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LINEAR RESOLVENT GROWTH OF A WEAK CONTRACTION DOES NOT IMPLY ITS SIMILARITY TO A NORMAL OPERATOR

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ABSTRACT. It was shown in [1] that if T is a contraction in a Hilbert space with finite defect (i.e., $\|T\| \leq 1$ and $\operatorname{rank}(I - T^*T) < \infty$), and if the spectrum $\sigma(T)$ does not coincide with the closed unit disk $\overline{\mathbb{D}}$, then the Linear Resolvent Growth condition

$$\|(\lambda I - T)^{-1}\| \leq \frac{C}{\operatorname{dist}(\lambda, \sigma(T))},\ \lambda \in \mathbb{C}\backslash\sigma(T)$$

implies that T is similar to a normal operator.

The condition $\operatorname{rank}(I - T^*T) < \infty$ measures how close T is to a unitary operator. A natural question is whether this condition can be relaxed. For example, it was conjectured in [1] that this condition can be replaced by the condition $I - T^*T \in \mathfrak{S}_1$, where $\mathfrak{S}_1$ denotes the trace class. In this note we show that this conjecture is not true, and that, in fact, one cannot replace the condition $\operatorname{rank}(I - T^*T) < \infty$ by any reasonable condition of closeness to a unitary operator.

Notation

We denote by $\mathbb{D}$ the unit disk $\{z \in \mathbb{C} : |z| < 1\}$ in the complex plane $\mathbb{C}$.

We write $s_n(A)$ for the singular number of the operator A, defined by

$$s_n(A) = \inf\{\|A - K\| : \operatorname{rank} K \leq n\}, \quad s_0(A) = \|A\|.$$

For a compact operator A, the sequence $s_k(A)^2$, $k = 0, 1, 2, \ldots$, is exactly the system of eigenvalues of A^*A (counting multiplicities) taken in decreasing order.

For $p > 0$, we denote by $\mathfrak{S}_p$ the Schatten–von-Neumann class of compact operators A such that $\sum_{k=1}^{\infty} s_k(A)^p < \infty$, and we write $\|A\|_{\mathfrak{S}_p} := \left(\sum_0^{\infty} s_n(A)^p\right)^{1/p}$ for the norm in $\mathfrak{S}_p$.

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0. Introduction and main results

In this note we are concerned with the question of similarity of an operator to a normal operator. We recall that two operators A and B are similar if there exists a (bounded) invertible operator R such that $A = RBR^{-1}$. Similarity of an operator T to a normal operator means that the operator T admits a rich functional calculus, so that, for example, $f(T)$ is well defined for any continuous function f on the complex plane $\mathbb{C}$.

We first give a brief overview of the history of this question. Probably the first criterion for the similarity of a contraction to a unitary operator was given in a paper by B. Sz.-Nagy and C. Foias [10]. (Recall that an operator T is called a contraction if $\|T\| \leq 1$.) This result was transformed into a resolvent test by I. Gohberg and M. Krein [5]. Further progress on the subject was made by N. Nikolski and S. Khruschev [8] who obtained a counterpart of the Gohberg–Klein result for contractions with spectra inside the unit disk $\mathbb{D}$ and defect operators of rank one. In [1], N.E. Benarama and N. Nikolski generalized this test to contractions of arbitrary finite defects.

Since for a normal operator N the norm of the resolvent can be computed as

$$\|(N - \lambda I)^{-1}\| = \frac{1}{\operatorname{dist}(\lambda, \sigma(N))},$$

the condition

$$\|(T - \lambda I)^{-1}\| \leq \frac{C}{\operatorname{dist}(\lambda, \sigma(T))}, \tag{0.1}$$

which we will call the *Linear Resolvent Growth* (LRG) condition, is necessary for the operator T to be similar to a normal operator. However, this condition is clearly not sufficient for similarity to a normal operator: multiplication by the independent variable z on the Hardy space H^2 clearly satisfies (0.1), but the similarity property does not hold.

However, if the spectrum of an operator is "thin" and the operator is close to a "good" operator, one can expect that the LRG condition (0.1) is sufficient for similarity to a normal operator.

In [1] it was shown that, if a contraction T is close to a unitary operator in the sense that it has a finite rank defect $I - T^*T$, and its spectrum does not coincide with the closed unit disk $\overline{\mathbb{D}}$, then LRG implies similarity to a normal operator. It was also shown that for a contraction T the condition $I - T^*T \in \mathfrak{S}_p$, $p > 1$, where $\mathfrak{S}_p$ stands for the Schatten–von-Neumann class, is not sufficient, and it was conjectured that the condition $I - T^*T \in \mathfrak{S}_1$ (together with the assumption that the spectrum is not the whole closed unit disk $\overline{\mathbb{D}}$) guarantees the equivalence of LRG and similarity to a normal operator.

We will show in this note that this is not the case, i.e., that one can find a contraction T, with simple countable spectrum and such that $I - T^*T \in \mathfrak{S}_1$

(or even $I - T^*T \in \cap_{p>0}\mathfrak{S}_p$), which satisfies LRG, but is not similar to a normal operator. Furthermore, we will show that no reasonable condition of closeness to a unitary operator (except for the finite rank defect of $I - T^*T$) implies that LRG is equivalent to similarity to a normal operator.

Let us explain what we mean by a "reasonable" condition. Suppose we have a function Φ (that measures how small an operator (defect) is) with values in $\mathbb{R}_+ \cup \{\infty\}$, which is defined on the set of non-negative operators in a Hilbert space H, satisfies $\Phi(\mathbf{0}) = 0$ and has the following properties:

(1) Φ is increasing, i.e., $\Phi(A) \le \Phi(B)$ if $A \le B$;
(2) $\Phi(A) < \infty$ if $\operatorname{rank} A < \infty$;
(3) Φ is upper semicontinuous, i.e., if $A_n \nearrow A$ (that is, $A_n \le A$ and $\|A_n - A\| \to 0$), then $\Phi(A) \le \lim_n \Phi(A_n)$;
(4) Φ is lower semicontinuous in the following weak sense: if $\operatorname{rank} A < \infty$, and $\operatorname{rank} A_n \le N$ for some $N < \infty$, and $\lim_n \|A_n\| = 0$, then $\lim_n \Phi(A \oplus A_n) = \Phi(A)$ (where $A \oplus B$ means that $\operatorname{range} A \perp \operatorname{range} B$ and $(\operatorname{Ker} A)^\perp \perp (\operatorname{Ker} B)^\perp$).

We extend Φ to non-selfadjoint operators by putting $\Phi(A) := \Phi((A^*A)^{1/2})$.

The following are examples of functions Φ of this type:

(1) $\Phi(A) = \|A\|_{\mathfrak{S}_p} = \left(\sum s_n(A)^p\right)^{1/p}$, where $s_n(A)$ is nth singular value of the operator A. In this case $\Phi(A) < \infty$ means exactly $A \in \mathfrak{S}_p$;
(2) $\Phi(A) := \sum_{n=1}^\infty 2^{-n} \|A\|_{\mathfrak{S}_{1/n}} / (1 + \|A\|_{\mathfrak{S}_{1/n}})$; in this case, $\Phi(A) < \infty$ if and only if $A \in \bigcap_{p>0} \mathfrak{S}_p$;
(3) Any weighted sum of singular numbers, such as

$$\Phi(A) = \sum_1^\infty 2^{2^n} s_n(A);$$

(4) The function

$$\Phi_\psi(A) := \sum_0^\infty \psi(s_n(A)),$$

where $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ is increasing, continuous at 0, and satisfies $\psi(0) = 0$. The condition $\Phi_\psi(A) < \infty$ characterizes the class $\mathfrak{S}_\psi$, introduced in [1], i.e., we have $A \in \mathfrak{S}_\psi$ if and only if $\Phi_\psi(A) < \infty$. Note that if we allow $\psi(0)$ to be positive, then for any ψ satisfying $\psi(0) > 0$ the class $\mathfrak{S}_\psi$ is just the ideal of finite rank operators.

Our main result is the following theorem.

THEOREM 0.1. *Let Φ be a function satisfying the conditions* (1)–(4) *above. Given $\varepsilon > 0$, there exists a contraction T on a Hilbert space H with the following properties.*

(1) *The spectrum $\sigma(T)$ is a countable subset of the closed unit disk $\overline{\mathbb{D}}$;*

(2) *$T = I + K$, where $\Phi(K) \le \varepsilon$ and $\Phi(K^*) \le \varepsilon$;*
(3) *$\Phi(I - T^*T) \le \varepsilon$ and $\Phi(I - TT^*) \le \varepsilon$;*
(4) *T satisfies the Linear Resolvent Growth condition*

$$\|(T - \lambda I)^{-1}\| \le \frac{C}{\operatorname{dist}(\lambda, \sigma(T))};$$

(5) *T is not similar to a normal operator.*

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1. Proof of the main result

1.1. Preliminaries about bases. Before proceeding to the proof, we recall some well-known facts about bases in a Hilbert space. An exhaustive treatment of the subject can be found on pages 131–133 and 135–142 of the monograph [7]. (See also the papers [12, 13, 14].)

Let $\{f_n\}_1^\infty$ be a complete system of vectors in a Hilbert space H. The system is called a *basis* if any vector $f \in H$ admits a unique decomposition

$$f = \sum_1^\infty c_n f_n,$$

where the series converges (in the norm of H), and the system is called an *unconditional basis* if it is a basis and the series converges unconditionally (i.e., converges for any reordering).

A complete system is called a Riesz basis if it is equivalent to the orthonormal basis, i.e., if there exists a bounded invertible operator R (the so-called *orthogonalizer*) such that $Rf_n = e_n$ for all n, where $\{e_n : n = 1, 2, \ldots\}$ is some orthonormal basis. Clearly, an orthogonalizer is unique up to a unitary factor on the left. The quantity $r(\{f_n\}) := \|R\| \cdot \|R^{-1}\|$ is therefore well defined and could serve as a measure of non-orthonormality of the Riesz basis $\{f_n\}$.

Clearly, a Riesz basis is an unconditional basis. Although we do not need this in this paper, we note that the converse is also true: a theorem due to Köthe and Töplitz states that a normalized unconditional basis (with $0 < \inf \|f_n\| \le \sup \|f_n\| < \infty$) is a Riesz basis.

We also mention the connection between Riesz bases and similarity to normal operators. It is a trivial observation that if T is an operator with simple eigenvalues and with a complete system of eigenvectors f_n, $n = 1, 2, \ldots$, then T is similar to a normal operator if and only if the system of eigenvectors is a Riesz basis. In this case the similarity transformation is given by an orthogonalizer R, and RTR^{-1} is a normal operator.

1.2. Global construction. Suppose we have constructed a sequence of finite rank operators $A_n : \mathbb{C}^n \to \mathbb{C}^n$, with simple spectrum, and let $\{f_k^n\}_{k=1}^n$ be the system of normalized (i.e., $\|f_k^n\| = 1$) eigenvectors of A_n. Suppose, moreover, that the operators A_n (which we do not require to be contractions) have the following properties:

(1) The operators A_n satisfy LRG uniformly, i.e., we have

$$\|(A_n - \lambda I)^{-1}\| \le \frac{C}{\operatorname{dist}(\lambda, \sigma(A_n))},$$

where the constant C does not depend on n.

(2) We have $\lim_n r(\mathcal{F}_n) = \infty$, where $r(\mathcal{F}_n) = \|R_{\mathcal{F}_n}\| \cdot \|R_{\mathcal{F}_n}^{-1}\|$ is the measure of non-orthogonality of the system $\mathcal{F}_n = \{f_k^n\}_{k=1}^{N_n}$ of the eigenvectors of A_n. (Recall that $R_{\mathcal{F}_n}$ is the *orthogonalizer* of the system $\mathcal{F}_n$.)

We now show that this implies the assertion of Theorem 0.1.

We construct an operator $T = \oplus_{n=1}^{\infty}\big(a_n A_n + b_n I\big)$, where $|b_n| < 1$, $\lim_n b_n = 1$ and $\lim_n a_n = 0$. We choose the numbers a_n and b_n such that the spectra of the summands $a_n A_n + b_n I$ do not intersect, so that the resulting operator has a simple spectrum.

Since the linear transformation $A \mapsto aA + bI$ does not change the LRG condition, and, moreover, does not change the constant in this condition (we leave the proof of this fact as a simple exercise for the reader), the operator T satisfies $\|(T - \lambda I)^{-1}\| \le C/\operatorname{dist}(\lambda, \sigma(T))$.

Furthermore, since the same linear transformation does not change the system of eigenvectors, we can conclude that the system $\mathcal{F}$ of eigenvectors of T is the direct sum of eigenvectors of all A_n, i.e., $\mathcal{F} := \oplus_{n=1}^{\infty}\mathcal{F}_n$.

Since $r(\mathcal{F}_n) \to \infty$ by Property (2) of A_n, the system $\mathcal{F}$ of eigenvectors of T is not a Riesz basis, and therefore (since T has simple spectrum) T is not similar to a normal operator.

It remains to show that one can choose numbers a_n and b_n such that the operator T is close to a unitary operator, in the sense that $\Phi(I - T) \le \varepsilon$, $\Phi(I - T)^* \le \varepsilon$, $\Phi(I - T^*T) \le \varepsilon$, and $\Phi(I - TT^*) \le \varepsilon$.

We will construct the numbers a_n, b_n by induction. We will always take a_n to satisfy $|a_n| \cdot \|A_n\| < 1 - |b_n|$. Under this assumption we have

$$\|I - T_n\| < 1 - |b_n| + |1 - b_n| \le 2 \cdot |1 - b_n|.$$

The simple identity $(I-\Delta)^*(I-\Delta) = I - \Delta - \Delta^* - \Delta^*\Delta$ (applied to $\Delta = I - T_n$, $\Delta = I - T_n^*$) implies that in this case

$$\|I - T^*T\|,\ \|I - TT^*\| < 6 \cdot |1 - b_n|,$$

if $|1 - b_n| \le 1/2$.

Therefore, by taking b_n sufficiently close to 1 (and a_n so that $|a_n| \cdot \|A_n\| < 1 - |b_n|$ holds) we can make the norms of the finite rank operators $I - T_n$, $I - T_n^* T_n$, and $I - T_n T_n^*$, where $T_n = a_n A_n + b_n I$, as small as we want.

Since $\Phi(\mathbf{0}) = 0$, Property (4) of Φ implies that we can choose a contraction $T_1 = a_1 A_1 + b_1 I$ such that

$$\Phi(I - T_1) \le \varepsilon/2, \qquad \Phi(I - T_1)^* \le \varepsilon/2,$$
$$\Phi(I - T_1^* T_1) \le \varepsilon/2, \qquad \Phi(I - T_1 T_1^*) \le \varepsilon/2.$$

Assume we have constructed the finite rank contractions $T_k = a_k A_k + b_k I$, $k = 1, 2, \ldots, n-1$, such that the operator $T^{(n-1)} = T_1 \oplus T_2 \oplus \ldots \oplus T_{n-1}$ satisfies $\|T^{(n-1)}\| < 1$, has simple spectrum, and satisfies

$$\Phi(I - T^{(n-1)}) \le (1 - 2^{-(n-1)})\varepsilon,$$
$$\Phi(I - T^{(n-1)*}) \le (1 - 2^{-(n-1)})\varepsilon,$$
$$\Phi(I - T^{(n-1)*} T^{(n-1)}) \le (1 - 2^{-(n-1)})\varepsilon,$$
$$\Phi(I - T^{(n-1)} T^{(n-1)*}) \le (1 - 2^{-(n-1)})\varepsilon.$$

By making the norm $\|I - T_n\|$ sufficiently small we can guarantee that the operator $T^{(n)} = T_1 \oplus T_2 \oplus \ldots \oplus T_n$ has simple spectrum and satisfies $\|T^{(n)}\| < 1$. Moreover, Property (4) of Φ implies that one can choose $T^{(n)}$ so that, in addition,

$$\Phi(I - T^{(n)}) \le (1 - 2^{-n})\varepsilon,$$
$$\Phi(I - T^{(n)*}) \le (1 - 2^{-n})\varepsilon,$$
$$\Phi(I - T^{(n)*} T^{(n)}) \le (1 - 2^{-n})\varepsilon,$$
$$\Phi(I - T^{(n)} T^{(n)*}) \le (1 - 2^{-n})\varepsilon.$$

Property (3) of Φ implies that the operator $T = \oplus_{n=1}^{\infty} T_n$ satisfies

$$\Phi(I - T) \le \varepsilon, \qquad \Phi(I - T^*) \le \varepsilon$$
$$\Phi(I - T^* T) \le \varepsilon, \qquad \Phi(I - T T^*) \le \varepsilon.$$

This completes the proof of Theorem 0.1, modulo the constructing of A_n.

1.3. More preliminaries about bases. We will need more information about bases. Let f_n, $n = 1, 2, \ldots$, be a linearly independent sequence of vectors. Let P_n denote the projection onto the first n vectors of the system, defined by $P_n \sum c_k f_k = \sum_1^n c_k f_k$. (The operators P_n are well defined on finite linear combinations of f_k.) The following characterization of bases is well-known; see, for example, [11, pp. 46–47], or [15, pp. 37–39].

THEOREM 1.1 (Banach Basis Theorem). *A complete system of vectors f_k, $k = 1, 2, \ldots$, is a basis if and only if* $\sup_n \|P_n\| =: K < \infty$.

If one *a priori* assumes that the projections P_n are bounded, then the theorem is just the Banach–Steinhaus Theorem.

We will need the following corollary characterizing the bases in terms of so-called *multipliers*. For a numerical sequence $\alpha := \{\alpha_n\}_1^\infty$, let M_α be a *multiplier*, defined by

$$M_\alpha f_n = \alpha_n f_n, \qquad n = 1, 2, \dots$$

(A priori, M_α is defined only on finite linear combinations $\sum c_k f_k$.) For a sequence α its variation $\operatorname{var}(\alpha)$ is defined by

$$\operatorname{var}\alpha := \sum_1^\infty |a_k - a_{k+1}|.$$

Clearly, if $\operatorname{var}\alpha < \infty$, the limit $\lim_n \alpha_n =: \alpha_\infty$ exists and is finite.

Corollary 1.2. *Let a system of vectors f_n, $n = 1, 2, \dots$, be a basis. If for a numerical sequence $\alpha = \{\alpha_n\}_1^\infty$ we have $\operatorname{var}\alpha < \infty$, then*

$$\|M_\alpha\| \le K \operatorname{var}\alpha + |\alpha_\infty|,$$

where K is the constant from the Banach Basis Theorem (Theorem 1.1), and $\alpha_\infty := \lim_n \alpha_n$.

Proof. The result follows immediately from the formula

$$M_\alpha = \sum_{n=1}^\infty (\alpha_n - \alpha_{n+1}) P_n + \alpha_\infty I,$$

where the operators P_n are the projections in the Banach Basis Theorem. □

Remark 1.3. The above corollary holds for bases in finite-dimensional spaces as well: one simply has to extend the finite sequence α to an infinite sequence, by adding zeroes.

Remark 1.4. Although we do not need this fact here, we mention that the converse of Corollary 1.2 is also true. Namely, a system of vectors f_n, $n = 1, 2, \dots$, is a basis if and only if for any numerical sequence α of bounded variation the corresponding multiplier M_α is bounded. The proof is quite easy; see [7, 11].

1.4. Construction of the operators A_n. To construct the operators A_n described in Section 1.2, consider a normalized ($\|f_n\| = 1$) system of vectors $\mathcal{F} := \{f_n\}_1^\infty$, which is a basis but not a Riesz basis. Such systems do exist; an example is given in Section 2 below. The measure of non-orthogonality of this system is

$$r(\mathcal{F}) := \|R_\mathcal{F}\| \cdot \|R_\mathcal{F}^{-1}\| = \infty.$$

Therefore, for finite truncations $\mathcal{F}_n = \{f_k\}_{k=1}^n$ we have

$$r(\mathcal{F}_n) := \|R_{\mathcal{F}_n}\| \cdot \|R_{\mathcal{F}_n}^{-1}\| \to \infty \qquad \text{as } n \to \infty.$$

We define operators A_n as follows. Let $\{\lambda_n\}_1^\infty$ be a strictly increasing sequence of real numbers. Define an operator A_n on $\mathcal{L}\{f_k : k = 1, \dots N_n\}$ by $A_n f_k = \lambda_k f_k$. It is easy to see that the operator A_n has simple spectrum, and that Property (2) of A_n is satisfied.

We have to show that Property (1) holds, i.e., that

$$\|(A_n - \lambda I)^{-1}\| \le \frac{C}{\operatorname{dist}(\lambda, \sigma(A_n))} .$$

To estimate the norm $\|(A_n - \lambda I)^{-1}\|$ we will use Corollary 1.2. Namely, if we put $\alpha := \{\alpha_k\}_1^\infty$ with

$$\alpha_k = \begin{cases} (\lambda_k - \lambda)^{-1}, & k \le n, \\ 0, & k > n , \end{cases}$$

then

$$\|(A_n - \lambda I)^{-1}\| \le \|M_\alpha\| \le K \cdot \operatorname{var} \alpha.$$

Thus, we need to show that

$$\operatorname{var} \alpha \le \frac{C}{\operatorname{dist}(\lambda, \sigma(A_n))} .$$

Suppose first that $\lambda_m \le \operatorname{Re} \lambda < \lambda_{m+1}$ for some $m \in \{1, 2, \dots, n-1\}$. Then

$$\operatorname{var} \alpha = \sum_{k=1}^{m-1} |\alpha_k - \alpha_{k+1}| + \sum_{k=m+1}^{n-1} |\alpha_k - \alpha_{k+1}| \quad + \quad |\alpha_m - \alpha_{m+1}| + |\alpha_n|.$$

The last two terms are easy to estimate:

$$|\alpha_m - \alpha_{m+1}| + |\alpha_n| \le |\alpha_m| + |\alpha_{m+1}| + |\alpha_n| \le \frac{3}{\operatorname{dist}(\lambda, \sigma(A_n))} .$$

For the first term, we use the estimate

$$\begin{aligned} \sum_{k=1}^{m-1} |\alpha_k - \alpha_{k+1}| &\le \sum_{k=1}^{m-1} \left| \frac{1}{\lambda_k - \lambda} - \frac{1}{\lambda_{k+1} - \lambda} \right| \\ &= \sum_{k=1}^{m-1} \left| \int_{\lambda_k}^{\lambda_{k+1}} \frac{dz}{(z-\lambda)^2} \right| \le \int_{\lambda_1}^{\lambda_m} \frac{dz}{|z-\lambda|^2} \le \frac{C}{|\lambda - \lambda_m|} . \end{aligned}$$

Similarly, we have

$$\sum_{k=m+1}^{n-1} |\alpha_k - \alpha_{k+1}| \le \frac{C}{|\lambda - \lambda_m|},$$

and the desired estimate follows.

In the cases when $\operatorname{Re} \lambda < \lambda_1$ or $\operatorname{Re} \lambda \ge \lambda_n$, the same argument applies, with only one sum. Hence we are done. □

REMARK 1.5. The fact that the operators A_n satisfy LRG follows immediately from a more general result about operators with spectrum on Ahlfors curves, proved in [1]. We gave the proof here only for the reader's convenience.

Note that the above argument would also work if we consider different monotone sequences $\{\lambda_k^n\}_{k=1}^n$, $n = 1, 2, \ldots$, and put $A_n f_k := \lambda_k^n f_n$.

2. Nontrivial conditional bases

Let us consider the space $L^2(w)$, where $w(t)$ is a nonnegative measurable function on the unit circle $\mathbb{T} = \partial\mathbb{D}$ and

$$\|f\|^2_{L^2(w)} := \int_{-\pi}^{\pi} |f(e^{it})|^2 w(e^{it}) \frac{dt}{2\pi} \,.$$

We will study properties of the system of exponents $\{z^n\}_{n=0}^\infty$. We have the following result.

PROPOSITION 2.1 ([14]). *Consider the system of exponents $\{z^n\}_{n=0}^\infty$ in the closed linear span in $L^2(w)$ that it generates.*

(1) *$\{z_n\}$ is a basis if and only if the weight w satisfies the Muckenhoupt (A_2) condition*

$$\sup_I \left(\frac{1}{|I|}\int_I w\right)\cdot\left(\frac{1}{|I|}\int_I w^{-1}\right) < \infty.$$

(2) *$\{z_n\}$ is an unconditional (Riesz) basis if and only if $w \in L^\infty(\mathbb{T})$ and $1/w \in L^\infty(\mathbb{T})$.*

Direct computations show that a weight with power singularity, say $w(z) = |z-1|^\alpha$ satisfies the Muckenhoupt (A_2)-condition if and only if $-1 < \alpha < 1$. By choosing any non-zero α in this interval we get an example of a basis which is not an unconditional (Riesz) basis.

Proof of Proposition 2.1. The statement is probably well-known, and we present the proof only for the reader's convenience.

By the Banach Basis Theorem (Theorem 1.1 above) the system $\{z^n\}_{n=0}^\infty$ is a basis if and only if the projections P_n defined by $P_n(\sum c_k z^k) = \sum_{k=0}^n c_k z^k$ are uniformly bounded.

Consider the so-called Riesz projection P_+, defined by $P_+(\sum c_k z^k) = \sum_{k=0}^\infty c_k z^k$. Since for $f \in \mathcal{L}(z^n : n \geq 0)$

$$P_n f = f - z^{n+1} P_+(\overline{z}^{n+1} f),$$

and multiplication by the independent variable z is a unitary operator on $L^2(w)$, it is easy to show that the operators P_n are uniformly bounded (on the closed linear span of $\{z^n\}_{n=0}^\infty$ in $L^2(w)$) if and only if the operator P_+ is bounded on $L^2(w)$. The latter condition is equivalent to the boundedness of the Hilbert Transform T given by $T := -iP_+ i(I - P_+)$, and it is well known (see [6] or [3, p. 254]) that T is bounded on $L^2(w)$ if and only if the weight w satisfies the Muckenhoupt (A_2)-condition. This proves part (1) of Proposition 2.1.

To prove part (2), note that the system of exponents is a Riesz basis if, for any analytic polynomial $f = \sum_{k=0}^{N} c_k z^k$,

$$c\|f\|^2_{L^2(w)} \le \sum |c_k|^2 = \|f\|^2_{L^2} \le C\|f\|^2_{L^2(w)} .$$

Since the multiplication by z is a unitary operator on $L^2(w)$, the last estimate should hold for any *trigonometric* polynomial $f = \sum_{-N}^{N} c_k z^k$. This is possible if and only if w and $1/w$ belong to L^∞. □

3. Linear fractional transformations and the Linear Resolvent Growth condition

The main reason why Theorem 0.1 holds is that LRG and similarity to a normal operator are both “Möbius invariant”, while the conditions like $I - T^*T \in \mathfrak{S}_p$ are not, if one pays attention to constants.

Let us clarify this statement. First, note that if $T = RNR^{-1}$, then $\varphi(T) = R\,\varphi(N)R^{-1}$ for any function φ that is analytic in a neighborhood of $\sigma(T)$. Thus, similarity to a normal operator is preserved for $\varphi(T)$.

We next show that LRG is preserved under linear fractional transformations $\varphi(T) = (aT + bI)(cT + dI)^{-1}$.

Lemma 3.1. *Let $\varphi(z) = (az + b)/(cz + d)$ be a linear fractional transformation (which may be degenerate, i.e., $a = 0$ or $c = 0$). If an operator T (which does not have to be a contraction) satisfies the Linear Resolvent Growth condition*

$$\|(T - \lambda I)^{-1}\| \le \frac{C}{\operatorname{dist}(\lambda, \sigma(T))} , \tag{3.1}$$

then

$$\|\varphi(T)\| \le 10C \sup_{z\in\sigma(T)} |\varphi(z)|.$$

Corollary 3.2. *Let $\varphi(z) = (az+b)/(cz+d)$ be a linear fractional transformation. If an operator T satisfies the Linear Resolvent Growth condition (3.1), then the operator $\varphi(T)$ satisfies the same condition with constant $10C$, i.e.,*

$$\|(\varphi(T) - \lambda I)^{-1}\| \le \frac{10C}{\operatorname{dist}\{\lambda, \sigma(\varphi(T))\}}.$$

Proof. Consider the function $\tau(z) := 1/(z-\lambda)$. The composition $\varphi_1 := \tau\circ\varphi$ is a linear fractional transformation (as can be seen, for example, by noting that it is a conformal automorphism of the Riemann sphere $\hat{\mathbb{C}} := \mathbb{C} \cup \infty$). Therefore Lemma 3.1 implies

$$\begin{aligned}\|(\varphi(T) - \lambda I)^{-1}\| &= \|\tau(\varphi(T))\| = \|\varphi_1(T)\| \\ &\le 10C \sup_{z\in\sigma(T)} |\tau(\varphi(z))| \\ &= 10C \sup_{w\in\varphi(\sigma(T))} |\tau(w)| = \frac{10C}{\operatorname{dist}\{\lambda, \varphi(\sigma(T))\}}.\end{aligned}$$

To complete the proof it suffices to note that, by the Spectral Mapping Theorem (see [2, Theorem VII.3.11]), we have $\sigma(\varphi(T)) = \varphi(\sigma(T))$ for any function φ that is analytic in a neighborhood of $\sigma(T)$. □

Proof of Lemma 3.1. We first observe that a *linear* transformation $T \mapsto aT + b$ preserves LRG and, moreover, preserves the constant implicit in the LRG condition. This is indeed trivial for the shift $T \mapsto T + bI$, and for the transformation $T \mapsto aT$ it follows from the following chain of estimates:

$$\begin{aligned}\|(aT - \lambda I)^{-1}\| &= |a|^{-1} \left\| \left(T - \tfrac{\lambda}{a} I\right)^{-1} \right\| \\ &\le \frac{1}{|a|} \cdot \frac{C}{\operatorname{dist}(\frac{\lambda}{a}, \sigma(T))} = \frac{C}{\operatorname{dist}(\lambda, \sigma(aT))}.\end{aligned}$$

We now prove the lemma. Consider first the case when φ is a linear function. Since the LRG condition is preserved under linear transformations, we can assume, without loss of generality, that $\varphi(z) = z$. By the Riesz–Dunford formula we have

$$T = \frac{1}{2\pi i} \int_\gamma z \cdot (zI - T)^{-1} dz,$$

where γ is a contour surrounding $\sigma(T)$ in positive direction.

Take γ to be the circle with center at 0 of radius $R > \rho(T)$, where $\rho(T) = \sup_{z\in\sigma(T)} |z|$ is the spectral radius of T. Then

$$\|T\| \le \frac{1}{2\pi} \cdot 2\pi R \cdot \rho(T) \cdot \frac{C}{R - \rho(T)} = \rho(T) \cdot \frac{CR}{R - \rho(T)} .$$

Taking the limit as $R \to \infty$ we get

$$\|T\| \le C\rho(T) = C \sup_{z\in\sigma(T)} |z|.$$

Next, consider the case when φ is a proper rational function, i.e., $\varphi = a/(bz+c)$. In this case the conclusion of the lemma is just the LRG condition, so the conclusion trivially holds with the same constant C.

Finally consider the general case

$$\varphi = \frac{az + b}{cz + d}, \qquad a \ne 0, \quad c \ne 0.$$

Let τ be a linear transformation of $\mathbb{C}$ which maps -1 to $-b/a$ and 0 to $-d/c$. Then $\varphi\circ\tau = \alpha\cdot(z-1)/z$, where $\alpha\in\mathbb{C}$. Since linear transformations preserve the LRG property, it is enough to prove the result for the case $\varphi = (z-1)/z$.

Let

$$\delta := \sup_{z\in\sigma(T)} |\varphi(z)| = \sup_{z\in\sigma(T)} \left|\frac{z-1}{z}\right|$$

and consider first the case when $\delta \geq 1/2$. We write

$$\varphi(T) = \frac{1}{2\pi i}\int_\Gamma \varphi(z)(zI-T)^{-1}dz$$

with $\Gamma = \gamma_r \cup \gamma_R$, where γ_r and γ_R denote the circles $|z| = r$ and $|z| = R$ in negative and positive directions, respectively. Letting $r\to 0$ and $R\to\infty$, we have

$$\lim_{R\to\infty}\Big\|\int_{\gamma_R}\dots\Big\| \leq \lim_{R\to\infty}\frac{1}{2\pi}\cdot 2\pi R\cdot\frac{C}{R} = C$$

and

$$\lim_{r\to 0}\Big\|\int_{\gamma_r}\dots\Big\| \leq \lim_{r\to 0}\frac{1}{2\pi}\cdot 2\pi r\cdot\frac{1}{r}\cdot\frac{C}{\operatorname{dist}(0,\sigma(T))} = \frac{C}{\operatorname{dist}(0,\sigma(T))}\ .$$

One can easily see (by explicitly computing the level sets of $|\varphi|$) that the set $\{z : |\varphi(z)| \leq \delta\}$ lies outside the disk $\{z : |z| = 1/(1+\delta)\}$, so that $\operatorname{dist}(0,\sigma(T)) \geq 1/(1+\delta)$. Therefore,

$$\lim_{r\to 0}\Big\|\int_{\gamma_r}\dots\Big\| \leq C\cdot(1+\delta),$$

and so

$$\|\varphi(T)\| \leq C\cdot(2+\delta) \leq 5C\delta = 5C\sup_{z\in\sigma(T)}|\varphi(z)|$$

if $\delta \geq 1/2$.

Now consider the case $\delta \leq 1/2$. It is easy to check that for $\delta < 1$ the level set $\{z : |\varphi(z)| \leq \delta\}$ is the closed disk centered at $c = 1/(1-\delta^2)$ and of radius $r = \delta/(1-\delta^2)$. By the definition of δ, the spectrum $\sigma(T)$ is contained in this level set. As before, we can write

$$\varphi(T) = \frac{1}{2\pi i}\int_\Gamma \varphi(z)(zI-T)^{-1}dz,$$

where Γ is now the circle of radius $\frac{3}{2}r$ centered at $c = 1/(1-\delta^2)$. We have

$$\|\varphi(T)\| \leq \lim_{r\to 0}\frac{1}{2\pi}\cdot 2\pi\frac{3}{2}r\cdot\frac{C}{r/2}\cdot\sup_{z\in\Gamma}|\varphi(z)| = 3C\sup_{z\in\Gamma}|(z-1)/z|.$$

Note that the supremum $\sup_{z\in\Gamma}|\varphi(z)|$ is attained at the point $x = c - \frac{3}{2}r = \frac{1-3\delta/2}{1-\delta^2}$. Therefore

$$\sup_{z\in\Gamma}|\varphi(z)| = \frac{1-x}{x} = \delta\cdot\frac{3/2-\delta}{1-3\delta/2} \leq \delta\cdot\frac{3/2}{1-3/4} = 6\delta.$$

Hence $\|\varphi(T)\| \le 6C\delta$, and we are done. □

4. Conjectures and open questions

To conclude this paper, let us state some conjectures. Let T be a contraction, and let $\sigma(T) \neq \overline{\mathbb{D}}$. Denote by T_μ the "Möbius transformation" of T, i.e.,

$$T_\mu := (T - \mu I)(I - \overline{\mu} T)^{-1}, \qquad \mu \in \mathbb{D}.$$

Note that if $\|T\| \le 1$, then $\|T_\mu\| \le 1$ for all $\mu \in \mathbb{D}$. Recall that $\|A\|_{\mathfrak{S}_p}$ stands for the Schatten–von-Neumann norm of the operator A,

$$\|A\|_{\mathfrak{S}_p} = \Big(\sum_0^\infty s_n(A)^p \Big)^{1/p}.$$

In Section 3 we showed that LRG, as well as similarity to a normal operator, are invariant with respect to linear fractional transformations, and hence, in particular, with respect to the above "Möbius transformations". Since the "Möbius transformation" maps a contraction to a contraction, the following conjecture seems plausible.

Conjecture 4.1. *If $\|T\| \le 1$, $\sigma(T) \neq \overline{\mathbb{D}}$, and*

$$\sup_{\mu \in \mathbb{D}} \|I - T_\mu^* T_\mu\|_{\mathfrak{S}_1} < \infty, \tag{4.1}$$

then the LRG condition (0.1) *implies that T is similar to a normal operator.*

We believe that the trace class $\mathfrak{S}_1$ plays a critical role here.

Conjecture 4.2. *The condition* (4.1) *is sharp, i.e., given $p > 1$ one can find an operator T with $\|T\| \le 1$ and $\sigma(T) \neq \overline{\mathbb{D}}$, which satisfies LRG and*

$$\sup_{\mu \in \mathbb{D}} \|I - T_\mu^* T_\mu\|_{\mathfrak{S}_p} < \infty,$$

but which is not similar to a normal operator.

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