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DIVISIBILITY OF IDEALS AND BLOWING UP

JOHN ATWELL MOODY

Abstract. Let R be a Noetherian integral domain, let $V = \mathrm{Spec}(R)$, and let I, J be nonzero ideals of R. Clearly, if J is either a divisor of I or a power of I there is a map $Bl_I(V) \to Bl_J(V)$ of schemes over V. The purpose of this note is to prove, conversely, that if such a map exists, then J must be a fractional ideal divisor of some power of I.

Let R be a Noetherian integral domain and let $J, I \subset R$ be ideals. Let $V = \mathrm{Spec}(R)$. It is often useful to know when there is a map $Bl_I(V) \to Bl_J(V)$ making the diagram

$$\begin{tikzcd} Bl_I(V) \arrow[rr] \arrow[dr] & & Bl_J(V) \arrow[dl] \\ & V & \end{tikzcd}$$

commute. Such a map is called a *map of schemes over* V. There is at most one such map, and such a map exists if and only if J pulls back to a locally principal sheaf of ideals on $Bl_I(V)$. (These are elementary results which follow from [1, Capter II, Section 2] by setting $Bl_I(V) = \mathrm{Proj}(R \oplus I \oplus I^2 ...)$) . The ideal I itself pulls back to $O(-E)$, the structure sheaf of $Bl_I(V)$ twisted by the exceptional divisor E.

Let us make two observations. Firstly, if J is equal to a power I^α of I, then J does pull back to the locally principal sheaf $O(-\alpha E)$. Secondly, if J is equal to a divisor of I then J pulls back to a divisor of $O(-E)$, which is again locally principal. The aim of this note is to prove that combining these two trivial cases accounts for every possibility.

Theorem. *The ideal J pulls back to a locally principal sheaf of ideals on $Bl_I(V)$ if and only if, as a fractional ideal, J is a divisor of I^α for some number α.*

The following corollaries are immediate.

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Corollary 1. *There is a map $Bl_I(V) \to Bl_J(V)$ of schemes over V if and only if there is a number α and a fractional ideal K such that $JK = I^\alpha$.*

Corollary 2. *There is an isomorphism $Bl_I(V) \cong Bl_J(V)$ of schemes over V if and only if there exist positive integers α and γ and fractional ideals K and L such that $JK = I^\alpha$ and $IL = J^\gamma$.*

Proof of Theorem. Let $(f_1, ..., f_n)$ be a generating sequence for I. Suppose J is a divisor of I^α for some α. This means there is a fractional ideal L of R so that $JL = I^\alpha$. Cover $Bl_I(V)$ by coordinate charts $U_i = \mathrm{Spec}(A_i)$ where $A_i = \bigcup_{j=0}^{\infty}(I/f_i)^j$. Since JL contains f_i^α, the ideal $J\cdot(L/f_i^\alpha)\cdot A_i \subset A_i$ contains 1. This implies that the ideal A_iJ is invertible, i.e., locally free.

It remains to prove the converse. Suppose A_iJ is locally free for each i. Let K be the fraction field of R. Recall that a fractional ideal of R is by definition any finitely-generated R-submodule of K. Given two such fractional ideals A and B we may form a new fractional ideal $[B : A] = \{x \in K : xA \subset B\}$. Note that for any fractional ideals A, B, C we have $A[B : C] \subset [AB : C]$. Suppose we succeed in proving that for each number i between 1 and n there is a number β_i such that

$$f_i^{\beta_i} \in J[I^{\beta_i} : J].$$

Then taking $\beta = max(\beta_1, ..., \beta_n)$ and multiplying both sides of the equation by $I^{\beta-\beta_i}$ gives

$$f_i^\beta \in JI^{\beta-\beta_i}[I^{\beta_i} : J] \subset J[I^\beta : J].$$

Since this holds for all i we have

$$(f_1^\beta, ..., f_n^\beta) \subset J[I^\beta : J] \subset I^\beta.$$

Now consider the ideal $I^{n\beta}$, generated by all monomials of degree $n\beta$ in the f_i. An easy counting argument shows that each such monomial is divisible by f_i^β for some i. Therefore

$$I^{n\beta} = (f_1^\beta, ..., f_n^\beta)I^{(n-1)\beta}.$$

Multiplying both sides of the previous display by $I^{(n-1)\beta}$ therefore gives

$$I^{n\beta} = (f_1^\beta, ..., f_n^\beta)I^{(n-1)\beta} \subset J[I^\beta : J]I^{(n-1)\beta} \subset I^\beta I^{(n-1)\beta} = I^{n\beta},$$

proving $JL = I^\alpha$ for $\alpha = n\beta$, $L = [I^\beta : J]I^{(n-1)\beta}$, as desired.

It remains to produce the promised numbers β_i with $f_i^{\beta_i} \in J[I^{\beta_i} : J]$. Fix i. By hypothesis there is a fractional ideal H_i of A_i, so $A_iJH_i = A_i$. It follows that the evaluation homomorphism

$$ev : A_iJ \otimes_R \mathrm{Hom}_{A_i}(A_iJ, A_i) \to A_i$$

is surjective. There is a natural isomorphism $\mathrm{Hom}_{A_i}(A_iJ, A_i) \cong \mathrm{Hom}_R(J, A_i)$ and so our evaluation homomorphism gives rise to a homomorphism

$$A_iJ \otimes_R \mathrm{Hom}_R(J, A_i) \to A_i.$$

Denote the image of this R-module homomorphism by $A_iJ \bullet \mathrm{Hom}_R(J, A_i)$, and if $X \subset A_iJ$ and $Y \subset \mathrm{Hom}_R(J, A_i)$ are R-submodules, denote by $X \bullet Y$ the image of the tensor product $X \otimes_R Y$. We have

$$A_iJ = \bigcup_{j=0}^{\infty} (I/f_i)^j J,$$

and since J is a finitely-generated R-module we have

$$\mathrm{Hom}_R(J, A_i) = \bigcup_{j=0}^{\infty} \mathrm{Hom}_R(J, (I/f_i)^j).$$

Therefore

$$\begin{aligned} 1 \in A_iJ \bullet \mathrm{Hom}_R(J, A_i) &= \bigcup_{j=0}^{\infty} \bigcup_{k=0}^{\infty} ((I/f_i)^j J) \bullet \mathrm{Hom}_R(J, (I/f_i)^k) \\ &= \bigcup_{j=0}^{\infty} \bigcup_{k=0}^{\infty} (I/f_i)^j J[(I/f_i)^k : J]. \end{aligned}$$

It follows that for some fixed j and k we have

$$1 \in (I/f_i)^j J[(I/f_i)^k : J] \subset J[(I/f_i)^{j+k} : J].$$

Taking $\beta_i = j + k$ we have $f_i^{\beta_i} \in J[I^{\beta_i} : J]$ as needed. □

References

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Dept. of Mathematics, Univ. of Warwick, Coventry, England
E-mail address: moody@maths.warwick.ac.uk