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EQUIDIMENSIONAL SYMMETRIC ALGEBRAS AND RESIDUAL INTERSECTIONS

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ABSTRACT. For a finitely generated module M, over a universally catenary local ring, whose symmetric algebra is equidimensional, the ideals generated by the rows of a minimal presentation matrix are shown to have height at most $\mu(M) - \operatorname{rank} M$. Moreover, in the extremal case, they are Cohen-Macaulay ideals if the symmetric algebra is Cohen-Macaulay. Some applications are given to residual intersections of ideals.

1. Introduction

Let R be a Noetherian ring and let M be a finitely generated R-module. If

$$R^m \xrightarrow{\phi} R^n \to M \to 0$$

is a presentation of M, then one may define the symmetric algebra $S(M)$ of M as

$$S(M) = R[T_1, ..., T_n]/(\ell_1, ..., \ell_m),$$

where

$$[\ell_1, ..., \ell_m] = [T_1, ..., T_n] \cdot \phi.$$

Thus the properties of $S(M)$ are reflected in the presentation matrix ϕ of M.

The problem of determining when the symmetric algebra is a domain has been extensively studied over the years. Although a great deal is known, one does not have a definitive answer. In this paper, we consider the equidimensionality and the constraint it places on the matrix ϕ. Our main result is the following, which we state in the local case.

THEOREM 1. *Let R be a universally catenary local ring, let M be a finitely generated R-module with presentation $R^m \xrightarrow{\phi} R^n \to M \to 0$, and let $J \neq R$ be the ideal generated by a row of ϕ. If $S(M)$ is equidimensional then* $\operatorname{ht} J \leq n - \operatorname{rank} M$.

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Here by the rank of a module M we mean

$$\operatorname{rank} M = \min\{\mu(M_p) \mid p \text{ is a minimal prime of } R\}.$$

Thus, for example, if R is a domain, then this is the usual (torsionfree) rank.

Note that the result is sharp, as can be seen by considering modules M with projdim $M \leq 1$. Such a module has a minimal presentation

$$0 \longrightarrow R^{n-\operatorname{rank} M} \xrightarrow{\phi} R^n \to M \to 0,$$

and one may construct examples with the property that a row of ϕ forms part of a system of parameters of R; hence $\operatorname{ht} J = \mu(J) = n - \operatorname{rank} M$. (The fact that $\mu(M) - \operatorname{rank} M \leq \dim R$, in the context of Theorem 1, can be seen using [7]; see also the proof of Lemma 2 below.)

While this paper was being written, we learned that similar results have been shown by Kwieciński [10] and very recently by Eisenbud, Huneke and Ulrich. In our situation, Kwieciński gives the result that ht $I_1(\phi) \leq n(n - \operatorname{rank} M)$, which is an immediate consequence of the Theorem when R is regular. He proves this in case R is a regular domain which is finitely generated over a field of characteristic zero; as one might expect from the additional hypotheses, the method of proof is quite different from the current one, which makes use of a deformation argument of [9, Proof of Lemma 3.2]. In an early version, we stated the result only in the case for ideals, but the general case is no different.

Before we turn to the proof of Theorem 1, we need a basic fact ensuring that the symmetric algebra has the expected dimension.

LEMMA 2. *Let R be a Noetherian local ring and let M be a finitely generated R-module. If $S(M)$ is equidimensional then* $\dim S(M) = \dim R + \operatorname{rank} M$.

Proof. By [7], for any minimal prime p of R, there exists a (unique) minimal prime $T(p)$ of $S(M)$ lying over p, and of dimension

$$\dim\ S(M)/T(p) = \mu(M_p) + \dim\ R/p.$$

But as $S(M)$ is equidimensional we must have

$$\dim\ S(M) = \mu(M_p) + \dim\ R/p$$

for any minimal prime p of R. It suffices to show that the minimum of $\mu(M_p)$ occurs at a prime with $\dim R/p = \dim R$. But this is clear as the sum $\mu(M_p) + \dim\ R/p$ is independent of the minimal prime p. □

2. Proof of Theorem 1

We may assume that J is the ideal generated by the last row of ϕ. Write

$$S = S(M) = R[T_1, ..., T_n]/(\ell_1, ..., \ell_m),$$

where

$$[\ell_1, ..., \ell_m] = [T_1, ..., T_n] \cdot \phi,$$

and let $\underline{x} = x_1, ..., x_{n-1}$ be the images of $T_1, ..., T_{n-1}$ in S. Set $P = (mS, \underline{x}) \in \operatorname{Spec}(S(M))$, where m denotes the maximal ideal of R. Then there are isomorphisms

$$\begin{aligned} S_P/(\underline{x})S_P &\cong (R[T]/T \cdot JR[T])_{(m)} \\ &\cong (R[T]/JR[T])_{(m)} \\ &\cong (R/J)(T), \end{aligned}$$

where $T = T_n$. Now supposing that $\operatorname{ht} J > n - \operatorname{rank} M$, as $S(M)$ is equidimensional and catenary, we have that

$$\begin{aligned} \dim\ S_P &= \dim\ S_P/(\underline{x})S_P + \operatorname{ht}(\underline{x})S_P \\ &\leq \dim\ S_P/(\underline{x})S_P + n - 1 \\ &= \dim\ (R/J)(T) + n - 1 \\ &= \dim\ R/J + n - 1 \\ &\leq \dim\ R - \operatorname{ht} J + n - 1 \\ &< \dim\ R - (n - \operatorname{rank} M) + n - 1 \\ &= \dim\ R + \operatorname{rank} M - 1 \\ &= \dim\ S - 1 \\ &= \dim\ S - \dim\ S/P \\ &= \dim\ S_P. \end{aligned}$$

This contradiction shows that $\operatorname{ht} J \leq n - \operatorname{rank} M$. □

THEOREM 3. *With the notation as in Theorem 1, assume that $S(M)$ is Cohen-Macaulay and that* $\operatorname{ht} J \geq n - \operatorname{rank} M$. *Then R/J is Cohen-Macaulay.*

Proof. Theorem 1 implies that $\operatorname{ht} J = n - \operatorname{rank} M$. But now the string of inequalities in the proof are all equalities. Thus $\operatorname{ht}(\underline{x})S_P = n - 1$. Since S_P is Cohen-Macaulay, it follows that $\underline{x}$ is an S_P-sequence. Hence $(R/J)(T) \cong S_P/(\underline{x})S_P$ is Cohen-Macaulay, and thus, by faithfully flat descent, so is R/J. □

We note that Theorem 3 also holds if one replaces "Cohen-Macaulay" by "Gorenstein". A similar remark holds for "complete intersection", but this is a less interesting condition for the symmetric algebra, as it means, at least if R is regular, that projdim $M \leq 1$ (see [13]). In this case the result would be clear: trivially $\mu(J) \leq \#$ columns of $\phi = n - \operatorname{rank} M$; thus if $\operatorname{ht} J \geq n - \operatorname{rank} M$, then $\mu(J) = \operatorname{ht} J$, and hence J and R/J are complete intersections.

We make one final note about the proof. This is the observation that we did not need the full strength that $S(M)$ is Cohen-Macaulay, but that a certain localization has this property. We will make use of this remark in a moment.

3. Applications to residual intersections

In the case of ideals, the previous results can be conveniently described using the notion of residual intersection. We find it useful to give the translation into this language.

Recall that a proper ideal J is said to be an s-residual intersection of I if there are elements $a_1, ..., a_s$ in I such that $J = (a_1, ..., a_s) : I$ and $\operatorname{ht} J \geq s$. If in addition $\operatorname{ht} I + J > s$ then J is called a geometric s-residual intersection. (To avoid trivial cases, one usually also assumes that $s \geq \operatorname{ht} I$, but we will find it convenient here not to impose this convention.) To study upper bounds on the heights of colon ideals one may often restrict attention to residual intersections. Actually, one would even like to know when an s-residual intersection has the "expected height" s, and when it is Cohen-Macaulay (see [6],[5],[8],[11]).

We set $A = (a_1, ..., a_s)$ and consider residual intersections $J = A : I$ with I/A cyclic. Such a J may obviously be chosen as a row of a matrix presenting I. Hence Theorems 1 and 3 now yield the following result.

Corollary 4. *Let R be a universally catenary local ring, let I be an R-ideal with* $\operatorname{ht} I > 0$*, and let $J = A : I$ be an s-residual intersection of I with I/A cyclic. If $S(I)$ is equidimensional then* $\operatorname{ht} J = s$; *if in addition* $\operatorname{Proj}(S(I))$ *is Cohen-Macaulay then so is R/J.*

Of course both conditions in the corollary are satisfied whenever $S(I)$ is Cohen-Macaulay. (How much weaker the conditions in the corollary are, however, is not so clear to the author.)

As an application, we now wish to give a residual intersection characterization of the Cohen-Macaulayness of the Rees algebra, for ideals of linear type. This may be viewed as a partial generalization of a result [3, Corollary 6.4] on d-sequences.

Recall that an ideal I of a Noetherian ring is said to be of linear type if the canonical epimorphism $S(I) \longrightarrow \mathcal{R}(I)$ to the Rees algebra $\mathcal{R}(I) = \oplus_{i\geq 0} I^i \cong R[It]$ is an isomorphism; in particular, this implies an isomorphism $S_{R/I}(I/I^2) \cong \operatorname{gr}_I(R) = \oplus_{i\geq 0} I^i/I^{i+1}$. For an ideal of linear type and positive height in a local Cohen-Macaulay ring, it is known that the Cohen-Macaulay property of $\mathcal{R}(I)$ is equivalent to that of $\operatorname{gr}_I(R)$ (see, for example, [1]). Thus we may concentrate on the Cohen-Macaulayness of $\operatorname{gr}_I(R)$.

We will exploit a basic example (see [6, 4.3]), which shows that, in the case of an ideal of linear type in a local Cohen-Macaulay ring, the extended Rees algebra $R[It, t^{-1}]$ is defined by an ideal J in $R[T_1, ..., T_n, U]$ which is a residual intersection of (I, U). More precisely, if $n = \mu(I)$, then one has $J = A : (I, U)$, where J is a geometric n-residual intersection and $(I, U)/A$ is cyclic.

THEOREM 5. *Let R be a local Cohen-Macaulay ring and let I be an R-ideal of linear type. Then the following are equivalent.*
(a) $\operatorname{gr}_I(R)$ is Cohen-Macaulay;
(b) for any local Cohen-Macaulay faithfully flat extension R' of R, and any $x \in R'$ regular on $\operatorname{gr}_{IR'}(R')$, every residual intersection $J = A : (I, x)$ of (I, x) in R', with $(I, x)/A$ cyclic, is Cohen-Macaulay.

Proof. Suppose first that (b) holds. Taking

$$R' = R[T_1, ..., T_n, U]_{(m_R, T_1, ..., T_n, U)}$$

and $x = U$, we have, as mentioned above, that $R[It, t^{-1}] \cong R[T_1, ..., T_n, U]/J$ is Cohen-Macaulay (as it is Cohen-Macaulay locally at its unique graded maximal ideal). Hence $\operatorname{gr}_I(R) \cong R[It, t^{-1}]/(t^{-1})$ is Cohen-Macaulay.

Conversely, suppose that $\operatorname{gr}_I(R)$ is Cohen-Macaulay. We will show that $\operatorname{gr}_{(I,x)}(R')$ is Cohen-Macaulay. First, note that by the flatness of the morphism $R \longrightarrow R'$, IR' is still of linear type. Further, since $R \longrightarrow T$ is a faithfully flat map of Cohen-Macaulay rings, it follows that by base change (after localizing at the irrelevant maximal ideals) that $\operatorname{gr}_I(R) \longrightarrow \operatorname{gr}_{IR'}(R')$ has the same property; in particular, $\operatorname{gr}_{IR'}(R')$ is Cohen-Macaulay. Now (I, x) is still of linear type [12, Proposition 2.5], and $\operatorname{gr}_{(I,x)}(R') \cong \operatorname{gr}_{IR'}(R')/(x)\operatorname{gr}_{IR'}(R')$. Hence $\operatorname{gr}_{(I,x)}(R')$ is Cohen-Macaulay, proving the claim. Thus we may conclude that $\mathcal{R}((I, x))$ is Cohen-Macaulay. Now we may apply Corollary 4 to $(I, x) \subset R'$ as $S_{R'}((I, x)) \cong \mathcal{R}((I, x))$ is Cohen-Macaulay, to obtain the required result. □

We next consider the Gorenstein case. This partially generalizes [4, Theorem 1.3].

THEOREM 6. *Let R be a local Gorenstein ring and let I be an R-ideal of linear type. Then the following are equivalent.*
(a) $\operatorname{gr}_I(R)$ is Gorenstein ;
(b) for any local Cohen-Macaulay faithfully flat extension R' of R, and any $x \in R'$ regular on $\operatorname{gr}_{IR'}(R')$, every residual intersection $J = A : (I, x)$ of (I, x) in R', with $(I, x)/A$ cyclic, is Gorenstein.

Proof. If (b) holds then $R[It, t^{-1}] \cong R[T_1, ..., T_n, U]/J$ is Gorenstein; hence so is $\operatorname{gr}_I(R) \cong R[It, t^{-1}]/(t^{-1})$.

Conversely, suppose that $\operatorname{gr}_I(R)$ is Gorenstein. Then, as in the proof of Theorem 5, it follows that $\operatorname{gr}_{(I,x)}(R')$ is Gorenstein. Hence, as is well-known, the blow-up $\operatorname{Proj}(\mathcal{R}((I, x))$ is Gorenstein. Since (I, x) is of linear type, the result now follows from the Gorenstein version of Corollary 4. □

We now wish to give a partial extension of Corollary 4. As an s-residual intersection in general can have height greater than s, obviously one cannot simply drop the equidimensional assumption on the symmetric algebra. However, it turns out that if we know already that the residual intersection has height s, and is unmixed (and geometric as well), then we may replace the assumption that $S(I)$ is Cohen-Macaulay with the Cohen-Macaulayness of the Rees algebra. This is actually a partial generalization, because under these extra hypotheses, the Cohen-Macaulayness of the symmetric algebra would already imply that it is the Rees algebra ([2, 6.8]).

Proposition 7. *Let R be a local Cohen-Macaulay ring, let I be an R-ideal with $\operatorname{ht} I > 0$, and let $J = A : I$ be a geometric s-residual intersection of I with I/A cyclic. Assume that J is unmixed and that $\operatorname{ht} J = s$. If $\operatorname{Proj}(\mathcal{R}(I))$ is Cohen-Macaulay then so is R/J.*

Proof. Write $A = (f_1, ..., f_s), I = (f_1, ..., f_{s+1})$, and choose a presentation

$$\mathcal{R}(I) \cong R[T_1, ..., T_{s+1}]/Q,$$

where Q is the ideal generated by all forms $F(T_1, ..., T_{s+1})$ vanishing on $f_1, ..., f_{s+1}$. The proof will follow that of Theorem 1, once we show the claim that

$$Q \subset (T_1, ..., T_s, T_{s+1}J).$$

Indeed, as $T_{s+1}J$ is contained in Q modulo $(T_1, ..., T_s)$, this containment would imply that $(Q, T_1, ..., T_s) = (T_1, ..., T_s, T_{s+1}J)$, giving the usual isomorphism locally at P, modulo the sequence $\underline{x}$.

To verify the claim, let $F \in Q$ be a form of degree $k \geq 1$. Then modulo $(T_1, ..., T_s)$ we may write $F \equiv aT_{s+1}^k$ with $a \in R$. We must show that $a \in J$. By evaluating, we see that $af_{s+1}^k \in (f_1, ..., f_s)$; hence $af_{s+1}^{k-1} \in (f_1, ..., f_s) : (f_{s+1}) = J$. Now if $a \notin J$, then $a \notin q$ for some p-primary component q of J. But then, as $af_{s+1}^{k-1} \in J \subset q$, p would, in particular, contain f_{s+1}, and hence $I + J$. Thus, since J is geometric, we have $\operatorname{ht} p \geq \operatorname{ht} I + J > s$, which contradicts the unmixedness of J. This proves the claim.

Notice that (as in the original argument [9, 3.2]), since I has positive height, $\mathcal{R}$ is equidimensional and $\dim \mathcal{R} = \dim R + 1$. Furthermore, we have $P = (m, \underline{x}) \in \operatorname{Proj}(\mathcal{R}(I))$; in particular, the local ring S_P is Cohen-Macaulay. The result now follows exactly as in the proof of Theorems 1 and 3. □

The assumption that the blow-up $\operatorname{Proj}(\mathcal{R}(I))$ is Cohen-Macaulay holds of course if $\mathcal{R}(I)$ is Cohen-Macaulay; more generally, it is well-known that the blow-up is Cohen-Macaulay if the associated graded ring $\operatorname{gr}_I(R)$ is Cohen-Macaulay.

We should point out again that Theorem 5 (as well as Corollary 4) holds if we replace the condition "Cohen-Macaulay" by "Gorenstein" or "complete intersection." One should note, however, that it is false for "regular"; this is due to the fact that regularity is not preserved under specialization. If the blow-up is smooth though, one can show, in some cases, that a *sufficiently general* residual intersection is regular ([9, 4.1]).

We should also say that, of course, the proposition does not hold without the assumption that J is unmixed; i.e., even if $\mathcal{R}(I)$ is Cohen-Macaulay, a geometric s-residual intersection of height s need not be Cohen-Macaulay. One way to construct such examples is to use again the result [6, 4.3], which (more generally without the assumption of linear type) gives the so-called extended symmetric algebra as a geometric n-residual intersection of height n.

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