With the above notations, we have*

$$\mathcal{H}_E = \mathcal{U}_{M_E}\mathcal{U}_N = \mathcal{N}_E \cap G_{M_E}G_N = \mathcal{N}_E \cap \mathcal{Z}_E.$$

Proof. Let $w \in \mathcal{N}_E \cap \mathcal{Z}_E$. Then $w = mn$, with $m \in G_{M_E}, n \in G_N$. Using the polar decomposition of m and n and the fact that $M_E \subseteq N' \cap M$, we have

$$w = v_m|m| \cdot v_n|n| = v_m v_n|m||n| = v_m v_n|mn| = v_n v_m|w| = v_n v_m,$$

where $v_m \in \mathcal{U}_{M_E}, v_n \in \mathcal{U}_N$. Hence $w \in \mathcal{H}_E$. □

We now have the technical tools we need to prove that the two Weyl groups coincide.

Theorem 3.5. *Let M be a von Neumann algebra, and let $E \in \mathcal{E}(M)$. Then the Weyl group obtained by the unitary construction and the Weyl group obtained by the invertible construction are isomorphic, i.e., we have $W_1(E) = W(E)$.*

Proof. Let $\varphi : W(E) \to W_1(E)$ be given by $\varphi([u]_{W(E)}) = [u]_{W_1(E)}$, for $u \in \mathcal{N}_E$. We claim that this map is well defined and an isomorphism. That φ is well defined is clear, since $\mathcal{H}_E \subseteq \mathcal{Z}_E$. To show that φ is an isomorphism, note first that, by Lemma 3.4, φ is injective. To see that φ is onto, let $g \in I_E$. We must find a unitary element $u \in \mathcal{N}_E$ such that $[g]_{W_1(E)} = [u]_{W_1(E)}$.

Since $g \in I_E$, we have $(g^*)^{-1} \in I_E$ (by adjoining and using the fact that E is $*$-linear), and so $g^* \in I_E$, since I_E is a group. Therefore $g^*g \in I_E$ and, by Proposition 3.1, we have $g^*g \in \mathcal{Z}_E$. Hence there exist $m \in M_E$, $n \in N$ with $g^*g = mn$. Using again the polar decompositions $m = v_m|m|$ and $n = v_n|n|$ with $v_m \in \mathcal{U}_{M_E}$ and $v_n \in \mathcal{U}_N$, we obtain

$$\begin{aligned} g^*g = mn &= v_m|m| \cdot v_n|n| = v_m v_n|m||n| \\ &= v_m v_n|mn| = v_m v_n|g^*g| = v_m v_n g^*g. \end{aligned}$$

This implies that $v_m v_n = 1$, and so we can write $g^*g = mn$ with $m \in M_E^+$, $n \in N^+$. Hence $|g| = m^{1/2}n^{1/2} \in \mathcal{Z}_E$. By the polar decomposition of g, there is a unitary $u \in \mathcal{U}_M$ with $g = u|g|$. Since $|g| \in \mathcal{Z}_E$ and $g \in I_E$, it follows that $u \in \mathcal{U}_M \cap I_E = \mathcal{N}_E$, and so $[g] = [u]$. Finally, since both groups are discrete, the isomorphism φ is also a homeomorphism. □

Remark 3.6. Nearly all constructions given in this paper can be extended trivially to C* algebras. However, in Proposition 3.3 problems appear, since the invertible group of a C* algebra need not be connected.

4. Differential geometry of $\mathcal{S}(E)$

In this section we only consider von Neumann algebras with separable predual in order to ensure the existence of faithful normal states.

Let $N \subseteq M$ be von Neumann algebras, and let $E \in \mathcal{E}(M,N)$ and $\mathcal{S}(E) = \{gE(g^{-1} \cdot g)g^{-1} : g \in G_M\}$. The differential geometry of the orbit $\mathcal{S}(E)$ has been studied by Larotonda and Recht [21] under the assumption that $N' \cap M \subseteq N$. Larotonda and Recht showed that, in this case, $\mathcal{S}(E)$ admits a differentiable structure and the map $\Pi_E : G_M \to \mathcal{S}(E)$ defines a reductive structure on $\mathcal{S}(E)$.

The aim of this section is to remove this hypothesis. We will show that the orbit $\mathcal{S}(E)$ can always be given a differentiable structure, and even a unique reductive structure if N is finite. We will also show that the existence of reductive structures for all conditional expectations $E \in \mathcal{E}(M)$ forces the algebra M to be finite.

4.1. Differentiable structure.

We state here some definitions and three classical theorems from Banach-Lie group theory that will be used in the sequel. For a general reference on this subject, see, for example, [20] or [19].

Definition 4.1. Given a Lie-Banach group G (which may be complex analytic, real analytic, or C^∞), we denote by $L(G)$ the Lie algebra of G, which will always be identified (as a complex or real Banach space) with the tangent space $T_1(G)$ of G at the identity. A subgroup H of G is called a *regular* subgroup if it is also a Lie-Banach group (of the same type) and if T_1H is closed and complemented in T_1G.

Theorem 4.2. *Let G be a Lie group and let $H \subseteq G$ be a subgroup such that there exist open sets U, V with $0 \in U$, $1 \in V$ and a decomposition $T_1(G) = X \oplus Y$ (as a Banach space) satisfying*

(1) $\exp : U \to V$ *is a diffeomorphism,*
(2) $H \cap V = \exp(X \cap U)$.

Then H is a regular subgroup of G and $T_1(H) = X$.

Theorem 4.3. *Let G be a Lie group, and let $H \subseteq G$ be a regular subgroup. Then we have:*

1. *G/H has a unique structure of differentiable manifold such that $G \to G/H$ is a submersion.*
2. *$G \to G/H$ is a principal bundle with structure group H.*
3. *The action $G \times G/H \to G/H$ is smooth.*

Theorem 4.4. *If H is a subgroup of a Lie group G and the connected component H_1 of 1 in H is a regular subgroup of G, then H is a regular subgroup of G if and only if H_1 is open in H.*

In the following result we construct a conditional expectation that will be essential in order to characterize the tangent space of $\mathcal{S}(E)$ (see also [9]).

Proposition 4.5. *Let $N \subseteq M$ be von Neumann algebras and let $E \in \mathcal{E}(M,N)$. Fix a faithful normal state φ on N, and set $\psi = \varphi \circ E$. Then there exists a unique conditional expectation $F \in \mathcal{E}(M, M_E)$ such that $EF = FE$ and $\psi \circ F = \psi$.*

Proof. Denote by σ_t^ψ, $t \in \mathbb{R}$, the modular group of M induced by ψ. Since $\psi = \varphi \circ E = \psi \circ E$, we have $\sigma_t^\psi \circ E = E \circ \sigma_t^\psi$ for all $t \in \mathbb{R}$ (see [9] or [28]). By direct computation we deduce that $\sigma_t^\psi(M_E) = M_E$ for every $t \in \mathbb{R}$. We take $F \in \mathcal{E}(M, M_E)$ to be the unique expectation with $\psi \circ F = \psi$ obtained by Takesaki's theorem on the existence of conditional expectations (see [28]). Since $E|_{M_E} \in \mathcal{E}(M_E, \mathcal{Z}(N))$, we have $E \circ F \in \mathcal{E}(M, \mathcal{Z}(N))$ and $\psi \circ (E \circ F) = \psi$. Representing M as usual in $L^2(M,\psi)$, the three conditional expectations $E, F, E \circ F$ give rise to three orthogonal projections e, f, g with $g = ef$. Since $g = g^*$, we have $ef = fe$, and so $EF = FE$. □

Using the expectation $F : M \to M_E$ from Proposition 4.5, we can define

$$\Delta = E + F - EF \in \mathcal{B}(M). \tag{16}$$

Note that Δ is a projection, since E and F commute. The image of Δ is the closed subspace $M_E + N$ of M, which can also be written as a direct sum:

$$\operatorname{Im} \Delta = (M_E \cap \ker E) \oplus N.$$

Proposition 4.6. *With the preceding notations, $\mathcal{Z}_E$ is a regular subgroup of G_M and $T_1\mathcal{Z}_E = (M_E \cap \ker E) \oplus N$.*

Proof. In order to use Theorem 4.2, we need a decomposition

$$T_1 G_M = X \oplus Y$$

with $X = M_E + N$, the natural candidate for $T_1\mathcal{Z}_E$. Such a decomposition exists because, as G_M is open in M, we have $T_1 G_M = M$ and so the projection Δ introduced above gives the desired decomposition.

Note that the exponential map of the Banach-Lie group G_M coincides with the usual exponential map $m \mapsto e^m$, if we identify $L(G_M)$ with M. Since exp is a local diffeomorphism, we can fix an open set $0 \in U$ such that $\exp : U \to V = \exp(U)$ is a diffeomorphism. Let $0 \in U' \subseteq U$ be an open set and let $x \in U' \cap X$. Then $x = a + b$ with $a \in M_E$, $b \in N$, and since a and b commute, we have $\exp(a+b) = \exp(a)\exp(b)$ with $\exp(a) \in G_{M_E}$ and $\exp(b) \in G_N$. This shows that $\exp(U' \cap X) \subseteq \exp(U') \cap \mathcal{Z}_E$.

Let $0 < \delta < 1/2$ be such that

$$B_M(1, \delta) = \{y \in M : \|y - 1\| < \delta\} \subseteq V.$$

Let $y \in B_M(1,\delta) \cap \mathcal{Z}_E$, and let $g \in M_E$ and $h \in N$ be such that $y = gh$. Note that $F(h)$ is in $G_{\mathcal{Z}(N)}$. Indeed, since $h \in N$, we have $F(h) \in \mathcal{Z}(N)$. To see that $F(h)$ is invertible note that

$$\|gF(h) - 1\| = \|F(gh - 1)\| \leq \|gh - 1\| < \delta < 1.$$

Now write $y = gh = (gF(h))(F(h)^{-1}h)$. Then $\|gF(h) - 1\| < \delta$ as before and by Lemma 3.2 it follows that

$$\|F(h)^{-1}g^{-1} - 1\| < 2\delta.$$

Note also that $\|gh - 1\| < \delta < 1$ implies $\|gh\| < 2$. Collecting these estimates, we obtain

$$\begin{aligned} \|F(h)^{-1}h - 1\| &= \|F(h)^{-1}g^{-1}gh - 1\| \\ &\leq \|gh\|\|F(h)^{-1}g^{-1} - 1\| + \|gh - 1\| \\ &< 4\delta + \delta = 5\delta. \end{aligned}$$

Let $\varepsilon > 0$ and δ be small enough such that $B_M(0, 2\varepsilon) \subseteq U$ and

$$\exp^{-1}(B_M(1, 5\delta)) \subseteq B_M(0, \varepsilon).$$

Set $V' = B_M(1,\delta) \subseteq V$ and $U' = \exp^{-1}(V') \subseteq U$. Let $y \in V' \cap \mathcal{Z}_E$. Then $\exp^{-1}(y) \in U'$ and, since $gF(h)$ and $F(h)^{-1}h$ are in $B_M(1, 5\delta)$, their preimages $a = \exp^{-1}(gF(h)) \in M_E$ and $b = \exp^{-1}(F(h)^{-1}h) \in N$ satisfy $a + b \in U \cap X$. Since $\exp(a+b) = y = \exp(\exp^{-1}(y))$ and exp is injective in U, this implies $a + b = \exp^{-1}(y) \in U'$. Hence $\exp(U' \cap X) = V' \cap \mathcal{Z}_E$. □

Corollary 4.7. *Let M be a von Neumann algebra and let $E \in \mathcal{E}(M)$. Then, with the preceding notations, the isotropy group I_E is a regular subgroup of G_M.*

Proof. We already know that $\mathcal{Z}_E$ is a regular subgroup, that it is the connected component of 1 in I_E and that it is open in I_E. Thus, by Theorem 4.4, I_E is a regular subgroup. □

Theorem 4.8. *Let M be a von Neumann algebra and let $E \in \mathcal{E}(M)$ be a faithful normal conditional expectation. Then the similarity orbit $\mathcal{S}(E) \simeq G_M/I_E$, with the quotient topology of the norm topology of G_M, can be given a* *unique complex analytic manifold structure such that it is a homogeneous space (i.e., the map* $\Pi_E : G_M \to \mathcal{S}(E)$ *is a principal bundle with group structure* I_E *and* $\Pi_E : G_M \to \mathcal{S}(E)$ *is a submersion).*

Proof. Apply Corollary 4.7 and Theorem 4.3. □

Remark 4.9. The analogue of Theorem 4.8 remains true (with complex analytic replaced by real analytic) for the unitary orbit $\mathcal{U}(E) \simeq \mathcal{U}_M/\mathcal{N}_E$ under the action of the real analytic Banach-Lie group $\mathcal{U}_M$.

4.2. Reductive Structure. We now consider conditions under which we can find a reductive structure in $\mathcal{S}(E)$ and characterize such a structure. We first recall the definition of Homogeneous Reductive Spaces (see also [23]):

Definition 4.10. A Homogeneous Reductive Space (HRS) is a differentiable manifold $\mathcal{Q}$ and a smooth transitive action of a Banach-Lie group G on $\mathcal{Q}$, $L : G \times \mathcal{Q} \to \mathcal{Q}$ with the following properties:

(1) *Homogeneous Structure:* For each $\rho \in \mathcal{Q}$ the map
$$\begin{array}{rcl} \Pi_\rho : G & \to & \mathcal{Q} \\ g & \mapsto & L_g\rho \end{array}$$
is a principal bundle with structure group $I_\rho = \{g \in G : L_g\rho = \rho\}$ (called the isotropy group of ρ).

(2) *Reductive Structure:* For each $\rho \in \mathcal{Q}$ there exists a closed linear subspace H_ρ of the Lie algebra $L(G)$ of G such that $L(G) = H_\rho \oplus L(I_\rho)$, which is invariant under the natural action of I_ρ and such that the distribution $\rho \mapsto H_\rho$ is smooth.

In order to give an HRS structure to the orbit $\mathcal{S}(E)$ under the action of G_M, we must find a decomposition
$$L(G_M) = L(I_E) \oplus \mathcal{K}_E$$
such that the "horizontal" space $\mathcal{K}_E$ verifies
$$g(\mathcal{K}_E)g^{-1} = \mathcal{K}_E \quad \text{for all} \quad g \in I_E. \tag{17}$$
Recall that $L(G_M)$ can be identified with M, because G_M is open in M. Also, $L(I_E)$ can be regarded as T_1I_E, and also as $T_1(I_E)_1$ (where $(I_E)_1$ is the connected component of I_E at 1). Since, by Proposition 3.3, the connected component of I_E at 1 is $\mathcal{Z}_E$, we have $T_1I_E = T_1\mathcal{Z}_E = L(\mathcal{Z}_E)$. Also, by Proposition 4.6, we have $L(\mathcal{Z}_E) = M_E + N$. Hence, by (16), such a decomposition of M can be found. It remains to show that we can find a complement of $M_E + N$ verifying the equivariance property (17).

Lemma 4.11. *Let* $B \subseteq A$ *be algebras, let* $P : A \to B$ *be a linear projection and let* $g \in G_A$ *be such that* $g(\ker P)g^{-1} \subseteq \ker P$ *and* $gBg^{-1} = B$. *Then we have* $P(gxg^{-1}) = gP(x)g^{-1}$ *for every* $x \in A$.

The proof of this result is straightforward.

LEMMA 4.12. *Let $N \subseteq M$ be von Neumann algebras, and let $E \in \mathcal{E}(M,N)$ be a faithful normal conditional expectation. Suppose that there exists a faithful normal tracial state φ of N. Let $\psi = \varphi \circ E$ and let $F \in \mathcal{E}(M, M_E)$ be as in Proposition 4.5. Then we have:*

1. *The expectation F is unique in the sense that for any other faithful normal tracial state ρ in N, the expectation $F_\rho \in \mathcal{E}(M, M_E)$ induced by $\rho \circ E$ is equal to F.*
2. *We have $I_E \subseteq I_F = \{g \in G_M : gF(\cdot)g^{-1} = F(g \cdot g^{-1})\}$.*

Proof. We first show the uniqueness of F. Let ρ be a faithful normal tracial state of N, and let $F_\rho \in \mathcal{E}(M, M_E)$ be the corresponding expectation given by Proposition 4.5. Then $F_\rho|_N \in \mathcal{E}(N, \mathcal{Z}(N))$ is the center valued trace of N, since $\rho \circ F_\rho|_N = \rho$ (see, for example, [17, 8.3.10]). Then

$$\psi \circ F_\rho = \psi \circ E \circ F_\rho = \psi \circ F_\rho \circ E = \psi \circ F_\rho|_N \circ E = \psi \circ F|_N \circ E = \psi \circ F,$$

and so $F_\rho = F$.

To prove (2), fix $g \in I_E$. It is easy to see that $gM_Eg^{-1} = M_E$. Taking the polar decomposition $g = |g^*|u$, we see by Proposition 3.1 and the proof of Theorem 3.5 that $u \in I_E \cap \mathcal{U}_M = \mathcal{N}_E$ and $|g^*| \in \mathcal{Z}_E$. Let us first verify that $u \in I_F$. Indeed the expectation $F_u = L_u(F) = uF(u^* \cdot u)u^*$ satisfies $F_u \in \mathcal{E}(M, M_E)$, $F_u \circ E = E \circ F_u$, and $\varphi(u \cdot u^*) \circ E \circ F_u = \varphi(u \cdot u^*) \circ E$. Thus, F_u is the expectation which corresponds by Proposition 4.5 to the trace $\varphi(u \cdot u^*)$ of N. By part (1) of the lemma, it follows that $F_u = F$, and so $u \in I_F$. Therefore it suffices to show that $\mathcal{Z}_E \subseteq I_F$ and $G_N \subseteq I_F$ (since $\mathcal{Z}_E = G_{M_E}G_N$).

Let $g \in G_N$, $y \in M_E$, and $x \in \ker F$. Using the fact that $\psi \circ F = \psi$, we obtain

$$\begin{aligned} \psi(F(gxg^{-1})y) &= \psi(F(gxg^{-1}y)) = \psi(gxg^{-1}y) \\ &= \varphi(E(gxg^{-1}y)) = \varphi(gE(xg^{-1}yg)g^{-1}) \\ &= \varphi(E(xg^{-1}yg)) = \psi(F(xg^{-1}yg)) \\ &= \psi(F(x)g^{-1}yg) = 0. \end{aligned}$$

Since $F(gxg^{-1}) \in M_E$ and ψ is faithful, it follows that $F(gxg^{-1}) = 0$, and thus $g(\ker F)g^{-1} \subseteq \ker F$. By Lemma 4.11, we conclude that $I_E \subseteq I_F$. This proves part (2) of the lemma. □

PROPOSITION 4.13. *Let $N \subseteq M$ be von Neumann algebras, let $E \in \mathcal{E}(M,N)$ be a faithful normal conditional expectation, and assume that N is finite. Then the similarity orbit $\mathcal{S}(E)$ has a unique HRS structure under the action of G_M.*

Proof. To find a reductive structure, we need to construct a decomposition $L(G_M) = L(I_E) \oplus \mathcal{K}_E$, where $\mathcal{K}_E$ is invariant by inner conjugation of elements of I_E. Fix a faithful normal tracial state φ of N and let $F \in \mathcal{E}(M, M_E)$ be induced by φ as in Proposition 4.5 and Lemma 4.12. By the remarks preceding Proposition 4.6, it is clear that the projection $\Delta = I - (I - E)(I - F)$ gives the desired decomposition, i.e., we have $\mathcal{K}_E = \ker \Delta$.

It remains to show that I_E leaves $\mathcal{K}_E$ invariant, and that the distribution $L_g E \mapsto g\mathcal{K}_E$ is smooth. The first assertion holds since $\mathcal{K}_E = \ker F \cap \ker E$ and, by Lemma 4.12, $I_E \subseteq I_F$.

To see that the distribution is smooth, note that the projection onto $\mathcal{K}_E$ with kernel $M_E + N$ is $I - \Delta = D = (1 - E)(1 - F)$. By Lemma 4.12, the map $\eta : \mathcal{S}(E) \to \mathcal{B}(M)$ given by

$$\eta(\Pi_E(g)) = L_g D = (1 - L_g E)(1 - L_g F), \quad g \in G_M$$

is well defined and gives the desired decomposition for all $\Pi_E(g) \in \mathcal{S}(E)$. Consider the commutative diagram

$$\begin{array}{ccc} G_M & \xrightarrow{Ad} & Gl(\mathcal{B}(M)) \\ \Big\downarrow{\scriptstyle \Pi_E} & & \Big\downarrow{\scriptstyle \Pi_D} \\ \mathcal{S}(E) & \xrightarrow{\eta} & \mathcal{B}(M) \end{array}$$

where $\Pi_D(\alpha) = \alpha \circ D \circ \alpha^{-1}$, $\alpha \in Gl(\mathcal{B}(M))$. Since, by Theorem 4.8, Π_E has analytic local cross sections, the map η is clearly analytic.

The uniqueness follows from the fact that $\mathcal{K}_E$ (actually, the expectation F) does not depend on the tracial state φ. Indeed, it is easy to see that, for every faithful normal tracial state ρ of N with corresponding expectation $F_\rho \in \mathcal{E}(M, M_E)$ given by Proposition 4.5, the restriction $F_\rho|_N \in \mathcal{E}(N, \mathcal{Z}(N))$ is the center valued trace of N, since $\rho \circ F_\rho|_N = \rho$. Hence $F_\rho = F$. □

Remark 4.14. Let $N \subseteq M$ be von Neumann algebras, let $E \in \mathcal{E}(M, N)$ be a faithful normal conditional expectation, and assume that $M_E \subseteq N$, but that N is not necessarily finite. Then the assertion of Theorem 4.13 holds with the same proof. Indeed, in this case we have $\mathcal{Z}_E = G_N$, and one does not need a tracial state of N since $\Delta = E$. This result was given in [21] under the slightly more restrictive hypothesis that $N' \cap M \subseteq N$.

4.15. Let M be an infinite von Neumann algebra. Then there exists a properly infinite projection $p \in \mathcal{Z}(M)$ such that pM is properly infinite and $(1 - p)M$ is finite. Let τ be a faithful normal trace in $(1 - p)M$. Since p is properly infinite, it can halved, i.e., there exists a projection $q \in M$ such that $q \sim p - q \sim p$, where $\sim$ denotes the von Neumann equivalence of projections. Using this projection q, we can identify pM with $qMq \otimes M_2(\mathbb{C})$. Thus we identify M with $(qMq) \otimes M_2(\mathbb{C})) \oplus (1 - p)M$.

Let N be the subalgebra $(qMq \otimes 1) \oplus (1-p)\mathbb{C}$ of M. Consider the expectation $E \in \mathcal{E}(M, N)$ given by

$$E = (\mathrm{id} \otimes \mathrm{tr}_2) \oplus \tau.$$

In matrix form this expectation can be represented by

$$E\left(\begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & x \end{pmatrix}\right) = \begin{pmatrix} (a+d)/2 & 0 & 0 \\ 0 & (a+d)/2 & 0 \\ 0 & 0 & \tau(x) \end{pmatrix}.$$

Straightforward calculations show that

$$N' \cap M = \mathcal{Z}(qMq) \otimes M_2(\mathbb{C}) \oplus (1-p)M$$

and

$$M_E = N' \cap M.$$

If $\mathcal{S}(E)$ admits a Homogeneous Reductive Structure, then there exists a bounded linear projection $P : M \to N + M_E$ with $g(\ker P)g^{-1} = \ker P$ for all $g \in I_E$. Since $\mathcal{U}_N \subseteq I_E$, we have

$$P(uxu^*) = uP(x)u^* \text{ for every } u \in \mathcal{U}_N$$

by Lemma 4.11. Note that, since $(N + M_E)^* = N + M_E$, we can assume that P is $*$-linear. Indeed, if P is not $*$-linear, we can replace P by

$$P'(x) = \frac{1}{2}(P(x) + P(x^*)^*), \quad x \in M,$$

which is also a projection onto $N + M_E$ and satisfies

$$P'(uxu^*) = uP'(x)u^* \quad \text{for every} \quad u \in \mathcal{U}_N.$$

Since

$$(18) \quad N + M_E = \left\{ \begin{pmatrix} n & z_2 & 0 \\ z_3 & n + z_1 & 0 \\ 0 & 0 & m \end{pmatrix} : n \in qMq, z_i \in \mathcal{Z}(qMq), m \in (1-p)M \right\},$$

it is clear that the elements located at coordinates 21 and 12 of the image of P belong to $\mathcal{Z}(qMq)$. Consider the linear map $T : qMq \to \mathcal{Z}(qMq)$ given by

$$T(n) = \frac{1}{2}\left(P\begin{pmatrix} 0 & n & 0 \\ n & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}_{21} + P\begin{pmatrix} 0 & n & 0 \\ n & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}_{12} \right), \ n \in qMq,$$

where $(\cdot)_{ij}$ denotes the matrix entry with coordinates ij. We now establish the properties of T that we will be of interest to us.

PROPOSITION 4.16. *Let M be an infinite von Neumann algebra. Let p, q, N and $E \in \mathcal{E}(M,N)$ be as in 4.15. Assume that the orbit $\mathcal{S}(E)$ admits a Homogeneous Reductive Structure. Consider the linear maps P and T defined above. Then we have:*

(1) *$T : qMq \to \mathcal{Z}(qMq)$ is a $*$-linear mapping.*
(2) *T is a projection onto $\mathcal{Z}(qMq)$.*
(3) *If $u \in \mathcal{U}_{qMq}$, then $T(unu^*) = T(n)$ for every $n \in qMq$.*
(4) *$T(xy) = T(yx)$ for every $x, y \in qMq$.*

Proof. (1) That the image of T is in $\mathcal{Z}(qMq)$ can be seen from (18). The $*$-linearity is clear since, by assumption, P is $*$-linear.

(2) If $s \in \mathcal{Z}(qMq)$, then the matrix

$$\begin{pmatrix} 0 & s & 0 \\ s & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

is clearly in $N + M_E$, and hence is left invariant by P, and $T(s) = s$.

(3) Let $u \in \mathcal{U}_{qMq}$ and consider

$$U = \begin{pmatrix} u & 0 & 0 \\ 0 & u & 0 \\ 0 & 0 & 1 \end{pmatrix} \in \mathcal{U}_N \subseteq I_E.$$

The basic property of P is that $P(UmU^*) = UP(m)U^*$ for every $m \in M$. But this clearly implies that $T(unu^*) = uT(n)u^* = T(n)$ for every $n \in qMq$.

(4) This follows from (3) since the unitaries generate the entire algebra. □

THEOREM 4.17. *Let M be a von Neumann algebra. Then the following conditions are equivalent:*

(1) *The similarity orbit $\mathcal{S}(E)$ of any expectation $E \in \mathcal{E}(M)$ can be given an HRS structure under the action of G_M.*
(2) *M is a finite von Neumann algebra.*

Proof. Let p be the largest projection in $\mathcal{Z}(M)$ such that pM is properly infinite, and let q be a subprojection of p that halves p, that is $q \sim p - q \sim p$. We will use the notations of 4.15 and the conditional expectation $E \in \mathcal{E}(M)$ considered there. If condition (1) holds, then, using Proposition 4.16 and 4.15, we can construct a "tracial" bounded projection $T : qMq \to \mathcal{Z}(qMq)$. Since q is also properly infinite, there is a projection $r \in qMq$ such that $r \sim q - r \sim q$ in qMq. Using the "traciality" of T, we have

$$T(q) = T(q-r) = T(q) - T(r) = T(q) - T(q) = 0. \tag{19}$$

Recall that, by Proposition 4.16, we have $T(q) = q$. By (19) it follows that $q = 0$, which in turn implies $p = 0$. Thus M is a finite von Neumann algebra.

Conversely, suppose that M is finite and $E \in \mathcal{E}(M)$. Then $N = E(M)$ is a finite von Neumann algebra and we can apply Proposition 4.13. □

REMARK 4.18. Let $N \subseteq M$ be von Neumann algebras and let $E \in \mathcal{E}(M, N)$ be such that $\mathcal{S}(E)$ has a structure of HRS. We will describe explicitly the geometrical invariants of $\mathcal{S}(E)$. We first compute the tangent map at 1 of the fibration $\Pi_E : G_M \to \mathcal{S}(E)$. For simplicity we consider $\mathcal{S}(E) \subseteq \mathcal{B}(M)$, despite the fact that the topology of $\mathcal{S}(E)$ is, in general, not that induced by $\mathcal{B}(M)$. In this sense, for $x \in M$, we have

$$(\mathrm{T}\ \Pi_E)_1(x) = [x, E(\cdot)] - E([x, \cdot]),$$

where $[x, y] = xy - yx$ for $x, y \in M$. Indeed, let $x \in M$ and consider the curve $\alpha(t) = e^{tx}$. Note that $\alpha(0) = 1$ and $\dot{\alpha}(0) = x$. Then

$$\begin{aligned}
(\mathrm{T}\ \Pi_E)_1(x) &= \frac{d}{dt}(\Pi_E(e^{tx}))|_{t=0} \\
&= \frac{d}{dt}\left(\mathrm{Ad}(e^{tx}) \circ E \circ \mathrm{Ad}(e^{-tx})\right)|_{t=0} \\
&= \left((\mathrm{Ad}(e^{tx}))' \circ E \circ \mathrm{Ad}(e^{-tx}) + \mathrm{Ad}(e^{tx}) \circ E \circ (\mathrm{Ad}(e^{-tx}))'\right)|_{t=0} \\
&= \left((\mathrm{Ad}(e^{tx}))([x, E \circ \mathrm{Ad}(e^{-tx})]) + \mathrm{Ad}(e^{tx}) \circ E \circ (\mathrm{Ad}(e^{-tx})([\cdot, x])\right)|_{t=0} \\
&= [x, E(\cdot)] - E([x, \cdot]).
\end{aligned}$$

An interesting computation using this formula shows, as it must, that

$$\ker(\mathrm{T}\ \Pi_E)_1 = M_E + N = L(I_E).$$

On the other hand, if $\mathcal{K}_E = \ker \Delta$ is the horizontal space at E of $\mathcal{S}(E)$, then

$$(\mathrm{T}\ \Pi_E)_1|_{\mathcal{K}_E} : \mathcal{K}_E \to T(\mathcal{S}(E))_E$$

is an isomorphism. It is usual to consider the inverse map $K_E : T(\mathcal{S}(E))_E \to \mathcal{K}_E$ in order to identify tangent vectors with elements of M (see, for instance, [23]). With this convention we now describe the torsion and curvature tensors, T and R, respectively. Let V, W and $Z \in T(\mathcal{S}(E))_E$. Then we have:

(1) $T(V, W) = (\mathrm{T}\ \Pi_E)_1([K_E(V), K_E(W)])$.
(2) $R(V, W)Z = (\mathrm{T}\ \Pi_E)_1([K_E(Z), \Delta([K_E(V), K_E(W)])])$.
(3) The unique geodesic γ at E such that $\dot{\gamma}(0) = V$ is given by

$$\gamma(t) = L_{e^{tK_E(V)}} E.$$

(4) The exponential map of $\mathcal{S}(E)$ is given by

$$\exp_E(X) = L_{e^{K_E(X)}} E \quad \text{for} \quad X \in T(\mathcal{S}(E))_E.$$

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