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SEPARABLE LIFTING PROPERTY AND EXTENSIONS OF LOCAL REFLEXIVITY

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Abstract. A Banach space X is said to have the *separable lifting property* if for every subspace Y of X^{**} containing X and such that Y/X is separable there exists a bounded linear lifting from Y/X to Y. We show that if a sequence of Banach spaces $E_1, E_2, \ldots$ has the joint uniform approximation property and E_n is c-complemented in E_n^{**} for every n (with c fixed), then $\left(\sum_n E_n\right)_0$ has the separable lifting property. In particular, if E_n is a $\mathcal{L}_{p_n,\lambda}$-space for every n ($1 < p_n < \infty$, λ independent of n), an L_∞ or an L_1 space, then $\left(\sum_n E_n\right)_0$ has the separable lifting property. We also show that there exists a Banach space X which is not extendably locally reflexive; moreover, for every n there exists an n-dimensional subspace $E \hookrightarrow X^{**}$ such that if $u : X^{**} \to X^{**}$ is an operator (= bounded linear operator) such that $u(E) \subset X$, then $||(u|_E)^{-1}|| \cdot ||u|| \geq c\sqrt{n}$, where c is a numerical constant.

1. Introduction

At the root of this investigation lies the principle of local reflexivity, formulated by J. Lindenstrauss and H. Rosenthal in [15] (see also Theorem 8.16 of [4]). It states:

Theorem 1.1. *If E and F are finite dimensional subspaces of X^{**} and X^*, respectively, and ε is a positive number, then there exists an operator $T : E \to X$ such that $||T||, ||T^{-1}|| < 1 + \varepsilon$, $T|_{E\cap X} = I_{E\cap X}$ and $f(ue) = f(e)$ for any $e \in E$ and $f \in F$.*

If S is a closed subspace of a Banach space X, we say that S is *complemented* (resp. *c-complemented*) in X if there exists a projection (= idempotent operator) from X onto S (resp. a projection whose norm does not exceed c).

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Suppose X and Z are Banach spaces, S is a closed subspace of X, $q : X \to X/S$ is the quotient map and $u : Z \to X/S$ is a linear operator. Then $\tilde{u} : Z \to X$ is called a *lifting* of u if $u = q \circ \tilde{u}$. Note that if $Z = X/S$, $u = I_{X/S}$ (the identity on X/S) and $\tilde{u}$ is a lifting of u, then $P = \tilde{u} \circ q$ is a projection on X such that $\ker P = S$ and $||P|| = ||\tilde{u}||$. Conversely, if such a projection P exists, then there exists a lifting $\tilde{u}$ of the identity on X/S such that $||\tilde{u}|| \leq ||P||$. Clearly, the identity on X/S lifts to X if and only if S is complemented in X.

If X is a Banach space, the identity on X^{**}/X need not lift to X^{**}. However, by [6], the principle of local reflexivity implies that the identity on X^{**}/X "locally lifts".

Theorem 1.2. *Suppose X and Y are Banach spaces such that Y is a subspace of X^{**} containing X and $\dim Y/X < \infty$. Let $q : Y \to Y/X$ be the quotient map. Then for every $\varepsilon > 0$ there exists a lifting $T : Y/X \to Y$ of the identity map on Y/X such that $||T|| < 2 + \varepsilon$. Consequently, there exists a projection P from Y onto X with norm not exceeding $3 + \varepsilon$.*

In this paper we consider two problems:

(1) Suppose $X \hookrightarrow Y \hookrightarrow X^{**}$ and Y/X is separable. Under what conditions does there exist a lifting of $I_{Y/X}$ to Y? In other words, under what conditions can Theorem 1.2 be generalized to the case of Y/X separable? This is the subject of Section 2. We show, in particular, that such a lifting exists if $X = (\sum_k E_k)_0$, where either $\sup_k \dim E_k < \infty$ or E_k is an L_p space with $1 < p < \infty$ (Proposition 2.2). Here and below, $\big(\sum_k X_k\big)_p$ (resp. $\big(\sum_k X_k\big)_0$) denotes the ℓ_p (resp. c_0) direct sum of Banach spaces X_k.

(2) Is it possible to strengthen the principle of local reflexivity? This question will be treated in Sections 3 and 4. We show that if Y is a subspace of X^{**} containing X as a subspace of finite codimension, F is a finite dimensional subspace of Y^* and $\varepsilon > 0$, then there exists a projection P from Y onto X such that $||P|| < 5 + \varepsilon$ and $f(Py) = f(y)$ for every $y \in Y$ and $f \in F$ (Proposition 4.1). We also show that the operators T mentioned in the statement of Theorem 1.1 cannot, in general, be extended to the whole of X^{**} (Theorem 3.1, Corollary 3.2). Moreover, there exists a Banach space X and a numerical constant c such that for every positive integer n there exists an n-dimensional subspace $E \hookrightarrow X^{**}$ such that if $u : X^{**} \to X^{**}$ is an operator and $u(E) \subset X$, then $||(u|_E)^{-1}|| \cdot ||u|| \geq c\sqrt{n}$ (Theorem 3.3). Other possible strengthenings of the principle of local reflexivity are discussed in Section 4.

Throughout the paper we shall use standard Banach space terminology which can be found, for instance, in [4] and [16]. We make some peripheral remarks about operator spaces; see [23] or [27] for an introduction to that subject. We say that a Banach space X has the *λ-approximation property* (*λ-AP*) if for every finite dimensional space $E \hookrightarrow X$ and $\varepsilon > 0$ there exists a finite rank map $u : X \to X$ such that $u|_E = I_E$ and $||u|| \le \lambda + \varepsilon$. If X has the λ-AP for some λ, we say X has the *bounded approximation property* (*BAP*). The space X is said to have the λ-*uniform approximation property* (λ-*UAP* in short) if there exists a function $f : \mathbb{N} \to \mathbb{N}$ (called a *uniformity function*) such that for every n-dimensional subspace $E \hookrightarrow X$ there exists an operator $u : X \to X$ such that $u|_E = I_E$, $||u|| \le \lambda$, and $\operatorname{rank} u \le f(n)$. A collection of Banach spaces $(X_i)_{i\in I}$ is said to have the λ-*joint UAP* (λ-*JUAP*) if these spaces have the λ-UAP with the same uniformity function. A Banach space X or a collection of Banach spaces $(X_i)_{i\in I}$ is said to have the λ+*UAP* (λ+*JUAP*) if it has the $(\lambda+\varepsilon)$-UAP (resp. $(\lambda+\varepsilon)$-JUAP) for every $\varepsilon > 0$. Clearly, the λ+UAP implies the λ-AP. It is known (see [7] or [19]) that a Banach space X has λ-UAP if and only if X^* does. Moreover, if f is a uniformity function for X, then it is also a uniformity function for X^{**}.

We say that an increasing sequence (E_n) of finite dimensional spaces is a *paving* of a Banach space X if $X = \overline{\cup_k E_k}$. A family $\mathcal{F}$ of finite dimensional spaces *paves* X if for any $\varepsilon > 0$ and for any finite dimensional subspace $E \hookrightarrow X$ there exist a subspace G of X containing E and $F \in \mathcal{F}$ such that $d(F, G) < 1 + \varepsilon$. (Here $d(\cdot,\cdot)$ denotes the Banach-Mazur distance.)

2. The separable lifting property

We say that a Banach space X has the λ-*separable lifting property* (λ-*SLP* in short) if for every subspace Y of X^{**} containing X and such that Y/X is separable there exists a linear lifting of the identity on Y/X to Y, with norm not exceeding λ. It is known (see [6]) that if Y/X is finite dimensional, then for every $\varepsilon > 0$ there exists such a lifting whose norm does not exceed $2 + \varepsilon$. The space X is said to have the *SLP* if it has the λ-SLP for some λ.

The main result of this section is as follows.

THEOREM 2.1. *Suppose $E_1, E_2, \dots$ is a sequence of Banach spaces having the λ+joint uniform approximation property and Y is a subspace of $\big(\sum_n E_n\big)_\infty$ containing $\big(\sum_n E_n\big)_0$ and such that $Y/\big(\sum_n E_n\big)_0$ is separable. Then there exists a projection $P : Y \to Y$ such that $\ker P = \big(\sum_n E_n\big)_0$ and $||P|| \le \lambda$.*

Below we will use the notion of M-ideal. A closed subspace J of a Banach space X is called an *L-summand* (resp. *M-summand*) in X if $X = J \oplus_1 J'$ (resp. $X = J \oplus_\infty J'$) for some subspace $J' \hookrightarrow X$. We say J as above is an *M-ideal* if its annihilator $J^\perp$ is an L-summand in X^*. We refer the reader to [9] for a detailed investigation of M-ideals.

Apparently, the first paper where the properties of M-ideals are used to solve lifting problems is [2]. There T. Andersen proved that if B is a C^*-algebra, I a closed two-sided ideal in it such that the C^*-algebra B/I has the positive unital approximation property (that is, the identity of of B/I can be approximated by positive unital finite rank maps) and B/I is separable, then there exists a positive unital (and therefore contractive) lifting $T : B/I \to B$ such that $Q \circ T = I_{B/I}$. (Here $Q : B \to B/I$ is the quotient map.) This result was later extended to n-positive maps by G. Robertson and R. Smith (see [28]).

Theorem 2.1 implies the following result.

Proposition 2.2. *Suppose $E_1, E_2, \dots$ is a sequence of reflexive Banach spaces having the $\lambda+$uniform approximation property. Then $\left(\sum_n E_n\right)_0$ has the λ-separable lifting property. Consequently:*

(1) *If $\sup_n \dim E_n < \infty$, then $\left(\sum_n E_n\right)_0$ has the 1-separable lifting property.*
(2) *If E_n is a $\mathcal{L}_{p_n,\lambda+\varepsilon}$-space for every n and every $\varepsilon > 0$ (with $1 < p_n < \infty$), then $\left(\sum_n E_n\right)_0$ has the λ-separable lifting property.*

Corollary 2.3. *Suppose $E_1, E_2, \dots$ is a sequence of Banach spaces with the $\lambda+$joint uniform approximation property and such that E_n is c-complemented in E_n^{**} for every n. Suppose Y is a separable subspace of $\left(\sum_n E_n\right)_0^{**}$ containing $\left(\sum_n E_n\right)_0$. Then there exists a projection from Y onto $\left(\sum_n E_n\right)_0$ of norm not exceeding $c(\lambda + 1)$.*

Remark. A. Sobczyk [30] proved that c_0 is 2-complemented in every separable Banach space containing it. (For a modern proof of this fact see [32] or Theorem 2.f.5 in [16].) One should also note the remarkable result of M. Zippin [33] that a separable Banach space which is complemented in every separable space containing it must be isomorphic to c_0. Proposition 2.2 is a generalization of Sobczyk's theorem.

Proof of Proposition 2.2. The first statement follows directly from Theorem 2.1. By [24], any set of $\mathcal{L}_{p,\lambda}$-spaces (λ fixed, $1 \le p \le \infty$) has the $\lambda+$JUAP, and $\mathcal{L}_{p,\lambda}$-spaces are reflexive if $1 < p < \infty$. This implies the statement of part (2) of the proposition. □

However, for some sequences $E_1, E_2, \dots$ of finite dimensional spaces, $\left(\sum_n E_n\right)_0$ does not have the SLP.

Proposition 2.4. *Suppose X is a separable Banach space failing the bounded approximation property and $E_1 \hookrightarrow E_2 \hookrightarrow \dots$ is an increasing sequence of finite dimensional subspaces of X such that $X = \overline{\cup_n E_n}$. Then there exists a separable subspace $Y \hookrightarrow \left(\sum_n E_n\right)_\infty$ containing $\left(\sum_n E_n\right)_0$ and such that there is no bounded projection from Y onto $\left(\sum_n E_n\right)_0$.*

Proof. Proposition 2.4 follows from Lusky's construction in [18]. Let Y be the subspace of $\left(\sum_n E_n\right)_\infty$ consisting of sequences $(e_1, e_2, \dots)$ for which $\lim_{n\to\infty} e_n$ exists in X. Note that Y has the BAP. Indeed, for every positive integer n we define a contractive finite rank projection $P_n : Y \to Y$ by setting $P_n(e_1, e_2, \ldots) = (e_1, e_2, \ldots, e_{n-1}, e_n, e_n, \ldots)$. The sequence (P_n) converges to I_Y uniformly on compact sets. Note also that $X = Y/\big(\sum_n E_n\big)_0$. Therefore Y is separable and $\big(\sum_n E_n\big)_0$ is not complemented in Y. (Otherwise, X would have the BAP.) □

Remark 1. Lusky [18] was interested in a sort of converse to Proposition 2.4: if X is a separable Banach space with the BAP and (E_n) is an increasing sequence of finite dimensional subspaces of X such that $X = \overline{\cup E_n}$ and Y is defined as in the proof of Proposition 2.4, then X is complemented in Y and Y is isomorphic to $X \oplus \big(\sum_n E_n\big)_0$. Indeed, then there exist finite rank operators T_n on X with $T_n X \subset E_n$ and $T_n \to I_X$ strongly. Define $S : X \to Y$ by setting $Sx = (T_n x)_n$. Then S is a lifting of the quotient map $q : Y \to X$.

Remark 2. Note that if X, Y and (E_n) are as in the proof of Proposition 2.4 and $Q : Y \to X$ is the quotient map, then $Q^* : X^* \to Y^*$ is an isometric isomorphism from X^* onto a 1-complemented subspace of Y^*. This follows from Proposition 1 in [10] (see also Corollary 1.4 of [11]).

Combining Proposition 2.4 and Remark 2 with known facts, we obtain the following observation made, but not published, by G. Schechtman and the first author in 1996:

Corollary JS. *There is a subspace Y_1 of c_0 which has a basis such that Y_1^* fails the approximation property.*

Proof. Let X be a subspace of c_0 which fails the approximation property ([16], Theorem 2.d.6), let $E_1 \hookrightarrow E_2 \hookrightarrow \ldots$ be an increasing sequence of finite dimensional subspaces of X such that $X = \overline{\cup_n E_n}$, and define Y as in Proposition 2.4. The space Y embeds into c_0; indeed, it is an old observation of Lindenstrauss that if a Banach space Y has a subspace Y_0 so that Y_0 and Y/Y_0 both embed into c_0, then so does Y. (Use the fact that any embedding from Y_0 into c_0 extends to an operator from Y into c_0 to see that Y embeds into $c_0 \oplus c_0$.) By Remark 2, Y^* contains a complemented copy of X^* and hence fails the approximation property. Since the projections P_n defined in the proof of Proposition 2.4 commute, Y has a finite dimensional decomposition. It then follows from Theorem 1.e.13 in [16] that there is a sequence $G_1, G_2, \ldots$ of finite dimensional spaces so that $Y_1 := Y \oplus (\sum G_n)_0$ has a basis, and Y_1 embeds into c_0 because both of its summands do. □

To prove Theorem 2.1, we need two lemmas. The first of these is essentially contained in [7].

LEMMA 2.5. *A sequence of Banach spaces $E_1, E_2, \ldots$ has the $\lambda+$joint uniform approximation property if and only if $\left(\sum_n E_n\right)_\infty$ has the $\lambda+$uniform approximation property.*

The following lemma is a known piece of Banach space lore; we include the proof for the sake of completeness.

LEMMA 2.6. *Suppose Z is a Banach space with the $\lambda+$uniform approximation property and Y_0 is a separable subspace of Z. Then there exists a separable subspace $Y \hookrightarrow Z$ containing Y_0 and having the $\lambda+$uniform approximation property.*

Proof. By the definition of the $\lambda+$UAP, for every $k \in \mathbb{N}$ there exists a function $f_k : \mathbb{N} \to \mathbb{N}$ such that for every n-dimensional subspace F of Z there exists an operator $u : Z \to Z$ such that $u|_F = I_F$, $||u|| \leq \lambda + 2^{-k}$ and $\operatorname{rank} u \leq f_k(n)$. Let $(E_i)_{i=1}^\infty$ be a sequence of finite dimensional subspaces of Y_0 such that for any $\varepsilon > 0$ and any finite dimensional $E \hookrightarrow Y_0$ there exists i such that the Hausdorff distance $d_H(B_E, B_{E_i}) < \varepsilon$. (Here B_F stands for the unit ball of F, viewed as a subset of Y_0.) Let $Y_1 = \overline{\operatorname{span}[u_{ik}(E_i)]}$, where $u_{ik} : Z \to Z$ is such that $u_{ik}|_{E_i} = I_{E_i}$, $||u_{ik}|| \leq \lambda + 2^{-k}$ and $\operatorname{rank} u_{ik} \leq f_k(\dim E_i)$. Then Y_1 is separable and for every finite dimensional subspace E of Y_0 and every positive integer k there exists an operator $u : Z \to Y_1$ such that $u|_E = I_E$, $||u|| \leq \lambda + 2^{-(k-1)}$ and $\operatorname{rank} u \leq f_k(\dim E)$.

Similarly, we construct a separable $Y_2 \hookrightarrow Z$ containing Y_1 and such that for every finite dimensional subspace E of Y_1 and every positive integer k there exists an operator $u : Z \to Y_2$ such that $u|_E = I_E$, $||u|| \leq \lambda + 2^{-(k-1)}$ and $\operatorname{rank} u \leq f_k(\dim E)$. In the same manner we find Y_3, Y_4, etc. Let $Y = \overline{\cup_k Y_k}$. Then Y is a separable subspace of Z and for every finite dimensional $E \hookrightarrow Y$ there exists an operator $u : Y \to Y$ such that $u|_E = I_E$, $\operatorname{rank} u \leq f_k(\dim E)$ and $||u|| \leq \lambda + 2^{-(k-2)}$. $\square$

Proof of Theorem 2.1. For notational simplicity let $W_0 = \left(\sum_n E_n\right)_0$ and $W_\infty = \left(\sum_n E_n\right)_\infty$. Then W_∞ has the $\lambda+$UAP by Lemma 2.5, and hence so does W_∞^{**}. It follows from Theorem I.2.2 of [9] that W_0 is an M-ideal in W_∞. By duality, there exists a subspace $W \hookrightarrow W_\infty^{**}$ such that $W_\infty^{**} = W_0^{**} \oplus_\infty W$. Thus, $W = \left(W_\infty / W_0\right)^{**}$ has the $\lambda+$UAP, and therefore W_∞ / W_0 has the $\lambda+$UAP. Since Y/W_0 is a subspace of W_∞ / W_0, Lemma 2.6 implies that there exists a separable Banach space Z with the $\lambda+$UAP and such that $Y/W_0 \hookrightarrow Z \hookrightarrow W_\infty / W_0$. Note that $Z = Y_1 / W_0$, where Y_1 is a subspace of W_∞. By Theorem II.2.1 of [9], there exists an operator $T : Y_1/X \to Y_1$ such that $QT = I_{Y_1/X}$ (where $Q : Y_1 \to Y_1/X$ is the quotient map) and $||T|| \leq \lambda$. Clearly, T maps Y/X into Y. $\square$

The space $C([0, 1])$ is an example of a separable Banach spaces with the 1+UAP which fails the SLP. Indeed, there exists a separable Banach space Z

containing $C([0,1])$ and such that there is no bounded projection from Z onto $C([0,1])$ (see [1]). Let j be the natural embedding of $C([0,1])$ into $C([0,1])^{**}$. Since $C([0,1])^{**}$ is 1-injective (see [12] or [14]), j has a contractive extension $\tilde{j} : Z \to C([0,1])^{**}$. Let $Y = \overline{\tilde{j}Z}$. Then Y is a separable subspace of $C([0,1])^{**}$ containing $C([0,1])$ and there is no bounded projection from Y onto $C([0,1])$. (Otherwise there would have existed a projection from Z onto $C([0,1])$.)

Suppose $E_1, E_2, \ldots$ are Banach spaces and Y is a subspace of $\big(\sum_n E_n\big)_\infty$ which contains $\big(\sum_n E_n\big)_0$ as a subspace of finite codimension. By Theorem 1.2, for every $\varepsilon > 0$ there exists a projection $P : Y \to Y$ such that $||P|| < 2 + \varepsilon$ and $\ker P = \big(\sum_n E_n\big)_0$. However, in this case a modification of a technique of Sobczyk [30] yields a sharper result:

Proposition 2.7. *Suppose $E_1, E_2, \ldots$ are Banach spaces and Y is a subspace of $\big(\sum_n E_n\big)_\infty$ which contains $X := \big(\sum_n E_n\big)_0$ as a subspace of finite codimension. Then there exists a projection $P : Y \to Y$ such that $||P|| = 1$ and $\ker P = X$.*

Proof. Let $m = \dim Y/X$ and find $y_1, \ldots, y_m \in Y$ ($y_k = (e_{1k}, e_{2k}, \ldots)$) so that $Y = \operatorname{span}[X, y_1, \ldots, y_m]$. For every positive integer n find N_n such that for every $i > N_n$ and all scalars $a_1, \ldots, a_m$

$$||\sum_{k=1}^m a_k e_{ik}|| \le (1 - 2^{-n})^{-1} \limsup_j ||\sum_{k=1}^m a_k e_{jk}||.$$

We can assume without loss of generality that $N_n < N_{n+1}$ for every n. For $1 \le k \le m$, let $\tilde{y}_k = (\alpha_1 e_{1k}, \alpha_2 e_{2k}, \ldots)$, where $\alpha_i = 1 - 2^{-n}$ if $N_n < i \le N_{n+1}$ and $\alpha_i = 0$ if $i \le N_1$. Clearly, $Y = \operatorname{span}[\big(\sum_n E_n\big)_0, \tilde{y}_1, \ldots, \tilde{y}_m]$. Let P be the projection satisfying $P|_X = 0$ and $P\tilde{y}_k = \tilde{y}_k$ for $1 \le k \le m$. Then $||P|| = 1$. $\square$

Remark. The starting point of this investigation was a question of Kirchberg: if K is the space of compact operators acting on ℓ_2 (with its natural operator space structure) and Y is a separable operator space containing K, does there exist a *bounded* projection from Y onto K? It is known that a *completely bounded* projection from Y onto K need not exist. By the Stinespring extension theorem (see, e.g., Theorem 7.3 of [23] or Theorem 3.6 of [26]) this question is equivalent to the following: if Y is a separable subspace of $B(\ell_2)$ containing K, does there exist a projection from Y onto K? By writing $Y = \operatorname{span}[K, y_1, y_2, \ldots]$ and cutting off the "off-diagonal" parts of $y_1, y_2, \ldots$ we can reformulate the problem as follows: if Y is a separable subspace of $\big(\sum_n M_n\big)_\infty$ containing $\big(\sum_n M_n\big)_0$, does there exist a bounded projection from Y onto $\big(\sum_n M_n\big)_0$? Here, $M_n = B(\ell_2^n)$ is the space of $n \times n$ matrices.

The question of Kirchberg is discussed in more detail in [21] and [29].

By Proposition 2.3, if there exists a separable Banach space X without the approximation property paved by the family $(M_n)_{n=1}^{\infty}$, then the answer to the last question is negative.

We know that $B(\ell_2)$ fails the approximation property (see [31]), and thus has separable subspaces without the approximation property. This leads to another open question: does the family $(M_n)_{n=1}^{\infty}$ pave $B(\ell_2)$? More generally, if $X = \overline{\cup E_i}$ is a paving of X, does the family $(E_i)_{i=1}^{\infty}$ pave X^{**}? Note that the positive answer to the first question will imply that the answer to Kirchberg's question is negative.

3. Extendable local reflexivity

Following [21], we say that a Banach space X is *C-extendably locally reflexive* (*C-ELR*) if for every finite dimensional $E \hookrightarrow X^{**}$ and $F \hookrightarrow X^{*}$ and for every $\varepsilon > 0$ there exists an operator $u : X^{**} \to X^{**}$ such that $u(E) \subset X$, $||u|| \leq C + \varepsilon$ and $f(e) = f(ue)$ for every $e \in E$ and $f \in F$. Note that given any $\varepsilon > 0$, we can guarantee that $||(u|_E)^{-1}|| < 1+\varepsilon$ by choosing F to be large enough. We say that X is *extendably locally reflexive* (*ELR*) if it is C-ELR for some C.

Rosenthal asked whether every Banach space is ELR. Below we give a negative answer to this question.

THEOREM 3.1. *Suppose X is a Banach space with the bounded approximation property. Then X is extendably locally reflexive if and only if X^* has the bounded approximation property. More precisely:*

1. *If X has the C_1-approximation property and is C_2-extendably locally reflexive, then X^* has the C_1C_2-approximation property.*
2. *If X^* has the C-approximation property, then X is C-extendably locally reflexive (and has the C-approximation property).*

REMARK. It is a well known consequence of the local reflexivity principle that if X^* has the C-AP, then so does X. Part (2) was proved by Rosenthal.

COROLLARY 3.2. *The space T of trace class operators on ℓ_2 is not extendably locally reflexive.*

Proof. If T is ELR, then, by Theorem 3.1, $T^* = B(\ell_2)$ has the BAP, which contradicts [31]. □

THEOREM 3.3. *There exists a separable Banach space X with the following property: for any $n \in \mathbb{N}$ there exists an n-dimensional subspace E of X^{**} such that if $u : X^{**} \to X^{**}$ is an operator and $u(E) \subset X$, then $||(u|_E)^{-1}|| \cdot ||u|| \geq c\sqrt{n}$, where c is a numerical constant.*

Proof of Theorem 3.1. (1) Pick $\varepsilon > 0$ and finite dimensional spaces $E \hookrightarrow X^{**}$ and $F \hookrightarrow X^*$. Since X is C_1-ELR, there exists an operator $v : X^{**} \to$

X^{**} such that $v(E) \subset X$, $||v|| < C_1+\varepsilon$, and $f(ve) = f(e)$ for all $e \in E, f \in F$. Since X has the C_2-AP, there exists a finite rank operator $S : X \to X$ such that $||S|| < C_2 + \varepsilon$ and $S|_{v(E)} = I_{v(E)}$. Consider the finite rank map $u = S^{**}v : X^{**} \to X$. Note that $||u|| < (C_1+\varepsilon)(C_2+\varepsilon)$ and $f(ue) = f(e)$ for all $e \in E, f \in F$.

Let $G = u^*(X^*) \hookrightarrow X^{***}$. By local reflexivity, there exists $T : G \to X^*$ such that $(Tg)(e) = g(e)$ for all $e \in E, g \in G$, and $||T|| < 1 + \varepsilon$. Consider $w = Tu^* : X^* \to X^*$. Then $||w|| < (1+\varepsilon)(C_1+\varepsilon)(C_2+\varepsilon)$ and $(wf)(e) = f(e)$ for all $e \in E, f \in F$.

Let I be a set of all triples (E, F, ε), where $\varepsilon > 0$ and E and F are finite dimensional subspaces of X^{**} and X^*, respectively. We say that $(E, F, \varepsilon) \prec (E', F', \varepsilon')$ if $E \hookrightarrow E'$, $F \hookrightarrow F'$, and $\varepsilon' < \varepsilon$. The relation $\prec$ defines a partial order on I. By the reasoning above, there exists a net of finite rank operators $w_i : X^* \to X^*$ $(i \in I)$ such that $\lim_i ||w_i|| \leq C_1C_2$ and $w_i \longrightarrow I_{X^*}$ in the point-weak topology. By Mazur's theorem, there exists a net of finite rank operators $\tilde{w}_j : X^* \to X^*$ such that $\lim_j ||\tilde{w}_j|| \leq C_1C_2$ and $w_j \longrightarrow I_{X^*}$ in the point-norm topology. (In fact, the $\tilde{w}_j$'s are convex combinations of w_i's.) This shows that X^* has the C_1C_2-AP.

(2) Suppose X^* has the C-AP. It is well known that X also has the C-AP. To show that X is C-ELR, pick $\varepsilon > 0$ and E and F as in the definition of extendable local reflexivity. Then there exists a finite rank operator $u : X^* \to X^*$ such that $u|_F = I_F$ and $||u|| < C + \varepsilon$. Let $G = u^*(X^{**})$. By the principle of local reflexivity, there exists an operator $T : G \to X$ such that $||T|| < 1 + \varepsilon$ and $f(Tg) = f(g)$ for every $f \in F$ and $g \in G$. Then $Tu^* : X^{**} \to X$ is a finite rank map, $f(Tue) = f(e)$ for every $e \in E$ and $f \in F$ and $||Tu|| < (1 + \varepsilon)(C + \varepsilon)$. Since ε can be chosen to be arbitrarily small, we conclude that X is C-ELR. □

To prove Theorem 3.3 we follow Pisier's construction in Chapter 10 of [25].

Lemma 3.4. *There exists a constant $c > 0$ and a separable Banach space Z such that*

1. $\pi_2(u) \leq c||u||$ *for every finite rank operator* $u : Z \to Z$;
2. *if* $E_1, E_2, \ldots, E_n$ *are finite dimensional subspaces of* Z, *then for every* $\varepsilon > 0$ $\left(\sum_{k=1}^n E_k\right)_1$ *is* $(1 + \varepsilon)$*-isomorphic to a subspace of* Z.

Proof. By Theorem 10.4 of [25], there exist a numerical constant c satisfying the following property: if Y is a separable Banach space with $c_2(Y) \leq c$, then there exists a separable Banach space $\tilde{Y}$ containing Y such that $c_2(\tilde{Y}) \leq c$ and $\pi_2(u) \leq c||u||$ for every finite rank operator $u : \tilde{Y} \to \tilde{Y}$. Here $c_2(Y)$ is the infimum of all real numbers λ such that $\left(\sum_{k=1}^n ||y_k||^2\right)^{1/2} \leq \lambda \cdot Ave_{\pm}||\sum_{k=1}^n \pm y_k||$ for every $y_1, \ldots, y_n \in Y$. By the Khintchine-Kahane inequality (see, e.g., Theorem 11.1 in [4] or Theorem 1.e.13 in [17]), $c_2(Y)$ is equivalent to the cotype 2 constant of Y. Since the one-dimensional space satisfies the conditions imposed on Y, there exists a separable infinite dimensional Banach space X_0 satisfying the following properties: $\pi_2(u) \leq c||u||$ for every finite rank operator $u : X_0 \to X_0$ and $c_2(X_0) \leq c$.

By the definition of $c_2(\cdot)$ and by the Minkowski Inequality (Theorem 25 in [8]), $c_2(\ell_1(X_0)) = c_2(X_0) \leq c$. By Theorem 10.4 of [25], there exists a separable Banach space X_1 containing $\ell_1(X_0)$ such that $c_2(X_1) \leq c$ and $\pi_2(u) \leq c||u||$ for every finite rank operator $u : X_1 \to X_1$. Similarly, find a separable Banach space X_2 containing $\ell_1(X_1)$ such that $c_2(X_2) \leq c$ and $\pi_2(u) \leq c||u||$ for every finite rank operator $u : X_2 \to X_2$. Proceed further in the same manner. Let $Z = \overline{\cup X_k}$. Clearly the space Z has the properties required by the lemma. □

Proof of Theorem 3.3. Let Z be as in the statement of Lemma 3.3. Find an increasing sequence of subspaces $E_1 \hookrightarrow E_2 \hookrightarrow \ldots \hookrightarrow Z$ such that $\dim E_k = k$ and $Z = \overline{\cup_k E_k}$. Let $X = \left(\sum_{k=1}^{\infty} E_k\right)_1$. Then $X^* = \left(\sum_{k=1}^{\infty} E_k^*\right)_\infty$. Let J be the isometric embedding of Z^* into X^* defined by $Jf = (f|_{E_1}, f|_{E_2}, \ldots)$. Pick a free ultrafilter $\mathcal{U}$ on $\mathbb{N}$ and define a contractive projection P on X^* with $PX^* = JZ^*$ by setting $P((f_1, f_2, \ldots)) = J \lim_{\mathcal{U}} f_i$ $(f_i \in E_i)$. The functional $f = \lim_{\mathcal{U}} f_i$ is defined by letting $f(x) = \lim_{\mathcal{U}} f_i(x)$ for $x \in \cup E_i$ and extending it by continuity to an element of X^*. We can thus identify Z^* with JZ^* and Z with a subspace of $Z^{**} \simeq P^* X^{**}$.

Suppose $u : X^{**} \to X^{**}$ is an operator such that $u(E_n) \subset X$. Let $\lambda = ||(u|_E)^{-1}||$. We will show that $||u|| \geq \sqrt{n}/(c\lambda)$. By a small perturbation argument we can assume without loss of generality that $u(E_n) \subset \left(\sum_{k=1}^{m} E_k\right)_1$. Let $T : \left(\sum_{k=1}^{m} E_k\right)_1 \to Z$ be an operator such that $||T|| < 1+\varepsilon$ and $||T^{-1}|| < 1+\varepsilon$. Then $||(Tu|_{E_n})^{-1}|| < \lambda(1+\varepsilon)$. Therefore

$$\pi_2(Tu) \geq \pi_2(Tu|_{E_n}) \geq \frac{1}{\lambda(1+\epsilon)} \pi_2(I_{E_n}) = \frac{\sqrt{n}}{\lambda(1+\epsilon)}.$$

Thus,

$$||u|| \geq ||u|_{E_n}|| \geq \frac{1}{||T||} ||Tu|_{E_n}|| \geq \frac{1}{c(1+\varepsilon)} \pi_2(Tu) \geq \frac{\sqrt{n}}{c\lambda(1+\varepsilon)^2}.$$

Since ε can be chosen to be arbitrarily small, $||u|| \geq \sqrt{n}/(c\lambda)$. □

4. Other strengthenings of local reflexivity

First we combine Theorems 1.1 and 1.2 into one.

PROPOSITION 4.1. *Suppose X is a Banach space, Y a subspace of X^{**} containing X as a subspace of finite codimension, F a finite dimensional subspace of X^*, E a finite dimensional subspace of Y, and $0 < \varepsilon < 1/2$. Then there exists a projection P from Y onto X such that $||P|| < 5 + \varepsilon$,*

$||P|_E|| < 1+\varepsilon$, $||(P|_E)^{-1}|| < 1+\varepsilon$, and $f(y) = f(Py)$ for every $y \in Y$ and $f \in F$.

Proof. By Theorem 1.2, there exists a projection $Q : Y \to Y$ such that $||Q|| < 2+\varepsilon/5$ and $\ker Q = X$. Let $G = QY$. Then there exists a finite dimensional subspace H of X such that $E \hookrightarrow G+H$. By the principle of local reflexivity, there exists an operator $u : G+H \to K$ (where K is a finite dimensional subspace of X) such that $||u|| < 1+\varepsilon/5$, $||u^{-1}|| < 1+\varepsilon/5$, $u|_H = I_H$ and $f(y) = f(uy)$ for every $y \in G+H$ and $f \in F$. Define P as follows: $P|_X = I_X$, $P|_G = u|_G$. Then P is a projection onto X, $||P|| < 5+\varepsilon$, $P|_E = u|_E$, and $f(y) = f(Py)$ for every $y \in Y$ and $f \in F$. □

Remark. Theorem 1.2 implies that for every finite dimensional $E \hookrightarrow X^{**}$ and $\varepsilon > 0$ there exists a finite dimensional $E_1 \hookrightarrow X^{**}$ containing E and a projection from E_1 onto $E_1 \cap X$ of norm not exceeding $3+\varepsilon$. However, extending E is necessary: for every $C > 0$ there exists a finite dimensional subspace $E \hookrightarrow \ell_\infty$ such that any projection from E onto $E \cap c_0$ has norm exceeding C. Indeed, pick any n and let F be a $(1+\varepsilon)$-isomorphic copy of ℓ_2^n in c_0. Clearly, we can find a $(1+\varepsilon)$-isomorphic copy of ℓ_∞^N (call it $\tilde{E}$) containing F. Then $\tilde{E} = F + \tilde{G}$, where $F \cap \tilde{G} = \{0\}$. By pushing $\tilde{G}$ out of c_0 (and deforming it slightly) we obtain $G \hookrightarrow \ell_\infty$ such that $G \cap c_0 = \{0\}$ and the Hausdorff distance between the unit balls of G and $\tilde{G}$ does not exceed ε/N. Let $E = F + G$. Then E is $(1+3\varepsilon)$-isomorphic to ℓ_∞^N and, by Grothendieck's theorem (see Theorem 5.4 of [25]), every projection from E onto F has norm at least $\sqrt{2/\pi}\sqrt{n}/(1+\varepsilon)^4$.

Proposition 4.1 shows that the operator T from Theorem 1.1 can be "extended" from E to $\mathrm{span}[E, X]$ (with its norm increasing from $1+\varepsilon$ to $5+\varepsilon$). Can T be extended further? More precisely, we ask four questions:

(1) Does there exist a constant C with the following property: if X is a Banach space and E a finite dimensional subspace of X^{**}, then there exists an operator $T : X^{**} \to X^{**}$ such that $TE \subset X$, $||T|| \leq C$ and $T|_{E\cap X} = I_{E\cap X}$? Note that we do not require that the action of any functionals be preserved, or that $T|_E$ be an isomorphism.
(2) Same as (1), with the additional condition $TX \subset X$.
(3) Same as (1), with the additional condition $T|_X = I_X$.
(4) Same as (1), with the additional condition $TX^{**} \subset X$.

We shall show that the answers to questions (3) and (4) are negative (Propositions 4.2 and 4.3, respectively). The questions (1) and (2) are open.

Remark. Questions (1)–(4) above are motivated by some operator space problems. Recall that an operator space X is called *locally reflexive* if there exists a constant $\lambda > 0$ such that for every finite dimensional $E \hookrightarrow X$ and $F \hookrightarrow$

X^* there exists an operator $u : E \to X$ such that $||u||_{cb} < \lambda$, $u|_{E\cap X} = I_{E\cap X}$, and $f(e) = f(ue)$ for every $e \in E$ and $f \in F$. This notion was introduced in [5], where it was shown that the full C^*-algebra of a free group with two generators $C^*(\mathbb{F}_2)$ is not locally reflexive. We say that an operator space X has the *local extension property* (*LEP*) if there exists a constant $\lambda > 0$ such that for every finite dimensional $E \hookrightarrow X^{**}$ there exists an operator $u : E \to X$ such that $||u||_{cb} < \lambda$ and $u|_{E\cap X} = I_{E\cap X}$. It is not known whether every maximal operator space is locally reflexive, or even has the LEP (see [20] and [22] for further discussion on this topic). However, if the answer to (1) is positive, then every maximal operator space has the LEP.

Proposition 4.2. *Suppose Z is a Banach space such that Z^{**} has the bounded approximation property and Z^{**}/Z fails the bounded approximation property. Then for every $C > 0$ there exists a finite dimensional subspace $E \hookrightarrow Z^{**}$ with the following property: if $T : Z^{**} \to Z^{**}$ is an operator, $TE \subset Z$ and $T|_Z = I_Z$, then $||T|| \geq C$.*

Remark. It was shown in [13] (see also Theorem 1.d.3 in [16]) that if X is a separable Banach space, then there exists a Banach space Z such that Z^{**} is separable, has a monotone basis, and $X = Z^{**}/Z$.

Proof. Suppose Z^{**} has the λ-AP. Let $q : Z^{**} \to Z^{**}/Z = X$ be the quotient map. Suppose, for the sake of contradiction, that there exists $C > 0$ such that for every $E \hookrightarrow Z^{**}$ there exists $T : Z^{**} \to Z^{**}$ such that $TE \subset Z$, $T|_Z = I_Z$, and $||T|| \leq C$. Let F be a finite dimensional subspace of X. Then there exists an operator $u : F \to Z^{**}$ such that $qu = I_F$. By assumption, there exists $T : Z^{**} \to Z^{**}$ such that $TuF \subset Z$, $T|_Z = I_Z$ and $||T|| \leq C$. Let $S = I_{Z^{**}} - T$ and consider the operator $\tilde{S} : X \to Z^{**}$ which maps $z^{**} + Z$ into Sz^{**}. Clearly $\tilde{S}$ is well-defined and $||\tilde{S}|| = ||S|| \leq C + 1$. Moreover, $q\tilde{S}|_F = I_F$. Indeed, if $x \in F$, then $\tilde{S}x = Sux \in ux + Z$, and therefore $q\tilde{S} = qux = x$.

Now fix $\varepsilon > 0$. Then there exists a finite rank operator $v : Z^{**} \to Z^{**}$ such that $||v|| < \lambda + \varepsilon$ and $||(I - v)|_{\tilde{S}F}|| < \varepsilon$. Then $qv\tilde{S} : X \to X$ is a finite rank map, $||qv\tilde{S}|| \leq (C + 1)(\lambda + \varepsilon)$ and $||qv\tilde{S}|_F - I_F|| \leq \varepsilon$. This contradicts our assumption that X fails the BAP. □

Remark. If Z^{**} satisfies the assumptions of Proposition 4.2, there does not exist a net of finite rank linear operators $u_i : Z^{**} \to Z^{**}$ such that $||u_i|| < C$ for every i, $u_i \to I$ strongly and $u_i Z \subset Z$. For then (u_i) would induce finite rank operators on Z^{**}/Z which tend to the identity.

A Banach space X is said to have the *compact approximation property* (*CAP*) if I_X can be approximated by compact operators in the topology of uniform convergence on compact subsets of X. We say that X has the *(weakly) compact bounded approximation property* (*(W)CBAP*) if for every finite dimensional subspace $E \hookrightarrow X$ and every ε there exists a (weakly) compact map $u : X \to X$ such that $||u|_E - I_E|| < \varepsilon$ and $||u|| < \lambda$ (for some λ independent of E).

The CBAP and the WCBAP are equivalent for subspaces of c_0. This follows from the fact that every weakly null normalized sequence in c_0 contains a subsequence which is equivalent to the unit vector basis for c_0 ([16], Propositions 1.a.12 and 2.a.1). This property of c_0 implies that an operator whose domain is a subspace of c_0 is compact iff it is weakly compact iff it is strictly singular.

By the discussion on page 94 of [16] (see also [3]), there exists a subspace of c_0 which fails the CAP (and, therefore, also fails the (W)CBAP). Thus, the negative answer to the question (4) follows from the proposition below.

Proposition 4.3. *Let X be an infinite dimensional subspace of c_0. Then the following are equivalent:*

(1) *X fails the weakly compact bounded approximation property.*
(2) *For every $C > 0$ there exists a finite dimensional subspace $E \hookrightarrow X$ such that if $T : X^{**} \to X$ is an operator and $T|_E = I_E$, then $||T|| > C$.*

Proof. (1) $\Rightarrow$ (2): We show first that every operator from X^{**} to X is strictly singular. Indeed, otherwise there will exist an operator $T : X^{**} \to X$ and infinite dimensional subspaces Y and Z of X^{**} and X, respectively, such that $TY = Z$ and $T|_Y$ is an isomorphism. Since every infinite dimensional subspace of c_0 contains an isomorphic copy of c_0 (see Theorem 2.a.2 in [16]), we can assume that both Y and Z are isomorphic to c_0. By Sobczyk's theorem (see [30] or Proposition 2.2), there exists a projection P from X onto Z. Then $\tilde{P} = (T|_Y)^{-1}PT$ is a projection from X^{**} onto Y. Let $u : c_0 \to Z$ be an isomorphism. Consider $u^{**} : \ell_\infty \to X^{(4)}$ and the contractive projection $Q : X^{(4)} \to X^{**}$. Then $u^{-1}\tilde{P}Qu^{**} : \ell_\infty \to c_0$ is a projection. This is, however, impossible (see Theorem 2.a.7 of [16]).

Thus, every operator from $T : X^{**} \to X$ is strictly singular. A fortiori, $T|_X$ is strictly singular and hence compact since X is a subspace of c_0. However, since X fails the WCBAP, for every $C > 0$ there exists a finite dimensional subspace $E \hookrightarrow X$ such that if $u : X \to X$ is compact and $u|_E = I_E$, then $||u|| > C$. By the reasoning above, if $T : X^{**} \to X$ is such that $T|_E = I_E$, then $||T|| > C$.

(2) $\Rightarrow$ (1): This implication is true for all Banach spaces X, not only for subspaces of c_0. Suppose X has the WCBAP. Then there exists $\lambda > 0$ such that for every finite dimensional $E \hookrightarrow X$ there exists a weakly compact operator $T : X \to X$ so that $T|_E = I_E$ and $||T|| < \lambda$. Since T is weakly compact, $T^{**}X^{**} \subset X$. □

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References

[1] D. Amir, *Projections onto continuous function spaces*, Proc. Amer. Math. Soc. **15** (1964), 396–402.

[2] T. Andersen, *Linear extensions, projections, and split faces*, J. Funct. Anal. **17** (1974), 161–173.

[3] A. Davie, *The approximation problem for Banach spaces*, Bull. London Math. Soc. **5** (1973), 261–266.

[4] J. Diestel, H. Jarchow, and A. Tongue, *Absolutely summing operators*, Cambridge University Press, Cambridge, 1995.

[5] E. Effros and U. Haagerup, *Lifting problems and local reflexivity for C^*-algebras*, Duke Math. J. **52** (1985), 103–128.

[6] T. Figiel, W. Johnson, and L. Tzafriri, *On Banach lattices and spaces having local unconditional structure, with applications to Lorentz function spaces*, J. Approx. Theory **13** (1975), 27–48.

[7] S. Heinrich, *Ultraproducts in Banach space theory*, J. Reine Angew. Math. **313** (1980), 72–104.

[8] G. Hardy, J. Littlewood, and G. Polya, *Inequalities*, Cambridge University Press, Cambridge, 1934.

[9] P. Harmand, D. Werner, and W. Werner, *M-ideals in Banach spaces and Banach algebras*, Springer-Verlag, New York, 1993.

[10] W. Johnson, *A complementably universal conjugate Banach space and its relation to the approximation problem*, Israel J. Math. **13** (1972), 301–310.

[11] ———, *Extensions of c_0*, Positivity **1** (1997), 55–74.

[12] W. Johnson and J. Lindenstrauss, *Basic concepts in the geometry of Banach spaces*, Handbook on the geometry of Banach spaces, North Holland, Amsterdam, 1999.

[13] J. Lindenstrauss, *On James' paper "separable conjugate spaces"*, Israel J. Math. **9** (1971), 279–284.

[14] J. Lindenstrauss and A. Pełczyński, *Absolutely summing operators on $\mathcal{L}_p$ spaces and their applications*, Studia Math. **29** (1968), 275–326.

[15] J. Lindenstrauss and H. Rosenthal, *The $\mathcal{L}_p$ spaces*, Israel J. Math. **7** (1969), 325–349.

[16] J. Lindenstrauss and L. Tzafriri, *Classical Banach spaces I*, Springer-Verlag, New York, 1977.

[17] ———, *Classical Banach spaces II*, Springer-Verlag, New York, 1979.

[18] W. Lusky, *A note on Banach spaces containing c_0 or C_∞*, J. Funct. Anal. **62** (1985), 1–7.

[19] V. Mascioni, *On the duality of the uniform approximation property in Banach spaces*, Illinois J. Math. **35** (1991), 191–197.

[20] T. Oikhberg, *Maximal operator spaces*, preprint.

[21] T. Oikhberg and H. Rosenthal, *Extension properties for the space of compact operators*, J. Funct. Anal. **179** (2001), 251–308.

[22] N. Ozawa, *On lifting property for universal C^* algebras of operator spaces*, preprint.

[23] V. Paulsen, *Completely bounded maps and dilations*, Pitman Res. Notes Math. Ser., vol.146, Wiley, New York, 1986.

[24] A. Pełczyński and H. Rosenthal, *Localization techniques in l_p spaces*, Studia Math. **52** (1974/75), 263–289.

[25] G. Pisier, *Factorization of linear operators and geometry of Banach spaces*, CBMS Reg. Conf. Ser. in Math., no. 60, Amer. Math. Soc., Providence, 1986.

[26] ———, *Similarity problems and completely bounded maps*, Springer-Verlag, New York, 1996.

[27] ———, *An introduction to the theory of operator spaces*, preprint.

[28] G. Robertson and R. Smith, *Liftings and extensions of maps on C^*-algebras*, J. Operator Theory **21** (1989), 117–131.

[29] H. Rosenthal, *The complete separable extension property*, J. Operator Theory **43** (2000), 329–374.

[30] A. Sobczyk, *Projection of the space (m) on its subspace (c_0)*, Bull. Amer. Math. Soc. **47** (1941), 938–947.

[31] A. Szankowski, *$B(H)$ does not have the approximation property*, Acta Math. **147** (1981), 89–108.

[32] W. Veech, *Short proof of Sobczyk's theorem*, Proc. Amer. Math. Soc. **28** (1971), 271–289.

[33] M. Zippin, *The separable extension problem*, Israel J. Math. **26** (1977), 372–387.

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