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HOMOLOGY LENS SPACES IN TOPOLOGICAL 4-MANIFOLDS

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Abstract. For a closed 4-manifold X^4 and closed 3-manifold M^3 we investigate the smallest integer n (perhaps $n = \infty$) such that M^3 embeds in $\#_n X^4$, the connected sum of n copies of X^4. It is proven that any lens space (or homology lens space) embeds topologically locally flatly in $\#_2(\mathbf{C}P^2 \# \overline{\mathbf{C}P}^2)$, in $\#_4 S^2 \times S^2$ and in $\#_8 \mathbf{C}P^2$.

1. Introduction

For any closed 4-manifold X^4 and closed 3-manifold M^3 one can define a simple numerical invariant, the X^4-genus of M^3, denoted $g_{X^4}(M^3)$, by saying $g_{X^4}(M^3) \leq n$ provided M^3 embeds in $\#_n X^4$, the connected sum of n copies of X^4. It follows that $0 \leq g_{X^4}(M^3) \leq \infty$ and that $g_{X^4}(M^3) = 0$ if M^3 embeds in S^4, while $g_{X^4}(M^3) = \infty$ if M^3 embeds in no $\#_n X^4$ for any integer n. Here we understand our manifolds to be topological manifolds and our embeddings to be locally flat. There is also an analogous invariant $g_{X^4}^{DIFF}(M^3)$ where one requires X^4 and the embedding to be smooth, and we endow the 3-manifold with its essentially unique smooth structure. Certainly $g_{X^4}(M^3) \leq g_{X^4}^{DIFF}(M^3)$ for any smooth 4-manifold.

For example consider the case $X = S^2 \times S^2$. It is known that any closed orientable 3-manifold M^3 embeds smoothly in some $\#_n S^2 \times S^2$, so that $g_{S^2\times S^2}^{DIFF}(M^3) < \infty$. It will be shown here that $g_{S^2\times S^2}(L(p,q)) \leq 4$ for every lens space $L(p,q)$. We doubt that the DIFF $S^2 \times S^2$-genus of lens spaces is bounded.

It is also known that any lens space $L(p,q)$ embeds smoothly in some $\#_n \mathbf{C}P^2$, so that $g_{\mathbf{C}P^2}^{DIFF}(L(p,q)) < \infty$. But not every 3-manifold embeds smoothly in some $\#_n \mathbf{C}P^2$, by gauge-theoretic considerations. For example, the Poincaré homology 3-sphere does not embed smoothly in $\#_n \mathbf{C}P^2$ for any positive integer n. It will be shown here that $g_{\mathbf{C}P^2}(L(p,q)) \leq 8$. It is not

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known whether this is a sharp bound, but it is known [2] that $g_{\mathbf{C}P^2}(L(8k,1)) = 5$ for $k > 2$. Again, we doubt very much that $g^{DIFF}_{\mathbf{C}P^2}(L(p,q))$ is bounded.

One interest in the embedding questions considered here stems from work of F. Fang [3] who showed that if a 3-manifold M^3 embeds in $\#_n\mathbf{C}P^2$, then the open 4-manifold $M^3 \times \mathbf{R}$ admits uncountably many smooth structures. See [3] for more references to the problem of constructing uncountably many smooth structures on suitable open 4-manifolds.

Now when a 3-manifold M^3 embeds in a closed, simply connected 4-manifold X^4, M^3 bounds in X^4 (since $H^3(X^4) \approx H_1(X^4) = 0$), splitting X^4 into two compact submanifolds U and V with $\partial U = M = \partial V$. Taking into account orientations, assuming that U and V inherit orientations from one on X^4 and that M^3 (which must then admit an orientation) is oriented, then we can assume that $\partial U = M^3$, while $\partial V = -M^3$. Thus, to show that M^3 embeds in any particular X^4 it suffices to show that M^3 and $-M^3$ bound appropriate (preferably simply connected) manifolds U and V such that $U \bigcup_M V \cong X^4$. To recognize $U \bigcup_M V$ as X^4 it helps to be in the topological category, where one can apply Freedman's classification in terms of the intersection pairing (and the Kirby-Siebenmann triangulation obstruction).

Our starting point will be the following result.

Theorem 1.1. *Any 3-dimensional homology lens space $L(p,q)$ bounds a compact, simply connected, topological 4-manifold with $b_2 \leq 2$.*

By exercising due care we can show that a homology lens space always bounds suitable 4-manifolds to show that it embeds in certain relatively small connected sums.

Theorem 1.2. *Any 3-dimensional homology lens space $L(p,q)$ embeds topologically locally flatly in $\#_2(\mathbf{C}P^2 \# \overline{\mathbf{C}P}^2)$, in $\#_4 S^2 \times S^2$, and in $\#_8\mathbf{C}P^2$.*

If p is odd or if $q \equiv \pm 1 \mod p$, then $L(p,q)$ actually embeds in $\#_5\mathbf{C}P^2$. But, as noted above, it is known that $L(8k,1)$ does not embed in $\#_4\mathbf{C}P^2$ when $k > 2$. (See [2].)

Conjecture 1.3. *For any simply connected 4-manifold X^4 and for any homology lens space $L(p,q)$, the X^4-genus $g_{X^4}(L(p,q)) < \infty$.*

For X^4 indefinite, this follows easily from the present work, so it essentially reduces to considering X^4 with a definite intersection pairing that does not split nontrivially as an orthogonal sum. The same conjecture may be posed for any rational homology 3-sphere in place of the lens space $L(p,q)$. In contrast F. Fang [3] has shown that there exist 3-manifolds with large first Betti number b_1 that do not embed in any positive definite, simply connected, 4-manifold.

All the results about topological embeddings of lens spaces derived in this paper apply equally well to any homology lens space, that is, a 3-manifold M^3 with $H_1(M^3)$ finite cyclic. In particular, a "homology lens space $L(p, q)$" refers to any 3-manifold with $H_1 = \mathbf{Z}_p$ and linking form equivalent to $((p - q)/q)$, or, after changing orientation, (q/p).

2. Intersection pairings and linking forms

The crucial invariants we must study are the linking form associated with a 3-manifold, the intersection pairing associated to a 4-manifold bounded by the 3-manifold, and the relationship between the two.

2.1. Intersection numbers and linking numbers. We refer to the classic book of Seifert and Threlfall [1934], Sections 73-77, for generalities and basic definitions of intersection numbers and linking numbers, in the context of polyhedral manifolds.

2.2. Intersection pairings and linking forms. By an *abstract intersection pairing* we understand a finitely generated free abelian group F together with a symmetric bilinear mapping $\mathcal{S} : F \times F \to \mathbf{Z}$ ($\mathcal{S}$ for "Schnittzahlen"). We will only be concerned with nondegenerate intersection pairings, such that the associated adjoint homomorphism $\operatorname{ad}\mathcal{S} : F \to \operatorname{Hom}(F, \mathbf{Z})$ has nonzero determinant, or equivalently has finite index image. It is often convenient to describe such pairings by square integer matrices that give the adjoint $\operatorname{ad}\mathcal{S}$ with respect to some basis of F and the corresponding dual basis for $\operatorname{Hom}(F, \mathbf{Z})$. Our main geometric example of such an intersection pairing is the second homology of a simply connected 4-manifold with connected boundary, where the boundary is a rational homology 3-sphere, under the usual intersection number. On the algebraic side any symmetric integer matrix with nonzero determinant determines such an intersection pairing. And all such algebraic intersection pairings can be realized by smooth 4-manifolds, by attaching 2-handles to the 4-ball along a suitable framed link.

By an *abstract linking form* we understand a finite abelian group G together with a symmetric bilinear mapping $\mathcal{V} : G \times G \to \mathbf{Q}/\mathbf{Z}$ ($\mathcal{V}$ for "Verschlingungszahlen"). We will only be concerned with nonsingular linking forms, such that the associated adjoint homomorphism $\operatorname{ad}\mathcal{V} : G \to \operatorname{Hom}(G, \mathbf{Q}/\mathbf{Z})$ is an isomorphism. One can describe such a linking form by a suitable matrix of rational numbers, by choosing, say, generators for G, and indicating the pairings of pairs of these generators. See also the end of this section.

Our main geometric example of such a linking form is the linking form of a 3-manifold. Let M^3 be a closed oriented 3-manifold and let $T_1(M^3)$ denote the torsion subgroup of the first homology $H_1(M^3)$, with integer coefficients. We define the classical linking form $\mathcal{V}_{M^3} : T_1(M^3) \times T_1(M^3) \to \mathbf{Q}/\mathbf{Z}$ as follows: Suppose that $\alpha, \beta \in T_1(M^3)$. Represent α and β by disjoint 1-cycles, or even simple closed curves A and B. There is a positive integer n such that $n\beta = 0$ in $H_1(M^3)$. Thus there is a 2-cycle C with $\partial C = nB$. Then $\mathcal{V}_{M^3}(\alpha,\beta) = A\cdot C/n$ in $\mathbf{Q}/\mathbf{Z}$. One argues that the linking form is well-defined, independent of all the choices made in the course of its definition. The linking form is symmetric and is nonsingular, in the sense that the associated homomorphism $T_1(M^3) \to \mathrm{Hom}(T_1(M^3), \mathbf{Q}/\mathbf{Z})$ is an isomorphism, by Poincaré Duality and Universal Coefficients.

2.3. Presentation of linking forms. Suppose that $M^3 = \partial W^4$, where W^4 is a compact oriented *simply connected* 4-manifold. Every closed oriented 3-manifold bounds such a 4-manifold. Consider the homology long exact sequence of the pair (W^4, M^3):

$$0 \longrightarrow H_2(M^3) \xrightarrow{i} H_2(W^4) \xrightarrow{j} H_2(W^4, M^3) \xrightarrow{\partial} H_1(M^3) \longrightarrow 0$$

Lemma 2.1. *If $j(\xi_i) = n_i\alpha_i$ (i=1,2) where $n_1n_2 \neq 0$, then $\mathcal{V}_{M^3}(\partial\alpha_1, \partial\alpha_2) = \frac{-1}{n_1n_2}\mathcal{S}(\xi_1,\xi_2)$ in $\mathbf{Q}/\mathbf{Z}$, where $\mathcal{S}$ denotes the intersection pairing on $H_2(W^4)$. Also, if $u \in H_2(M^3)$ and $\eta \in H^2(W^4) = H_2(W^4, M^3)$, then $\partial(\eta)\cdot u = \eta(i(u))$.*

The last condition is null in the case the boundary is a rational homology sphere or, equivalently, the intersection pairing in nondegenerate. One says that $H_2(W^4)$ together with the intersection pairing $\mathcal{S}$ *presents* $H_1(M^3)$. In general a nondegenerate intersection pairing $(F, \mathcal{S})$ *presents* a linking form $(G, \mathcal{V})$ if there is a short exact sequence

$$0 \longrightarrow F \xrightarrow{\mathrm{ad}\,\mathcal{S}} \mathrm{Hom}(F, Z) \xrightarrow{\partial} G \longrightarrow 0$$

such that if $\mathrm{ad}\,\mathcal{S}(\xi_i) = n_i\alpha_i$ (i=1,2) where $n_1n_2 \neq 0$, then $\mathcal{V}(\partial\alpha_1, \partial\alpha_2) = \frac{-1}{n_1n_2}\mathcal{S}(\xi_1,\xi_2)$ in $\mathbf{Q}/Z$. Another way of writing this (compare Turaev [1984], Section 3) is $\mathcal{V}(\partial\alpha_1, \partial\alpha_2) = -\mathcal{S}^{-1}(\alpha_1, \alpha_2)$ in $\mathbf{Q}/\mathbf{Z}$. Here $\mathcal{S}^{-1}$ denotes the restriction of the rational bilinear pairing whose associated homomorphism is $(\mathrm{ad}\,\mathcal{S} \otimes \mathbf{Q})^{-1} : \mathrm{Hom}(F, \mathbf{Q}) = \mathrm{Hom}(F, \mathbf{Z}) \otimes \mathbf{Q} \to F \otimes \mathbf{Q}$.

One consequence of Wall's analysis [13] of linking forms is that any abstract linking form is presented by some abstract intersection pairing. For example, (q/p) is presented by a matrix whose size depends on a certain continued fraction decomposition of q/p. Geometrically this corresponds to expressing a lens space as the boundary of an appropriate plumbing manifold, whose second Betti number depends on the length of the corresponding continued fraction. One of our goals is to bound the rank of such presentations.

2.4. Topological realization of algebraic presentations. Here we describe known conditions for translating a presentation into a 4-manifold with given boundary.

Given an abstract intersection pairing, described, say, by a symmetric integer matrix with nonvanishing determinant, then this intersection pairing can easily be realized by a compact simply connected smooth 4-manifold obtained by attaching 2-handles to the 4-ball along any framed link whose linking matrix is the given symmetric integer matrix. In this way one hardly controls the corresponding boundary, except to say that its linking pairing is determined as above. It turns out that in the topological category one can say much more, actually prescribing the boundary in advance. Boyer [1] and Stong [11] have independently proven the following result, which extends Freedman's original realization result for closed simply connected 4-manifolds.

THEOREM 2.2. *If the geometric linking form* $(H_1(M^3), \mathcal{V})$ *is presented by an abstract intersection pairing* $(F, \mathcal{S})$*, then* M^3 *is the boundary of a simply connected topological 4-manifold* X^4 *with* $H_2(X^4) = F$ *and* $\mathcal{S}$ *as intersection pairing.*

One corollary of this result is that if a given 3-manifold N^3 bounds a simply connected 4-manifold Y^4, then any other 3-manifold M^3 with the same linking form bounds a topological 4-manifold X^4 with the same intersection pairing as Y^4. Going further, the work of Boyer actually characterized when two simply connected 4-manifolds with given boundary are homeomorphic.

So, in the topological category, the question of what kinds of simply connected 4-manifolds have a given boundary is in fact reduced to a purely algebraic one about existence of suitable presentations of linking forms.

REMARK 2.3. When the intersection pairing on F has odd type, the work of Boyer and of Stong shows that both a zero and a nonzero Kirby-Siebenman stable triangulation obstruction in $H^4(X^4, M^3; \mathbf{Z}_2)$ can be realized.

Both Boyer and Stong dealt with general closed oriented 3-manifolds, not just rational homology spheres, as considered here. In the present case a rather simpler proof is available, which we sketch for the reader's convenience (cf. Boyer [1], Section 8).

Sketch proof of Theorem 2.2. Given a presentation $(F, \mathcal{S})$ of the linking form on $H_1(M^3)$, we can realize $(F, \mathcal{S})$ by a framed link in the 3-sphere. Attaching 2-handles to the 4-ball along this framed link produces a smooth, compact, simply connected 4-manifold V^4 with intersection pairing $(F, \mathcal{S})$. The boundary $\partial V^4 = N^3$ is a 3-manifold with a linking form equivalent to that of the given 3-manifold M^3. Passing to the dual handle decomposition, we see that V^4 can be described as being obtained from $N^3 \times I$ by attaching 2-handles along a framed link in $N^3 = N^3 \times \{1\}$, and then capping off with a 4-handle. We can choose a framed link in M^3 that mirrors this link in N^3, in the sense that the elements of the link represent corresponding elements in first homology and all linking numbers and framings agree with those in N^3. If we add 2-handles to $M^3 \times I$ along this framed link in $M^3 \times \{0\}$, we obtain a compact, smooth 4-manifold W^4, with one boundary component M^3 and the other boundary component a homology 3-sphere Σ^3. By Freedman we can cap off Σ^3 with a compact contractible topological 4-manifold Δ^4. In particular $X^4 = W^4 \cup \Delta^4$ is a compact simply connected, topological 4-manifold with boundary M^3 and intersection pairing equivalent to $(F, \mathcal{S})$. $\square$

3. Minimal presentations of (q/p)

In this section we will derive smallest possible presentations of the indecomposable linking forms (q/p).

3.1. Rank 1 forms. Here we determine which rank 1 linking forms are presented by rank 1 intersection pairings.

A rank 1 linking form on $\mathbf{Z}/p$ can be described by a 1×1 matrix (q/p), where q is prime to p and q is well-defined up adding a multiple of p and multiplying by a square of a unit mod p. It is the linking form of the lens space $L(p,q)$. Such a linking form can always be presented by some intersection pairing. One way to do this is to develop a continued fraction expansion of $p/(p-q)$, as in Hirzebruch et al. [7]. Say it is $[a_1, \ldots, a_n]$. This defines a plumbing 4-manifold $P^4[a_1, \ldots, a_n]$, which has an intersection pairing of rank n and has oriented boundary $L(p,q)$. As p and q vary, the rank n does not stay bounded. We seek a way of controlling the rank.

The 1×1 intersection pairing (p) is positive definite and realizes the linking form $(-1/p)$. We state this as follows:

THEOREM 3.1. *The linking form (q/p) is realized by a rank 1 matrix $[a]$ if and only if $\mp q$ is a quadratic residue mod p (and $a = \pm p$).*

3.2. Rank 2 forms. We begin the study of rank 2 presentations with a general realization statement, which provided the original starting point for this paper.

THEOREM 3.2. *If p and q are relatively prime integers, then the abstract linking form $(q/p) : \mathbf{Z}/p \times \mathbf{Z}/p \to \mathbf{Q}/\mathbf{Z}$ is presented by a non-degenerate rank 2 abstract intersection pairing $\mathcal{S} : \mathbf{Z}^2 \times \mathbf{Z}^2 \to \mathbf{Z}$ of odd type.*

Proof. For the purposes of the proof we may assume that p is positive. Note that we may also replace q by $-q$ if we wish. If b and d are integers, then the matrix

$$G = \begin{bmatrix} q/p & b \\ b & d \end{bmatrix}$$

understood mod $\mathbf{Z}$ also gives the linking form (q/p). If one can choose the integers b and d so that $\det G = \pm 1/p$, then G^{-1} is integral and $\mathcal{S} = -G^{-1}$ presents (q/p).

Now $\det G = dq/p - b^2$, so this amounts to solving the equation $dq - pb^2 = \pm 1$ for b and d. Actually we have a little more freedom, since q is only defined modulo p.

One of q, $-q$, $p+q$, or $3p+q$ must be congruent to 3 mod 4. Thus we may assume that $q \equiv 3 \bmod 4$. Now consider the arithmetic progression $q+4np$, $n = 1, 2, \ldots$. By Dirichlet's theorem on primes in an arithmetic progression (see [8], for example), some $q+4np$ is prime. Therefore we can assume that q is a prime congruent to 3 mod 4. But then, since -1 is not a square mod q, either p or $-p$ must be a square mod q and the proposition follows. □

Remark 3.3. Note that we have shown that $(\pm q/p)$ is presented by the intersection pairing

$$\mathcal{S} = -G^{-1} = \begin{bmatrix} -dp & bp \\ bp & -q \end{bmatrix}$$

and in particular has a diagonal entry that is negative and odd.

4. First topological applications

Here we combine the theorem of Boyer and Stong with the algebraic result of the preceding section to find small coboundaries for lens spaces and other 3-manifolds.

Corollary 4.1. *If p is a positive integer and q is an integer prime to p, then any homology lens space $L(p,q)$ is the boundary of a simply connected topological 4-manifold, with $b_2 = 1$ if and only if $\pm q$ is a quadratic residue* mod p.

Corollary 4.1 is originally due to O. Saeki [9], who studied homology lens spaces that bound simply connected topological or differentiable 4-manifolds with $b_2 = 1$, without the present regard for orientations. In particular he found lens spaces that bound such 4-manifolds topologically but not smoothly, using first μ invariants and also Donaldson theory. Earlier R. Fintushel and R. Stern [4] studied the problem of when a lens space bounds such a 4-manifold *smoothly* and found both further obstructions and some explicit constructions.

Corollary 4.2. *If p is a positive integer and q is an integer prime to p, then any homology lens space $L(p,q)$ is the boundary of a simply connected topological 4-manifold, with $b_2 \leq 2$.*

We note that the rank 2 presentation matrix above is necessarily of odd type, as the diagonal entry $\pm q$ is odd.

Corollary 4.3. *Any homology lens space $L(p,q)$ admits a topological embedding in $\#_2(\mathbf{C}P^2 \# \overline{\mathbf{C}P}^2)$.*

Proof. We have seen that $L(p,q)$ bounds a simply connected 4-manifold of odd type and with $b_2 = 2$. The double of such a 4-manifold is precisely

$\#_2(\mathbf{C}P^2\# \overline{\mathbf{C}P}^2)$. This follows from Freedman's classification of simply connected topological 4-manifolds together with the observation that the mod 2 Kirby-Siebenmann invariant of the double vanishes by additivity. □

5. Even intersection pairings and applications

Here we investigate presenting a linking form by an *even* intersection pairing. We use an algebraic analog of the geometric notions of "blow-up" and "blow-down". If V is an intersection pairing, then we will refer to $V \oplus \langle +1 \rangle$ and $V \oplus \langle -1 \rangle$ as being blow-ups of V. If $v \in V$ and $v \cdot v = \pm 1$, then $v^\perp = \{u \in V : u \cdot v = 0\}$ is an orthogonal summand of V and we say that $v^\perp$ is obtained from V by blowing down v. Notice in particular that V and its blow-ups and blow-downs all present the same linking form, as would the orthogonal sum of V with any unimodular pairing.

Recall that if V is an intersection pairing, then an element $v \in V$ is said to be *characteristic* if one has $v \cdot w \equiv w \cdot w \mod 2$ for all $w \in V$. An element that is not characteristic is called *ordinary.* It is easy to see that characteristic elements always exist. (Geometrically, the mod 2 choices for characteristic elements correspond to spin structures on the boundary manifold.) A key point is that if $v \in V$ is characteristic, then the induced pairing on $v^\perp$ is of even type. Similarly, if V has odd type and $v \in V$ is not characteristic, then the induced pairing on $v^\perp$ is again of odd type.

Proposition 5.1. *The rank 1 linking form (q/p) is presented by a rank 4 intersection pairing of even type.*

Proof. We know that (q/p) is presented by a rank 2 pairing V of odd type. Now V contains characteristic elements $v \in V$. If one could choose v such that $v \cdot v = \pm 1$, then (q/p) would be presented by the rank 1 pairing $v^\perp$, which would be of even type. This cannot happen in general, since, in particular, we would need p even. But in any case consider $V \oplus \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$. Characteristic elements in $\left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$ are of the form $w = 2ke_1 + 2\ell e_2$ and $w \cdot w = 4k\ell$, which can be any multiple of 4. Thus in $V \oplus \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$ there are characteristic elements of the form $v + w$ such that $(v + w)^2 = -1$, 0, 1, or 2. If $(v + w)^2 = \pm 1$, then pass to the orthogonal complement $(v + w)^\perp$, which is even, of rank 3, and presents (q/p). Otherwise, first add on an additional $\langle +1 \rangle$ or $\langle -1 \rangle$ and an additional basis vector e to $v + w$ to get a characteristic element $u = v + w + e$ in $V \oplus \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right] \oplus (\pm 1)$, such that $u^2 = \pm 1$. Passing to $u^\perp$ then provides a rank 4 even pairing presenting (q/p), as required. □

Corollary 5.2. *If p is a positive integer and q is an integer prime to p, then any homology lens space $L(p, q)$ admits a topological embedding in $\#_n\, S^2 \times S^2$, $n \leq 4$.*

Proof. We have seen that $L(p, q)$ bounds a simply connected 4-manifold of even type and with $n = b_2 \leq 4$. The double of such a 4-manifold is precisely $\#_n\ S^2 \times S^2$, by Freedman's classification of simply connected topological 4-manifolds. □

6. Definite intersection pairings and applications

Here we investigate presenting a linking form by a *definite* intersection pairing. It is not too hard to do this, but we have to exert some effort to keep the resulting rank as small as possible. We begin with some lemmas useful in applying the blow-up/blow-down procedure introduced in the preceding section.

LEMMA 6.1. *Suppose that V is an odd intersection pairing that is not positive definite. Then there is $v \in V$ such that $v \cdot v < 0$ and $v \cdot v$ is odd.*

Proof. There is a $u \in V$ such that $u \cdot u < 0$ and there is $w \in V$ such that $w \cdot w$ is odd. We only need to consider further the case that $u \cdot u$ is even and $w \cdot w > 0$; otherwise we are done. Replacing w by $-w$ if necessary we may also assume that $u \cdot w \leq 0$.

Now let $v = w + ku$, $k \in \mathbf{Z}$. We compute that

$$v \cdot v = w \cdot w + 2ku \cdot w + k^2 u \cdot u$$

from which it is clear that $v \cdot v$ is odd and that for sufficiently large k we also have $v \cdot v < 0$. □

LEMMA 6.2. *Suppose that V is an odd intersection pairing that is not positive definite. Then there is $u \in V$ such that $u \cdot u = -n < 0$, u is ordinary (actually 2-divisible), and $n - 1$ is not of the form $4^a(8b + 7)$.*

Proof. There is a $v \in V$ such that $v \cdot v = -(2k + 1) < 0$ for some integer k. Set $u = 2v$. Then $u \cdot u = 4v \cdot v < 0$. Also u is ordinary, since there is some $w \in V$ such that $w \cdot w$ is odd, while $u \cdot w = 2v \cdot w$ is even. Finally, setting $n = -u \cdot u$, we have $n - 1 = 4(2k + 1) - 1 = 8k + 3$, which is never of the form $4^a(8b + 7)$. □

COROLLARY 6.3. *If V is an intersection pairing of odd type that is not positive definite, then there is $v \in V \oplus 3\langle +1 \rangle$ such that $v \cdot v = -1$ and v is ordinary.*

Proof. By the preceding result there is $u \in V$ such that $u \cdot u = -n < 0$, u is ordinary (and, in fact, 2-divisible), and $n - 1$ is not of the form $4^a(8b + 7)$. By number theory (see [6], for example), $n - 1$ can be written as a sum of 3 squares, $n - 1 = a_1^2 + a_2^2 + a_3^2$. Set $v = u + a_1 e_1 + a_2 e_2 + a_3 e_3$, where e_1, e_2, and e_3 form a standard orthonormal basis for $3\langle +1 \rangle$. Then $v \cdot v = u \cdot u + a_1^2 + a_2^2 + a_3^2 = -n + n - 1 = -1$ and v is ordinary since there is

$w \in V \subset V \oplus 3\langle +1 \rangle$ such that $w \cdot w$ is odd, but $v \cdot w = u \cdot w$ is even since u is 2-divisible. □

PROPOSITION 6.4. *The rank 1 linking form (q/p) is presented by a positive definite intersection pairing of rank ≤ 6 and of odd type.*

Proof. We know that (q/p) is presented by a rank 2 pairing V of odd type. If V happens to be positive definite, then we are done. There are two remaining cases, depending on whether V is indefinite or negative definite.

First suppose that V is indefinite. Then V contains an ordinary element $v \in V$ such that $v \cdot v = -n < 0$. If we could choose v such that $v \cdot v = -1$, then (q/p) would be presented by the rank 1 form $v^{\perp}$. In general this is impossible to achieve. We can, however, blow up the form, passing to $V \oplus 3\langle +1 \rangle$, which still represents (q/p). Here we can find an ordinary $v \in V \oplus 3\langle +1 \rangle$ such that $v \cdot v = -1$. Passing to $v^{\perp}$, we obtain a positive definite integral pairing of rank $2 + 3 - 1 = 4$ presenting (q/p).

Finally we must consider the case when (q/p) is presented by a negative definite rank 2 pairing V of odd type. In this situation we do much the same as before. We need to blow down twice, however. In order to do this, we must blow up $3\langle +1 \rangle$ *twice*. The net effect is to produce a positive definite pairing of rank $2+3+3-1-1 = 6$ and odd type presenting (q/p). It should be noted that in any particular case we seem to be able to do better than this, and we know of no case where the full rank 6 possibility is actually required. □

REMARK 6.5. If (q/p) is represented by a rank 2 positive definite odd pairing, then $(-q/p)$ is represented by a rank 2 negative definite odd pairing, hence by a rank ≤ 6 positive definite odd pairing. (And, of course, if (q/p) is represented by a rank 2 negative definite odd pairing, then $(-q/p)$ is represented by a rank 2 positive definite odd pairing, while $(-q/p)$ is then represented by a rank ≤ 6 positive definite odd pairing.) If (q/p) is represented by a rank 2 indefinite odd pairing, then $(-q/p)$ is also represented by a rank 2 indefinite odd pairing, hence both are represented by rank ≤ 5 positive definite odd pairings, according to the proof of the theorem.

COROLLARY 6.6. *If p is a positive integer and q is an integer prime to p, then any homology lens space $L(p, q)$ admits a topological embedding in $\#_n \mathbf{C}P^2$, for some $n \leq 8$.*

Proof. Replacing q by $-q$ if necessary, we know that either $L(p, q)$ bounds a simply connected 4-manifold of odd type and with positive definite intersection pairing and $b_2 = 2$, while $-L(p, q)$ bounds a simply connected 4-manifold with positive definite intersection pairing and $b_2 \leq 6$; or $L(p, q)$ and $-L(p, q)$ both bound simply connected 4-manifolds with positive definite intersection pairings of odd type and $b_2 \leq 4$. In either of the two cases, the union of the two 4-manifolds along the lens space yields a simply connected 4-manifold, with $b_2 \leq 8$. Thus in either case we obtain an embedding of $L(p,q)$ in a closed, simply connected 4-manifold with positive definite intersection pairing of odd type and rank at most 8. It follows from the classification of unimodular intersection pairings of low rank that the intersection pairing is diagonalizable. By Freedman's classification theorem, it only remains to be sure the Kirby-Siebenmann stable triangulation obstruction vanishes. This is taken care of by Remark 2.3 above, since we can, if necessary, change the Kirby-Siebenmann invariant of one of the two pieces to make the global invariant vanish. Then Freedman's classification theorem shows that we have $\#_n \mathbf{C}P^2$. □

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