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PROBABILITY MEASURES ON ALMOST CONNECTED AMENABLE LOCALLY COMPACT GROUPS AND SOME RELATED IDEALS IN GROUP ALGEBRAS

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Abstract. Given a locally compact group G let $\mathcal{J}_a(G)$ denote the set of all closed left ideals J in $L^1(G)$ which have the form $J = [L^1(G) * (\delta_e - \mu)]^{-}$ where μ is an absolutely continuous probability measure on G. We explore the order structure of $\mathcal{J}_a(G)$ when $\mathcal{J}_a(G)$ is ordered by inclusion. When G is connected and amenable we prove that every nonempty family $\mathcal{F} \subseteq \mathcal{J}_a(G)$ admits both a minimal and a maximal element; in particular, every ideal in $\mathcal{J}_a(G)$ contains an ideal that is minimal in $\mathcal{J}_a(G)$. Furthermore, we obtain that every chain in $\mathcal{J}_a(G)$ is necessarily finite. A natural generalization of these results to almost connected amenable groups is discussed. Our proofs use results from the theory of boundaries of random walks.

1. Introduction

Given a probability measure μ on a locally compact group G consider the set

$$J_\mu = \overline{\{\varphi - \varphi * \mu\,;\, \varphi \in L^1(G)\}}$$

where the bar means closure with respect to the L^1-norm. J_μ is a left ideal in the group algebra $L^1(G)$. As shown by Willis [15], ideals of this form appear naturally in connection with the theory of random walks on G, in the study of amenability, and in certain factorization questions in group algebras.

Let $P(G)$, $P_a(G)$, and $P_d(G)$ denote, respectively, the sets of probability measures on G, absolutely continuous probability measures on G, and discrete probability measures on G. Define

$$\mathcal{J}(G) = \{J_\mu\,;\, \mu \in P(G)\},\ \ \mathcal{J}_a(G) = \{J_\mu\,;\, \mu \in P_a(G)\},$$
$$\mathcal{J}_d(G) = \{J_\mu\,;\, \mu \in P_d(G)\}.$$

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Each of the sets $\mathcal{J}(G)$, $\mathcal{J}_a(G)$, and $\mathcal{J}_d(G)$ is partially ordered by inclusion. Willis [15] showed that for a locally compact second countable group every ideal in $\mathcal{J}(G)$, $\mathcal{J}_a(G)$, or $\mathcal{J}_d(G)$ is contained in a maximal one. When G is amenable there is only one maximal element in $\mathcal{J}(G)$, $\mathcal{J}_a(G)$, and $\mathcal{J}_d(G)$ – it is the ideal $L^1_0(G) = \{\varphi \in L^1(G)\,;\, \int_G \varphi = 0\}$. When G is not amenable, $L^1_0(G)$ is not a member of $\mathcal{J}(G)$ and maximal ideals are not unique. The purpose of the present work is to provide more details about the order structure of the ideal space $\mathcal{J}(G)$ in the case when G is an almost connected amenable locally compact second countable group. We hope that our results in this special case can help to uncover an interesting structure of $\mathcal{J}(G)$ in wider classes of groups.

The ideal space $\mathcal{J}(G)$ has, trivially, the smallest element $\{0\}$ corresponding to the point measure δ_e where e is the identity of G. Suppose now that G is connected and different from the singleton $\{e\}$. Then $\delta_e \notin P_a(G)$ and one can ask whether $\mathcal{J}_a(G)$ admits any *minimal* elements. We will show that, when G is amenable and second countable, the answer to this question is positive: every element of $\mathcal{J}_a(G)$ contains an element that is minimal in $\mathcal{J}_a(G)$. More is true: every nonempty family $\mathcal{F} \subseteq \mathcal{J}_a(G)$ admits both a minimal and a maximal element. Furthermore, every chain in $\mathcal{J}_a(G)$ is necessarily finite.

It is not difficult to see that, when G is only almost connected but not connected, then $\mathcal{J}_a(G)$ need not have any minimal elements (see Section 4). Recall that a probability measure on G is called *adapted* if the smallest closed subgroup of G containing the support $\operatorname{supp}\mu$ is G itself. Let

$$\mathcal{J}_{aa}(G) = \{J_\mu\,;\, \mu \in P_a(G),\, \mu \text{ is adapted}\}.$$

We will prove that the results mentioned above remain true for the ideal space $\mathcal{J}_{aa}(G)$ of an almost connected amenable locally compact second countable group. We note that every absolutely continuous probability measure on a connected locally compact group is automatically adapted; in this case we therefore have $\mathcal{J}_a(G) = \mathcal{J}_{aa}(G)$.

We conjecture that the above or similar results hold also without the restriction that G be amenable. This is so when G is a connected semisimple Lie group with finite centre. Some special examples of semisimple Lie groups with infinite centre also support our conjecture. However, as we explain in the sequel, semisimple Lie groups with possibly infinite centre and general nonamenable groups introduce certain nontrivial complications which call for a much more subtle and complex argument than the one used here.

The proofs that we provide below are based on a detailed knowledge of boundaries of random walks on almost connected locally compact groups [8]. The boundaries form a subclass of the class of what we call contractive homogeneous spaces. The results about the ideals J_μ follow from analogous results about contractive homogeneous spaces (which we obtain in Section 3). It would be interesting to know whether there is another route to our results, perhaps resembling the abstract approach of Willis [15] used to prove the existence of maximal elements in $\mathcal{J}(G)$.

Our paper is organized as follows. Section 2 contains preliminary material; for the convenience of the reader we collect here relevant results about boundaries, μ-boundaries, contractive homogeneous spaces, and strongly approximately transitive group actions, and explain the connection between those and the ideals J_μ. In most cases proofs are omitted as they can be found elsewhere. In Section 3 we study the class of contractive homogeneous spaces of an almost connected amenable locally compact group: the order relation '$\mathcal{Y}$ is an equivariant image of $\mathcal{X}$' and equivariant isometries $L^\infty(\mathcal{Y}) \to L^\infty(\mathcal{X})$ are investigated. The results of Section 3 are then used in Section 4 to study the order structure of the ideal space $\mathcal{J}_{aa}(G)$.

2. Preliminaries

2.1. Measures. Given a Borel space $\mathcal{X}$ we shall denote by $M(\mathcal{X})$ the space of complex measures on $\mathcal{X}$ and by $P(\mathcal{X})$ the set of probability measures. $\|\cdot\|$ will stand for the total variation norm. When $F : \mathcal{X} \to \mathcal{Y}$ is a Borel map from $\mathcal{X}$ into a Borel space $\mathcal{Y}$ and μ is a measure or a complex measure on $\mathcal{X}$, we shall write $F\mu$ for the measure $(F\mu)(B) = \mu\big(F^{-1}(B)\big)$ on $\mathcal{Y}$. When $\mathcal{X}$ is a topological space, by the weak topology on $M(\mathcal{X})$ we shall mean the $\sigma\big(M(\mathcal{X}), C_b(\mathcal{X})\big)$- topology where $C_b(\mathcal{X})$ is the algebra of bounded continuous functions on $\mathcal{X}$.

Given a measure space $(\mathcal{X}, \alpha)$ we shall always identify $L^1(\mathcal{X}) = L^1(\mathcal{X}, \alpha)$ with the space of complex measures that are absolutely continuous with respect to α. $L^1_1(\mathcal{X}) \subseteq L^1(\mathcal{X})$ will stand for the subset of probability measures. When $\varphi \in L^1(\mathcal{X})$ and $f \in L^\infty(\mathcal{X}) = L^\infty(\mathcal{X}, \alpha)$, we shall write $\langle \varphi, f\rangle$ for $\int f\, d\varphi$. By the weak* topology on $L^\infty(\mathcal{X})$ we shall mean the $\sigma(L^\infty, L^1)$-topology.

2.2. SAT *G*-spaces. Let G be a group. A Borel G-space $\mathcal{X}$ with a σ-finite quasi-invariant measure α is called *strongly approximately transitive* (SAT) if it admits a probability measure $\rho \ll \alpha$ such that the convex hull of the orbit $G\rho$ is norm dense in $L^1_1(\mathcal{X})$. A measure ρ with this property is called a *SAT measure*. The following result is proven in [10, Proposition 2.2]. Here $B(G)$ denotes the space of bounded complex functions on G equipped with the sup norm.

PROPOSITION 2.2.1. *Let $\mathcal{X}$ be a Borel G-space with a σ-finite quasi-invariant measure α. The following conditions are equivalent for a probability measure $\rho \in L^1_1(\mathcal{X})$:*

(i) *ρ is a SAT measure;*
(ii) *for every Borel set A with $\alpha(A) > 0$ and every $\varepsilon > 0$ there exists $g \in G$ such that $\rho(gA) > 1 - \varepsilon$;*

(iii) *the map* $R : L^\infty(\mathcal{X}) \to B(G)$ *given by* $(Rf)(g) = \langle g\rho, f\rangle$ *is an isometry.*

Let $\mathcal{X}$ be a homogeneous space of a locally compact second countable (lcsc) group G. A probability measure $\nu \in P(\mathcal{X})$ is called *contractible* if the weak closure of the orbit $G\nu$ contains a point measure δ_x. For the next result see [10, proof of Corollary 2.5].

Proposition 2.2.2. *An absolutely continuous probability measure* ν *on a homogeneous space* $\mathcal{X}$ *of a lcsc group* G *is a SAT measure if and only if it is contractible.*

2.3. Boundaries. We will call a homogeneous space $\mathcal{X}$ of a lcsc group G a *boundary* of G if every probability measure $\nu \in P(\mathcal{X})$ is contractible.

Proposition 2.3.1. *A homogeneous space* $\mathcal{X}$ *of a lcsc group* G *is a boundary of* G *if and only if it admits a quasi-invariant SAT measure.*

Proof. The 'only if' part is a trivial consequence of Proposition 2.2.2.

Suppose ρ is a quasi-invariant SAT measure on $\mathcal{X}$ and $\nu \in P(\mathcal{X})$. Since ρ is contractible, by Lemma 6.5 in [8] there exists $x_0 \in \mathcal{X}$, a Borel set $B \subseteq \mathcal{X}$ and a sequence $\{h_n\}_{n=1}^\infty$ such that $\rho(B) = 1$ and $\lim_{n\to\infty} h_n b = x_0$ for every $b \in B$. Let α_0 be a probability measure on G equivalent to the Haar measure. Then $\alpha = \alpha_0 * \nu$ is a quasi-invariant probability measure on $\mathcal{X}$, and hence equivalent to ρ. Consequently,

$$\alpha(B) = \int_G (g\nu)(B)\,\alpha_0(dg) = 1.$$

So $(g\nu)(B) = \nu\big(g^{-1}(B)\big) = 1$ for some $g \in G$. Put $g_n = h_n g$. Then $\lim_{n\to\infty} g_n x = x_0$ for every $x \in g^{-1}B$. This implies that $\text{w-}\lim_{n\to\infty} g_n\nu = \delta_{x_0}$. □

Proposition 2.3.2. *Let* $\mathcal{X}$ *be a boundary of a lcsc group* G*. Then the centre* C *of* G *and every compact normal subgroup* $K \subseteq G$ *stabilize every point of* $\mathcal{X}$*.*

Proof. Let ρ be a quasi-invariant SAT measure. Recall that the function $G \ni g \to \|g\rho - \rho\|$ is continuous with respect to the total variation norm. By the 0-2 law [10, Proposition 3.1], for every $g \in G$ the number $a(g) = \sup_{h\in G} \|h^{-1}gh\rho - \rho\|$ is either 0 or 2. When $g \in C$, $a(g) = \|g\rho - \rho\|$. When $g \in K$, using compactness and normalcy of K, and the continuity, it follows that $a(g) = \|g'\rho - \rho\|$ for some $g' \in K$. In any case, due to quasi-invariance of ρ we obtain that $g\rho = \rho$ for every $g \in CK$. Since the convex hull $\mathrm{co}(G\rho)$ is dense in $L^1_1(\mathcal{X})$, it follows that CK acts trivially on $L^1(\mathcal{X})$ and, hence, also on $L^\infty(\mathcal{X})$. As $\mathcal{X}$ is a homogeneous space this implies that elements of CK stabilize every point of $\mathcal{X}$. □

2.4. Contractive homogeneous spaces. A subset U of a homogeneous space $\mathcal{X}$ of a lcsc group G is called *contractible* if for every nonempty open $V \subseteq \mathcal{X}$ there exists $g \in G$ with $gU \subseteq V$. The homogeneous space $\mathcal{X}$ is called *contractive* if it admits a nonempty open contractible subset.

PROPOSITION 2.4.1. *A contractive homogeneous space of a lcsc group is SAT. A homogeneous space of an almost connected lcsc group is SAT if and only if it is contractive. In particular, every boundary of an almost connected lcsc group is contractive.*

Proof. The first statement is proven in [10, Corollary 2.5]. For a proof of the second see [8, Theorem 4.6]. □

The proofs of the next two results can be found in [8, Corollary 5.2 and Lemma 4.5].

PROPOSITION 2.4.2. *If $\mathcal{X}$ is a contractive homogeneous space of an almost connected lcsc group G then:*

(i) *$\mathcal{X}$ consists of finitely many open orbits of G_e , the connected component of the identity e.*
(ii) *Every orbit of G_e is a contractive homogeneous space of G_e.*
(iii) *Every SAT measure is supported on one of the orbits of G_e.*

PROPOSITION 2.4.3. *Let $\mathcal{X}$ and $\mathcal{Y}$ be contractive homogeneous spaces of an almost connected lcsc group G. Then there exists a compact normal subgroup $K \subseteq G$ such that G/K is a Lie group and that K stabilizes every point of $\mathcal{X}$ and every point of $\mathcal{Y}$.*

2.5. Contractive homogeneous spaces of connected amenable Lie groups. Let G be a connected amenable Lie group with Lie algebra $\mathfrak{g}$. Choose a Levi decomposition $\mathfrak{g} = \mathfrak{r} \oplus \mathfrak{s}$ where $\mathfrak{r}$ is the radical and $\mathfrak{s}$ a Levi subalgebra. Denote by R the radical of G and by S the Levi subgroup corresponding to $\mathfrak{s}$. Then $G = RS$ and S is compact (because G is amenable).

Let $\mathfrak{p}$ be a Cartan subalgebra [3, Chap. VII, §2.1] of $\mathcal{C}_{\mathfrak{r}}(\mathfrak{s})$, the centralizer of $\mathfrak{s}$ in $\mathfrak{r}$, and P the connected Lie subgroup corresponding to $\mathfrak{p}$. Consider the adjoint representation ρ of P on the complexification $\tilde{\mathfrak{g}}$ of $\mathfrak{g}$. Recall that a function $\xi : P \to \mathbb{C}$ is called a weight of ρ if $\tilde{\mathfrak{g}}^{\xi} = \bigcap_{g \in P} \operatorname{Ker}\big(\rho(g) - \xi(g)\big)^d \neq \{0\}$ where d is the dimension of $\mathfrak{g}$. Every weight is an analytic homomorphism of P into $\mathbb{C} - \{0\}$. Let Σ denote the set of weights. We have $\tilde{\mathfrak{g}} = \bigoplus_{\xi \in \Sigma} \tilde{\mathfrak{g}}^{\xi}$, $\overline{\tilde{\mathfrak{g}}^{\xi}} = \tilde{\mathfrak{g}}^{\overline{\xi}}$ where the bar means the complex conjugation, $[\tilde{\mathfrak{g}}^{\xi}, \tilde{\mathfrak{g}}^{\eta}] \subseteq \tilde{\mathfrak{g}}^{\xi\eta}$, and each $\tilde{\mathfrak{g}}^{\xi}$ is invariant under ρ. (These facts are immediate consequences of the well known results about representations of nilpotent Lie algebras [3, Chap. VII, §1].)

Given $p \in P$ let

$$\Sigma_+(p) = \{\xi \in \Sigma\,;\, |\xi(p)| \geq 1\} \quad \text{and} \quad \Sigma_-(p) = \{\xi \in \Sigma\,;\, |\xi(p)| < 1\}.$$

Define

$$\tilde{\mathfrak{g}}_\pm(p) = \bigoplus_{\xi \in \Sigma_\pm(p)} \tilde{\mathfrak{g}}^\xi.$$

Then $\tilde{\mathfrak{g}}_\pm$ are subalgebras of $\tilde{\mathfrak{g}}$ invariant under complex conjugation. There exist (unique) subalgebras $\mathfrak{g}_\pm(p) \subseteq \mathfrak{g}$ such that $\tilde{\mathfrak{g}}_\pm(p) = \mathfrak{g}_\pm(p) \oplus i\mathfrak{g}_\pm(p)$. Denote by $G_\pm(p)$ the connected Lie subgroups corresponding to $\mathfrak{g}_\pm(p)$. The following proposition is a special case of Proposition 3.5 in [8] (see also Remark 3.6 in [8]).

Proposition 2.5.1. *$G_+(p)$ and $G_-(p)$ are closed subgroups of G and the mapping $G_-(p) \times G_+(p) \ni (g_-, g_+) \to g_-g_+ \in G$ is a homeomorphism of $G_-(p) \times G_+(p)$ onto G. Moreover $G_-(p)$ is a simply connected subgroup of the nilradical of G, while $G_+(p)$ contains S and every connected nilpotent Lie subgroup containing P.*

For the next result see [8, Theorem 5.5 and Proposition 3.7].

Proposition 2.5.2. *A homogeneous space $\mathcal{X}$ of a connected amenable Lie group is contractive if and only if there exist $p \in P$ and $x_0 \in \mathcal{X}$ such that the stabilizer G_{x_0} of x_0 contains $G_+(p)$. If G_{x_0} contains $G_+(p)$ for some $p \in P$, then for every $x \in \mathcal{X}$, $\lim_{n\to\infty} p^n x = x_0$.*

Corollary 2.5.3. *A contractive homogeneous space of a connected amenable lcsc group is a boundary. A contractive homogeneous space of an almost connected amenable lcsc group G is a boundary if and only if it consists of a single orbit of G_e.*

Proof. By Proposition 2.4.3 a contractive homogeneous space of a connected amenable lcsc group can be considered as a contractive homogeneous space of a connected amenable Lie group. Then the first claim follows from the last statement of Proposition 2.5.2. The 'if' part of the second claim is then a direct consequence of Proposition 2.4.2(ii). The 'only if' part is a consequence of Proposition 2.3.1 and Proposition 2.4.2(iii). □

2.6. $\boldsymbol{\mu}$-boundaries and ideals $\boldsymbol{J_\mu}$. Let G be a lcsc group and μ a probability measure on G. A bounded Borel function $h : G \to \mathbb{C}$ is called *μ-harmonic* if it satisfies

$$h(g) = \int_G h(gg')\,\mu(dg'), \quad g \in G.$$

Let $\mathcal{H}_\mu$ denote the space of equivalence classes, modulo the Haar measure λ, of the bounded μ-harmonic functions.

The formula

$$(P_\mu f)(g) = \int f(gg')\,\mu(dg') \quad (\mathrm{mod}\lambda)$$

defines a positive, weak* continuous contraction $P_\mu : L^\infty(G) \to L^\infty(G)$. $\mathcal{H}_\mu$ is precisely the subspace of fixed points of P_μ [5, Proposition 2]. The adjoint of P_μ is a positive contraction $P_\mu^* : L^1(G) \to L^1(G)$ given by

$$P_\mu^* \varphi = \varphi * \mu.$$

The ideal J_μ is the norm closure of the range of the operator $I - P_\mu^*$. $\mathcal{H}_\mu$ is the anihilator of J_μ in $L^\infty(G)$:

$$\mathcal{H}_\mu = J_\mu^\perp = \{f \in L^\infty(G)\,;\, \langle \varphi, f\rangle = 0 \text{ for all } \varphi \in J_\mu\}.$$

A homogeneous space $\mathcal{X}_\mu$ is called a *μ-boundary* if there exists a probability measure ρ_μ on $\mathcal{X}_\mu$ such that the *Poisson formula*

$$h(g) = \int_{\mathcal{X}} f(gx)\,\rho_\mu(dx), \quad f \in L^\infty(\mathcal{X}_\mu), \tag{2.6.1}$$

defines an isometry, R_μ, of $L^\infty(\mathcal{X}_\mu)$ onto $\mathcal{H}_\mu$. We note that the measure ρ_μ, called the *Poisson kernel*, necessarily satisfies $\rho_\mu = \mu * \rho_\mu$. The μ-boundary, if exists, is unique up to an equivariant isomorphism [12, §3]. [1]

The isometry R_μ defined by (2.6.1) is weak* continuous, its adjoint $R_\mu^* : L^1(G) \to L^1(\mathcal{X}_\mu)$ is given by

$$R_\mu^* \varphi = \varphi * \rho_\mu. \tag{2.6.2}$$

Since, $J_\mu = \mathcal{H}_\mu^\perp$, it is clear that J_μ coincides with the kernel of R_μ^*. Thus,

$$J_\mu = \{\varphi \in L^1(G)\,;\, \varphi * \rho_\mu = 0\}. \tag{2.6.3}$$

Recall that a probability measure μ on G is called *spread out* if for some n the n-th convolution power μ^n is nonsingular with respect to the Haar measure. For the next two results see [11, §2.2] and [8, Corollary 4.7], respectively.

Proposition 2.6.1. *The μ-boundary of a spread out probability measure on a lcsc group is a SAT G-space and the Poisson kernel is a SAT measure.*

Proposition 2.6.2. *The μ-boundary exists for every spread out probability measure on an almost connected lcsc group G and is a contractive homogeneous space of G.*

Proposition 2.6.3. *The μ-boundary of an adapted spread out probability measure on an almost connected amenable lcsc group G is a boundary of G.*

[1] In this work the μ-boundary is, by definition, a homogeneous space. In general, such a μ-boundary need not exist; there always exists a μ-boundary in a wider sense, defined as a Borel G-space.

Proof. By Corollary 5.4 in [8] the μ-boundary of an adapted spread out probability measure on G consists of a single orbit of G_e. We conclude by Corollary 2.5.3. □

For a nondiscrete group the class of spread out probability measures is considerably larger than $P_a(G)$. However, due to a result of Willis [15, Proposition 2.6], from the point of view of the theory of the ideals J_μ spread out measures provide no greater generality than the absolutely continuous ones:

PROPOSITION 2.6.4. *For every lcsc group G,*

$$\mathcal{J}_a(G) = \{J_\mu\,;\, \mu \in P(G),\ \mu \textit{ is spread out}\},$$
$$\mathcal{J}_{aa}(G) = \{J_\mu\,;\, \mu \in P(G), \mu \textit{ is spread out and adapted}\}.$$

Proof. The first equality follows trivially from [15, Proposition 2.6]. To get the second one it then suffices to show that if $J_\mu = J_\nu$ and μ is adapted then ν is also adapted. Let H_ν be the closed subgroup generated by $\operatorname{supp}\nu$ and $\pi : G \to G/H_\nu$ denote the canonical mapping. Then for every $f \in C_b(G/H_\nu)$ the function $f \circ \pi$ is ν-harmonic and therefore also μ-harmonic. Hence, $f(\pi(e)) = \int_G f \circ \pi \, d\mu = \int_{G/H_\nu} f \, d(\pi\mu)$. Consequently, $\pi\mu = \delta_{H_\nu}$. So $\mu(H_\nu) = 1$ and $H_\nu = G$ by the adaptedness of μ. □

3. More on contractive homogeneous spaces

Given two homogeneous spaces $\mathcal{X}$ and $\mathcal{Y}$ of a lcsc group G we shall write $\mathcal{X} \succ \mathcal{Y}$ if $\mathcal{Y}$ is an equivariant image of $\mathcal{X}$, and $\mathcal{X} \cong \mathcal{Y}$ if $\mathcal{X}$ and $\mathcal{Y}$ are isomorphic (as G-spaces). In general, as the following elementary example shows, $\mathcal{X} \succ \mathcal{Y}$ and $\mathcal{Y} \succ \mathcal{X}$ does not imply $\mathcal{X} \cong \mathcal{Y}$. Our first goal will be to show that, in the class of contractive homogeneous spaces of an almost connected amenable lcsc group, $\mathcal{X} \succ \mathcal{Y}$ and $\mathcal{Y} \succ \mathcal{X}$ does imply $\mathcal{X} \cong \mathcal{Y}$.

EXAMPLE 3.1. Consider the semidirect product $G = \mathbb{Q} \times_\tau \mathbb{Z}$ where τ is the automorphism $\tau(x) = 4x$. For $k \in \mathbb{N}$, let $H_k = k\mathbb{Z} \times \{0\}$. Then the H_k's are subgroups of G and $H_8 = (0,1)H_2(0,1)^{-1} \subseteq H_4 \subseteq H_2$. Consequently, $G/H_2 \succ G/H_4$ and $G/H_4 \succ G/H_2$. But H_2 and H_4 are not conjugate; hence, $G/H_2 \ncong G/H_4$.

LEMMA 3.2. *Let $\mathcal{X}$ be a contractive homogeneous space of a connected amenable Lie group G. Then the stabilizer subgroups G_x, $x \in \mathcal{X}$, are connected and coincide with their normalizers.*

Proof. We will work in the setting of Section 2.5. By Proposition 2.5.2 there exist $x_0 \in \mathcal{X}$ and $p \in P$ such that $H = G_{x_0}$ contains $G_+(p)$. It suffices to show that H is connected and coincides with its normalizer $N_G(H)$. Denote by $\mathfrak{h}$ the Lie algebra of H. Clearly, $H \subseteq N_G(H) \subseteq N_G(H_e) = \mathfrak{n}_G(\mathfrak{h})$ where H_e denotes the connected component of e in H and $\mathfrak{n}_G(\mathfrak{h}) = \{g \in G\,;\, \mathrm{Ad}(g)\mathfrak{h} = \mathfrak{h}\}$. It suffices to show that $\mathfrak{n}_G(\mathfrak{h}) \subseteq H_e$.

Let L and $\mathfrak{l}$ denote the nilradicals of G and $\mathfrak{g}$, respectively. Since $\mathfrak{g} = \mathfrak{r} + \mathfrak{s}$ and $\mathfrak{r} = \mathfrak{l} + \mathfrak{p}$ [14, Lemme 1.9], we have $G = LPS$. Hence, as $H_e \supseteq SP$ (see Proposition 2.5.1), it suffices to show that if $g \in L$ and $\mathrm{Ad}(g)\mathfrak{h} = \mathfrak{h}$ then $g \in H_e$. But $g = \exp(X)$ for some $X \in \mathfrak{l}$, because L is a connected nilpotent Lie group. It follows that $(e^{\mathrm{ad}\,X} - I\,)\mathfrak{h} \subseteq \mathfrak{h}$. As $\mathrm{ad}\,X$ is nilpotent [2, Corollaire 7, p. 67], it can be expressed as a polynomial in $(e^{\mathrm{ad}\,X} - I\,)$. Hence, X normalizes $\mathfrak{h}$ and, consequently, also $\mathfrak{h} \cap \mathfrak{r}$. Now, by [8, Lemma 3.8], $\mathfrak{p}$ is contained in a Cartan subalgebra $\mathfrak{p}'$ of $\mathfrak{r}$. By Proposition 2.5.1, the connected Lie subgroup P' corresponding to $\mathfrak{p}'$ is contained in H, so $\mathfrak{p}' \subseteq \mathfrak{h} \cap \mathfrak{r}$. Then $X \in \mathfrak{h} \cap \mathfrak{r} \subseteq \mathfrak{h}$ by [3, Corollaire 4, p. 20], and therefore $g = \exp(X) \in H_e$. □

Corollary 3.3. *Let $\mathcal{X}$ be a contractive homogeneous space of an almost connected amenable Lie group G. If G has n connected components then each stabilizer subgroup G_x, $x \in \mathcal{X}$, has at most n connected components.*

Proof. By Proposition 2.4.2(ii) $G_e x$ is a contractive homogeneous space of G_e. So $(G_e)_x = G_e \cap G_x$ is connected . As $(G_x)_e \subseteq G_e \cap G_x$ it follows that $(G_x)_e = G_e \cap G_x$. Hence, $G_x/(G_x)_e = G_x/(G_e \cap G_x) \cong G_x/G_e \subseteq G/G_e$. □

Lemma 3.4. *Let $\mathcal{X}$ be a homogeneous space of a Lie group G and suppose that the stabilizers G_x, $x \in \mathcal{X}$, are almost connected. Then every equivariant map $F : \mathcal{X} \to \mathcal{X}$ is a bijection.*

Proof. F is surjective because G acts transitively on $\mathcal{X}$. To prove that F is injective choose $x \in \mathcal{X}$. Then $G_x \subseteq G_{F(x)}$. It suffices to show that $G_x = G_{F(x)}$. Now, $F(x) = gx$ for some $g \in G$. Thus $G_x \subseteq gG_xg^{-1}$. But G_x and gG_xg^{-1} have the same dimensions and the same (finite) number of connected components. Hence, $(G_x)_e = \big(gG_xg^{-1}\big)_e$ and, consequently, $G_x = gG_xg^{-1}$. □

Theorem 3.5. *Let $\mathcal{X}$ and $\mathcal{Y}$ be contractive homogeneous spaces of an almost connected amenable lcsc group G. Suppose there exist equivariant maps $F_1 : \mathcal{X} \to \mathcal{Y}$ and $F_2 : \mathcal{Y} \to \mathcal{X}$. Then both F_1 and F_2 are homeomorphism. In particular, $\mathcal{X} \succ \mathcal{Y}$ and $\mathcal{Y} \succ \mathcal{X}$ implies $\mathcal{X} \cong \mathcal{Y}$.*

Proof. F_1 and F_2 are surjective by transitivity and it is well known that they are continuous open mappings. Form $F = F_2 \circ F_1$. F is then an equivariant map of $\mathcal{X}$ into $\mathcal{X}$. By Proposition 2.4.3 $\mathcal{X}$ can be considered as a contractive homogeneous space of an almost connected amenable Lie group. Then Corollary 3.3 and Lemma 3.4 yield that F is a bijection. Hence, F_1 and F_2 must be injective. □

Given a family $\mathcal{F}$ of homogeneous spaces of a lcsc group G, an element $\mathcal{X} \in \mathcal{F}$ will be called *maximal* (resp., *minimal*) in $\mathcal{F}$ if for every $\mathcal{Y} \in \mathcal{F}$, $\mathcal{Y} \succ \mathcal{X}$ (resp., $\mathcal{Y} \prec \mathcal{X}$) implies $\mathcal{Y} \cong \mathcal{X}$.

Proposition 3.6. *Let $\mathcal{F}$ be a nonempty family of contractive homogeneous spaces of an almost connected amenable Lie group G. Then $\mathcal{F}$ admits both a minimal and a maximal element.*

Proof. Let us define

$$\mathcal{A} = \{H \subseteq G\,;\, H \text{ is a closed subgroup with } G/H \cong \mathcal{X} \text{ for some } \mathcal{X} \in \mathcal{F}\},$$

$$d_* = \min\{\dim H\,;\, H \in \mathcal{A}\}, \quad d^* = \max\{\dim H\,;\, H \in \mathcal{A}\},$$

$$\mathcal{A}_* = \{H \in \mathcal{A}\,;\, \dim H = d_*\}, \quad \mathcal{A}^* = \{H \in \mathcal{A}\,;\, \dim H = d^*\}.$$

By Corollary 3.3, $c_* = \min\{\#(H/H_e); H \in \mathcal{A}_*\}$ and $c^* = \max\{\#(H/H_e); H \in \mathcal{A}^*\}$ are well defined integers. Choose $H_* \in \mathcal{A}_*$ with $\#\big(H_*/(H_*)_e\big) = c_*$ and $H^* \in \mathcal{A}^*$ with $\#\big(H^*/(H^*)_e\big) = c^*$. By definition there exist $\mathcal{X}_* \in \mathcal{F}$ with $\mathcal{X}_* \cong G/H_*$ and $\mathcal{X}^* \in \mathcal{F}$ with $\mathcal{X}^* \cong G/H^*$. It is easy to see that $\mathcal{X}_*$ is maximal while $\mathcal{X}^*$ is minimal in $\mathcal{F}$. □

Corollary 3.7. *Every contractive homogeneous space of an almost connected amenable Lie group is an equivariant image of a maximal contractive homogeneous space.*

Proof. Apply Proposition 3.6 to the family

$$\{\mathcal{Y}\,;\, \mathcal{Y} \text{ is a contractive homogeneous space and } \mathcal{Y} \succ \mathcal{X}\},$$

where $\mathcal{X}$ is the given contractive homogeneous space. □

Proposition 3.8. *Let $\mathcal{F}$ be a nonempty family of contractive homogeneous spaces of an almost connected amenable Lie group G, linearly ordered by $\prec$. Then $\mathcal{F}/\cong$ is finite.*

Proof. For every $\mathcal{X} \in \mathcal{F}$ let $m(\mathcal{X})$ and $n(\mathcal{X})$ denote the dimension and the number of connected components of the stabilizer subgroups G_x, $x \in \mathcal{X}$, respectively. Define $\gamma : \mathcal{F} \to \mathbb{N} \times \mathbb{N}$ by $\gamma(\mathcal{X}) = \big(m(\mathcal{X}), n(\mathcal{X})\big)$. Due to Corollary 3.3, the range of γ is finite. Hence it suffices to show that if $\mathcal{X}_1, \mathcal{X}_2 \in \mathcal{F}$ and $\gamma(\mathcal{X}_1) = \gamma(\mathcal{X}_2)$ then $\mathcal{X}_1 \cong \mathcal{X}_2$.

Since $\mathcal{F}$ is linearly ordered we may assume that $\mathcal{X}_1 \succ \mathcal{X}_2$. But this means that $G_{x_1} \subseteq G_{x_2}$ for some $x_1 \in \mathcal{X}_1$ and $x_2 \in \mathcal{X}_2$. The equality $\gamma(\mathcal{X}_1) = \gamma(\mathcal{X}_2)$ implies (as in the proof of Lemma 3.4) that $G_{x_1} = G_{x_2}$. □

Theorem 3.9. *Let $\mathcal{F}$ be a nonempty family of boundaries of an almost connected amenable lcsc group G. Then $\mathcal{F}$ contains a minimal and a maximal element. If $\mathcal{F}$ is linearly ordered by $\prec$ then $\mathcal{F}/\cong$ is finite. Every boundary of G is an equivariant image of a maximal boundary.*

Proof. An almost connected locally compact group admits the largest compact normal subgroup K (which is the intersection of all maximal compact subgroups). By Proposition 2.3.2 we can consider $\mathcal{F}$ as a family of contractive homogeneous spaces of the Lie group G/K. Then Propositions 3.6 and 3.8 apply. □

Remark 3.10. Even for very elementary connected solvable Lie groups the class of boundaries can contain uncountably many nonisomorphic members [9, Example 5.10]. Finiteness of $\mathcal{F}/\cong$ in Proposition 3.8 and Theorem 3.9 is a nontrivial property of the contractive homogeneous spaces.

Remark 3.11. Our definition of the boundary is motivated by a more restrictive definition given by Furstenberg [6, 7]. He defines the boundary of G (let us call it the F-boundary) as a *compact* minimal G-space $\mathcal{X}$ with the property that every probability measure on $\mathcal{X}$ is contractible [7, Definition 4.1]. For an amenable locally compact group every F-boundary is a singleton [7, Proposition 4.3]. In general, every F-boundary of a locally compact group is an equivariant image of a maximal F-boundary (called the universal boundary in [7]) which is unique up to an isomorphism [7, Proposition 4.6]. For a connected semisimple Lie group our definition of the boundary and Furstenberg's coincide; in this case every boundary is an equivariant image of the unique maximal boundary G/MAN where MAN is a minimal parabolic subgroup. However, in general, maximal boundaries need not be unique: Example 5.4 in [9] shows that a connected solvable Lie group can have nonisomorphic maximal boundaries of different dimensions.

Remark 3.12. It is easy to see that Proposition 3.6, Corollary 3.7 and Proposition 3.8 do not hold for almost connected lcsc groups that are not Lie. When H is an open subgroup of a lcsc group, G/H is trivially a contractive homogeneous space. When G is abelian the only contractive homogeneous spaces possible are of this trivial form [10, Theorem 3.3]. Recall that for a totally disconnected G open-closed subgroups form a neighbourhood base at e. Hence, when G is infinite, abelian, totally disconnected, and compact, then G is almost connected but the family of contractive homogeneous spaces does not have any maximal elements and there exist infinite chains of nonisomorphic contractive homogeneous spaces.

Propositions 3.6 and 3.8 also fail for some connected nonamenable Lie groups. Let G be a connected semisimple Lie group with an Iwasawa decomposition $G = KAN$ and let M denote the centralizer of A in K. Then a homogeneous space G/H is contractive if and only if H contains a conjugate of M_eAN [8, Corollary 5.8]. Thus the class of contractive homogeneous spaces has the largest element G/M_eAN. However, consider the universal covering G of $\mathrm{SL}(n,\mathbb{R})$. In this case M coincides with the centre of G and is isomorphic to $\mathbb{Z}$ so that M_e is trivial. Since $\mathbb{Z}$ admits infinite descending chains of subgroups it is easy to see that there exist infinite chains of nonisomorphic contractive homogeneous spaces without any maximal elements.

Our next proposition is motivated by an analogous result of Furstenberg [7, Proposition 4.2] on the compact F-boundaries mentioned in Remark 3.11. It will lead us to Theorem 3.16 that will be crucial in Section 4 in establishing a connection between the order relation $\succ$ on the space of μ-boundaries and the order structure of the ideal space $\mathcal{J}_a(G)$.

Proposition 3.13. *Let $\mathcal{X}$ and $\mathcal{Y}$ be homogeneous spaces of an almost connected amenable lcsc group G such that $\mathcal{X}$ is contractive while $\mathcal{Y}$ is a boundary. Then every equivariant map $\varphi : \mathcal{X} \to P(\mathcal{Y})$ has $\delta_{\mathcal{Y}}$ as its range. Moreover, there exists at most one equivariant map $F : \mathcal{X} \to \mathcal{Y}$.*

Proof. By Proposition 2.4.3 it suffices to deal with the case when G is a Lie group. Consider the action of G_e. Working in the setting of Section 2.5 (with G replaced by G_e) and appealing to Proposition 2.4.2 there exist $x_0 \in \mathcal{X}$ and $y_0 \in \mathcal{Y}$ such that $P \subseteq (G_e)_{x_0}$ and $P \subseteq (G_e)_{y_0}$. Since $\mathcal{Y}$ is a boundary, by Corollary 2.5.3 and Proposition 2.5.2 there exists $p \in P$ such that $\lim_{n\to\infty} p^n y = y_0$ for every $y \in \mathcal{Y}$. Now, $\varphi(x_0) = \varphi(p^n x_0) = p^n\varphi(x_0)$. Hence, $\varphi(x_0) = \text{w-}\lim_{n\to\infty} p^n\varphi(x_0) = \delta_{y_0}$. Since G acts transitively on $\mathcal{X}$ this proves the first statement.

To obtain the second statement suppose that $F_i : \mathcal{X} \to \mathcal{Y}$ are equivariant maps. Then $\varphi(x) = \frac{1}{2}(\delta_{F_1(x)} + \delta_{F_2(x)})$ is an equivariant map into $P(\mathcal{Y})$. As $\varphi(\mathcal{X}) = \delta_{\mathcal{Y}}$ we get $F_1 = F_2$. □

Corollary 3.14. *Let $\mathcal{X}$ and $\mathcal{Y}$ be contractive homogeneous spaces of an almost connected amenable lcsc group G and $\varphi : \mathcal{X} \to P(\mathcal{Y})$ an equivariant map. If for some sequence $\{x_n\}_{n=1}^{\infty} \subseteq \mathcal{X}$ the sequence $\varphi(x_n)$ converges weakly to a point measure then $\varphi(\mathcal{X}) = \delta_{\mathcal{Y}}$.*

Proof. Again, it suffices to consider the case when G is a Lie group. Let $x_0 \in \mathcal{X}$. We can write $x_n = g_n x_0$ with $g_n \in G$. Since G/G_e is finite there exists a subsequence g_{n_k} such that $g_{n_k}G_e$ is constant, say $g_{n_k}G_e = g_0G_e$. Put $h_k = g_0^{-1}g_{n_k}$. Then $h_k \in G_e$ and $\text{w-}\lim_{k\to\infty} h_k\varphi(x_0) = \delta_{y_0}$ for some $y_0 \in \mathcal{Y}$.

Now, by Proposition 2.4.2 $\mathcal{Y}$ is a union of finitely many open orbits of G_e. We have $1 = \delta_{y_0}(G_e y_0) = \lim_{k\to\infty}\big(h_k\varphi(x_0)\big)(G_e y_0) = \varphi(x_0)(G_e y_0)$ because $h_k \in G_e$. It follows that the restriction of φ to $G_e x_0$ can be considered as an equivariant map into $P(G_e y_0)$. As $G_e y_0$ is a boundary of G_e (see Corollary 2.5.3), Proposition 3.13 and transitivity imply that $\varphi(\mathcal{X}) = \delta_{\mathcal{Y}}$. □

Lemma 3.15. *Let $\mathcal{X}$ and $\mathcal{Y}$ be homogeneous spaces of a lcsc group G and $\Phi : L^\infty(\mathcal{Y}) \to L^\infty(\mathcal{X})$ a positive, equivariant, weak* continuous, identity preserving contraction. Then there exists a transition probability T from $\mathcal{X}$ to*

$\mathcal{Y}$ such that

$$(\Phi f)(x) = \int_{\mathcal{Y}} T(x, dy)\, f(y) \quad \text{(a.e.)}$$

and such that the mapping $\mathcal{X} \ni x \to T(x, \cdot) \in P(\mathcal{Y})$ is equivariant. If Φ is an isometry then there is also a sequence $\{x_n\}_{n=1}^{\infty} \subseteq \mathcal{X}$ such that the sequence $T(x_n, \cdot)$ converges weakly to a point measure.

Proof. For a proof of the existence of T see, e.g., [13, Lemma 3.7]. Suppose Φ is an isometry. Let $y \in \mathcal{Y}$ and $\{V_n\}_{n=1}^{\infty}$ be a decreasing base of open neighbourhoods of y. Since the characteristic function χ_{V_n} has norm 1 in $L^{\infty}(\mathcal{Y})$, there is x_n with $T(x_n, V_n) > 1 - \frac{1}{n}$. Hence, $T(x_n, \cdot) \to \delta_y$. □

THEOREM 3.16. *Let $\mathcal{X}$ and $\mathcal{Y}$ be contractive homogeneous spaces of an almost connected amenable lcsc group and $\Phi : L^{\infty}(\mathcal{Y}) \to L^{\infty}(\mathcal{X})$ a positive, equivariant, identity preserving, weak* continuous isometry. Then there exists an equivariant map $F : \mathcal{X} \to \mathcal{Y}$ which induces Φ, i.e., which satisfies $\Phi f = f \circ F$ for every $f \in L^{\infty}(\mathcal{Y})$.*

Proof. Combine Lemma 3.15 with Corollary 3.14. □

REMARK 3.17. When $\mathcal{Y}$ is a boundary of G, Proposition 3.13 and Lemma 3.15 imply that every positive, equivariant, identity preserving, weak* continuous *contraction* $\Phi : L^{\infty}(\mathcal{Y}) \to L^{\infty}(\mathcal{X})$ is, in fact, an isometry. Therefore when $\mathcal{Y}$ is a boundary the conclusion of Theorem 3.16 remains in force under the weaker assumption that Φ be a contraction. Note also that in this case there exists at most one positive, equivariant, identity preserving, weak* continuous contraction $\Phi : L^{\infty}(\mathcal{Y}) \to L^{\infty}(\mathcal{X})$.

4. Ideals J_μ and μ-boundaries

Before turning to our main topic, the order structure of the ideal space $\mathcal{J}_{aa}(G) = \{J_\mu\,;\, \mu \in P_a(G),\ \mu \text{ is adapted}\}$ of an almost connected amenable lcsc group G, we wish to mention an interesting fact about the ideals in $\mathcal{J}_{aa}(G)$ which follows from Propositions 2.6.3 and 2.3.2.

Let H be a closed subgroup of a lcsc group G, and let $L_0^1(G, H)$ denote the kernel of the canonical mapping of $L^1(G)$ onto $L^1(G/H)$. Thus

$$L_0^1(G, H) = \{\varphi \in L^1(G)\,;\, \pi\varphi = 0\} = \{\varphi \in L^1(G)\,;\, \varphi * \delta_H = 0\},$$

where $\pi : G \to G/H$ is the canonical mapping. Recall that when H is normal then

$$L_0^1(G, H)^{\perp} = \{f \in L^{\infty}(G)\,;\, gf = f \text{ for every } g \in H\}. \tag{4.1}$$

LEMMA 4.1. *Let H be a closed normal subgroup of G, $\mu \in P(G)$, and $\mathcal{X}_\mu$ the μ-boundary. Then $L_0^1(G, H) \subseteq J_\mu$ if and only if H stabilizes every point of $\mathcal{X}_\mu$.*

Proof. $L_0^1(G,H) \subseteq J_\mu$ is equivalent to $\mathcal{H}_\mu = J_\mu^\perp \subseteq L_0^1(G,H)^\perp$. Hence, by (4.1) we have $L_0^1(G,H) \subseteq J_\mu$ if and only if every $h \in \mathcal{H}_\mu$ is invariant under the action of H. Since $\mathcal{H}_\mu \cong L^\infty(\mathcal{X}_\mu)$ this means that H acts trivially on $L^\infty(\mathcal{X}_\mu)$. As $\mathcal{X}_\mu$ is a homogeneous space the latter is equivalent to the condition that H stabilizes every point of $\mathcal{X}_\mu$. □

According to Proposition 5.2 in [15], given a compact normal subgroup K of a lcsc group G and an adapted probability measure $\mu \in P_a(G)$, there exists an open subgroup L of K, which is normal in G and such that $L_0^1(G,L) \subseteq J_\mu$. When G is almost connected and amenable a considerably stronger result holds:

THEOREM 4.2. *Let G be an almost connected amenable lcsc group, C the centre of G and K the largest compact normal subgroup. Then $L_0^1(G,CK) \subseteq J_\mu$ for every adapted spread out probability measure μ.*

Proof. Combine Lemma 4.1 with Propositions 2.3.2 and 2.6.3. □

REMARK 4.3. Theorem 4.2 fails without the assumption that G be amenable. It is well known that when μ is a spread out probability measure on a connected semisimple Lie group with finite centre C, then, in general, the centre (which is also a compact normal subgroup) does not stabilize points of the μ-boundary [1, 6].

LEMMA 4.4. *Let μ and ν be probability measures on a lcsc group G, $\mathcal{X}_\nu$ the ν-boundary, and ρ_ν the Poisson kernel. Then $J_\mu \subseteq J_\nu$ if and only if $\mu * \rho_\nu = \rho_\nu$.*

Proof. Let $\{\varepsilon_n\}_{n=1}^\infty$ be a sequence of probability measures in $L^1(G)$ weakly convergent to δ_e. If $J_\mu \subseteq J_\nu$ then by (2.6.3) and the definition of J_μ, $\varepsilon_n * \rho_\nu - \varepsilon_n * \mu * \rho_\nu = 0$ for every n. Since the mapping $M(G) \ni \varphi \to \varphi * \rho_\nu \in M(\mathcal{X}_\nu)$ is weakly continuous, the equality $\mu * \rho_\nu = \rho_\nu$ follows by taking the limit $n \to \infty$.

Conversely, if $\mu * \rho_\nu = \rho_\nu$ then $(\varphi - \varphi * \mu) * \rho_\nu = 0$ for every $\varphi \in L^1(G)$. Hence, $J_\mu \subseteq J_\nu$ by (2.6.3) and the definition of J_μ. □

LEMMA 4.5. *Let $(\mathcal{X},\alpha)$ and $(\mathcal{Y},\beta)$ be σ-finite measure spaces and $\Phi : L^\infty(\mathcal{X}) \to L^\infty(\mathcal{Y})$ a positive, weak* continuous, identity preserving isometry. Then the range $\mathcal{H}$ of Φ is weak* closed and the inverse $\Phi^{-1} : \mathcal{H} \to L^\infty(X)$ is positive, weak* continuous, and identity preserving.*

Proof. That the range of Φ is weak* closed and that Φ^{-1} is weak* continuous follows by a routine application of the Krein-Smulian Theorem (see [4, Theorem 7, Chap. V.5 and Corollary 11, Chap. V.3]). Next, since Φ preserves identity and complex conjugation, so does Φ^{-1}. Using the fact that a real element of an L^∞-space is positive if and only if $\|f\| = \|2f - \|f\|\|$, one concludes that Φ^{-1} preserves positivity too. □

LEMMA 4.6. *Let μ be a spread out probability measure on an almost connected amenable lcsc group G, $\mathcal{X}_\mu$ the μ-boundary, and ρ_μ the Poisson kernel. If $\mathcal{X}$ is a homogeneous space of G supporting a contractible probability measure ν such that $\nu = \mu * \nu$ then $\mathcal{X}$ is contractive and there exists an equivariant map $F : \mathcal{X}_\mu \to \mathcal{X}$ such that $F\rho_\mu = \nu$.*

Proof. Due to the equality $\nu = \mu * \nu$, for every bounded Borel function $f : \mathcal{X} \to \mathbb{C}$ the function $(f * \nu)(g) = \int_{\mathcal{X}} f(gx)\,\nu(dx)$ is μ-harmonic. Now, for spread out μ the μ-harmonic functions are continuous [1, Proposition I.6, p.23]. This implies that ν is absolutely continuous: indeed, when α_0 is a finite measure equivalent to the Haar measure then $\alpha_0 * \nu$ is a quasi-invariant measure on $\mathcal{X}$; if $(\alpha_0 * \nu)(A) = \int_G (\chi_A * \nu)(g)\,\alpha_0(dg) = 0$ then continuity of $\chi_A * \nu$ gives $(\chi_A * \nu)(e) = \nu(A) = 0$. Applying Proposition 2.2.2 we conclude that ν is a SAT measure. By Proposition 2.4.1 $\mathcal{X}$ is a contractive homogeneous space.

Next, since ν is SAT, the mapping $\Phi : L^\infty(\mathcal{X}) \to \mathcal{H}_\mu$ given by $(\Phi f) = f * \nu$ is a positive equivariant, weak* continuous, identity preserving isometry. Let $R_\mu : L^\infty(\mathcal{X}_\mu) \to \mathcal{H}_\mu$ be the isometry given by (2.6.1). Using Lemma 4.5 we get that $\Psi = R_\mu^{-1}\Phi : L^\infty(\mathcal{X}) \to L^\infty(\mathcal{X}_\mu)$ is a positive, equivariant, weak* continuous, identity preserving isometry. Then Theorem 3.16 shows that there exists an equivariant map $F : \mathcal{X}_\mu \to \mathcal{X}$ inducing Φ. For $f \in L^\infty(\mathcal{X})$ we have

$$\begin{aligned}\int_{\mathcal{X}} f(gx)\,\nu(dx) &= (\Phi f)(g) = (R_\mu \Psi f)(g) = \int_{\mathcal{X}_\mu} f\big(F(gx)\big)\,\rho_\mu(dx)\\ &= \int_{\mathcal{X}_\mu} f(gx)\,(F\rho_\mu)(dx) \quad \text{(a.e.)}.\end{aligned}$$

Using the fact that the μ-harmonic functions are continuous one easily concludes that $\nu = F\rho_\mu$. □

COROLLARY 4.7. *Let μ be a spread out probability measure on an almost connected amenable lcsc group and $\mathcal{X}$ a boundary of G. Then $\mathcal{X}$ supports at most one probability measure ν such that $\nu = \mu * \nu$. If $\mathcal{X}$ supports such ν then $\mathcal{X}$ is an equivariant image of the μ-boundary.*

Proof. By Proposition 3.13 there is at most one equivariant map $F : \mathcal{X}_\mu \to \mathcal{X}$. □

THEOREM 4.8. *Let μ and ν be spread out probability measures on an almost connected amenable lcsc group G, $\mathcal{X}_\mu$ the μ-boundary and $\mathcal{X}_\nu$ the ν-boundary. If $J_\mu \subseteq J_\nu$ then $\mathcal{X}_\mu \succ \mathcal{X}_\nu$.*

Proof. Let ρ_μ and ρ_ν be the Poisson kernels. By Lemma 4.4, $\mu * \rho_\nu = \rho_\nu$. Then Lemma 4.6 yields the desired conclusion. □

THEOREM 4.9. *Let μ and ν be spread out probability measures on an almost connected amenable lcsc group G, $\mathcal{X}_\mu$ the μ-boundary and $\mathcal{X}_\nu$ the ν-boundary. If $J_\mu \subseteq J_\nu$ and $\mathcal{X}_\mu \prec \mathcal{X}_\nu$ then $J_\mu = J_\nu$ and $\mathcal{X}_\mu \cong \mathcal{X}_\nu$.*

Proof. By Lemmas 4.4 and 4.6 there exists an equivariant function $F : \mathcal{X}_\mu \to \mathcal{X}_\nu$ such that $F\rho_\mu = \rho_\nu$. Since $\mathcal{X}_\mu \prec \mathcal{X}_\nu$ Theorem 3.5 yields that $\mathcal{X}_\mu \cong \mathcal{X}_\nu$ and that F is a homeomorphism. Thus $F^{-1}\rho_\nu = \rho_\mu$. Since F^{-1} is equivariant, it follows that $\nu * \rho_\mu = \rho_\mu$. Then $J_\nu \subseteq J_\mu$ by Lemma 4.4 and, consequently, $J_\mu = J_\nu$. □

THEOREM 4.10. *Let G be an almost connected amenable lcsc group and $\mathcal{F} \subseteq \mathcal{J}_{aa}(G)$ a nonempty family. Then $\mathcal{F}$ admits a maximal and a minimal element.*

Proof. For every $J_\mu \in \mathcal{F}$ let $\mathcal{X}_{J_\mu}$ be the corresponding μ-boundary. Then $\{\mathcal{X}_J\}_{J\in\mathcal{F}}$ is a family of boundaries of G (see Proposition 2.6.3). Hence, by Theorem 3.9, it admits a minimal element $\mathcal{X}_{J_*}$ and a maximal element $\mathcal{X}_{J^*}$. Theorems 4.8 and 4.9 imply that J_* is maximal while J^* is minimal in $\mathcal{F}$. □

COROLLARY 4.11. *If G is an almost connected amenable lcsc group then every ideal J in $\mathcal{J}_{aa}(G)$ contains an ideal that is minimal in $\mathcal{J}_{aa}(G)$.*

Proof. Apply Theorem 4.10 to the family $\{J' \in \mathcal{J}_{aa}(G)\,;\, J' \subseteq J\}$. □

THEOREM 4.12. *If G is an almost connected amenable lcsc group then every chain in $\mathcal{J}_{aa}(G)$ is finite.*

Proof. Let $\mathcal{C}$ be a chain in $\mathcal{J}_{aa}(G)$. For every $J_\mu \in \mathcal{C}$ let $\mathcal{X}_{J_\mu}$ be the corresponding μ-boundary. By Theorem 4.8 the family $\mathcal{F} = \{X_J\,;\, J \in \mathcal{C}\}$ is linearly ordered. Hence, by Theorem 3.9 $\mathcal{F}/\cong$ is finite. But by Theorem 4.9 distinct members of $\mathcal{F}$ give rise to distinct elements of $\mathcal{F}/\cong$. □

REMARK 4.13. When $\mathcal{X}$ is a maximal boundary of an almost connected amenable lcsc group G, it can be shown, similarly to [9, Proposition 5.8] that there exists an adapted $\mu \in P_a(G)$ such that $\mathcal{X}$ is the μ-boundary. It is clear from Theorems 4.8 and 4.9 that for such μ the ideal J_μ is minimal in $\mathcal{J}_{aa}(G)$. Although all examples of minimal ideals that we know correspond to maximal boundaries, it is not clear whether the μ-boundary of a minimal ideal is necessarily a maximal boundary.

REMARK 4.14. Amenability of G implies that $L^1_0(G)$ is the largest member $\mathcal{J}_{aa}(G)$ [15, Theorem 1.2]. Since a given maximal boundary can serve as the μ-boundary for infinitely many measures μ (see [9, Lemma 5.7 and Proposition 5.8]), it is not hard to see that a given member of $\mathcal{J}_{aa}(G)$ (e.g., $L_0^1(G)$) can contain infinitely many minimal ideals and that one can have chains of length 2. Example 5.10 in [9] can be used to construct chains of length 3. In general, examples of chains of arbitrary (finite) length can be found in connected solvable Lie groups. We refrain from going into the details here as this would require a lengthy digression into the theory of the μ-boundaries.

Remark 4.15. Using Propositions 3.6 and 3.8 one can see that Theorems 4.10, 4.12 and Corollary 4.11 remain valid for the ideal space $\mathcal{J}_a(G)$ of an almost connected amenable *Lie* group. However, this is no longer true when G is not a Lie group. When G is abelian, then for every $\mu \in P(G)$, $J_\mu = L_0^1(G, H_\mu)$ where H_μ is the closed subgroup generated by $\operatorname{supp}\mu$. Hence, if G is infinite, abelian, totally disconnected, and compact, then it is almost connected, but using a neighbourhood base at e consisting of open subgroups it is easy to see that there are no minimal elements in $\mathcal{J}_a(G)$ and one can have infinite descending chains of ideals in $\mathcal{J}_a(G)$.

Remark 4.16. It is natural to inquire whether or to what extent the results described in Theorems 4.10, 4.12 and Corollary 4.11 remain true without the assumption that G be amenable. When G is a connected semisimple Lie group the μ-boundaries of spread out probability measures are, up to isomorphisms, the homogeneous spaces G/H where H is a closed subgroup contained in the minimal parabolic subgroup MAN and containing the connected component $(MAN)_e$ [8, Corollary 5.8 and below]. When G has finite centre it is well known that this family of homogeneous spaces is finite modulo isomorphisms [1, 6]; Theorems 4.10, 4.12 and Corollary 4.11 can then be established proceeding similarly as in the present work. However, when G has infinite centre the family of μ-boundaries can be infinite, as, e.g., for the universal covering of $\mathrm{SL}(2,\mathbb{R})$. This particular group admits infinite chains of μ-boundaries (see Remark 3.11). Nevertheless, a rather lengthy and intricate proof shows that it is still true that every ideal in $\mathcal{J}_a(G)$ contains a minimal ideal.

Another difficulty in studying nonamenable groups stems from the fact that the μ-boundary of an adapted spread out probability measure on an almost connected nonamenable lcsc group need not be a boundary [1, 6], in contrast to what is described in Propositions 2.6.3. This, in particular, makes the reduction to the Lie group case more delicate than for amenable groups (where a "global" reduction is possible due to Propositions 2.3.2 and 2.6.3).

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